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SOME SLATER'S TYPE TRACE INEQUALITIES FOR CONVEX FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. Some trace inequalities of Slater type for convex functions of selfadjoint operators in Hilbert spaces under suitable assumptions for the involved operators are given. Applications for particular cases of interest are also provided.

1. Introduction

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f : I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x,\ y \in \mathring{I}$ and $x < y$, then $f'_-(x) \leq f'_+(x) \leq f'_-(y) \leq f'_+(y)$ which shows that both f'_- and f'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \to \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$f(x) \geq f(a) + (x-a)\varphi(a) \text{ for any } x, a \in I.$$

It is also well known that if f is convex on I, then ∂f is nonempty, f'_-, $f'_+ \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_-(x) \leq \varphi(x) \leq f'_+(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If f is differentiable and convex on $\mathring{I}$, then $\partial f = \{f'\}$.

The following result is well known in the literature as *Slater inequality:*

Theorem 1 (Slater, 1981, [34]). *If $f : I \to \mathbb{R}$ is a nonincreasing (nondecreasing) convex function, $x_i \in I$, $p_i \geq 0$ with $P_n := \sum_{i=1}^n p_i > 0$ and $\sum_{i=1}^n p_i\varphi(x_i) \neq 0$, where $\varphi \in \partial f$, then*

$$\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) \leq f\left(\frac{\sum_{i=1}^n p_i x_i\varphi(x_i)}{\sum_{i=1}^n p_i\varphi(x_i)}\right). \tag{1.1}$$

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As pointed out in [8, p. 208], the monotonicity assumption for the derivative φ can be replaced with the condition

$$\frac{\sum_{i=1}^{n} p_i x_i \varphi(x_i)}{\sum_{i=1}^{n} p_i \varphi(x_i)} \in I, \tag{1.2}$$

which is more general and can hold for suitable points in I and for not necessarily monotonic functions.

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle .,. \rangle)$. The *Gelfand map* establishes a $*$-isometric isomorphism Φ between the set $C(\mathrm{Sp}(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $\mathrm{Sp}(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [17, p. 3]):

For any $f, g \in C(\mathrm{Sp}(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;

(ii) $\Phi(fg) = \Phi(f) \Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in \mathrm{Sp}(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in \mathrm{Sp}(A)$.

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(\mathrm{Sp}(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $\mathrm{Sp}(A)$, then $f(t) \geq 0$ for any $t \in \mathrm{Sp}(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $\mathrm{Sp}(A)$ then the following important property holds:

$$f(t) \geq g(t) \text{ for any } t \in \mathrm{Sp}(A) \text{ implies that } f(A) \geq g(A) \tag{P}$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [17] and the references therein. For other results, see [28], [22] and [24].

The following result that provides an operator version for the Jensen inequality and can be found in Mond & Pečarić [26] (see also [17, p. 5]):

Theorem 2 (Jensen's inequality)**.** *Let A be a selfadjoint operator on the Hilbert space H and assume that $\mathrm{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$f(\langle Ax, x \rangle) \leq \langle f(A) x, x \rangle, \tag{MP}$$

for each $x \in H$ with $\|x\| = 1$.

As a special case of Theorem 2 we have the following Hölder-McCarthy inequality:

Theorem 3 (Hölder-McCarthy, 1967, [23])**.** *Let A be a selfadjoint positive operator on a Hilbert space H. Then*

(i) $\langle A^r x, x \rangle \geq \langle Ax, x \rangle^r$ for all $r > 1$ and $x \in H$ with $\|x\| = 1$;

(ii) $\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r$ for all $0 < r < 1$ and $x \in H$ with $\|x\| = 1$;

(iii) If A is invertible, then $\langle A^r x, x \rangle \geq \langle Ax, x \rangle^r$ for all $r < 0$ and $x \in H$ with $\|x\| = 1$.

The following result that provides a reverse of the Jensen inequality has been obtained in [11]:

Theorem 4 (Dragomir, 2008, [11])**.** *Let I be an interval and $f: I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operators on the Hilbert space H with* $\operatorname{Sp}(A) \subseteq [m, M] \subset \mathring{I}$, *then*

$$(0 \leq) \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \leq \langle f'(A)Ax, x\rangle - \langle Ax, x\rangle \langle f'(A)x, x\rangle, \tag{1.3}$$

for any $x \in H$ with $\|x\| = 1$.

Perhaps more convenient reverses of (MP) are the following inequalities that have been obtained in the same paper [11]:

Theorem 5 (Dragomir, 2008, [11])**.** *Let I be an interval and $f: I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operators on the Hilbert space H with* $\operatorname{Sp}(A) \subseteq [m, M] \subset \mathring{I}$, *then*

$$\begin{aligned} &(0 \leq) \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \\ &\leq \begin{cases} \frac{1}{2}(M-m)\left[\|f'(A)x\|^2 - \langle f'(A)x, x\rangle^2\right]^{1/2} \\ \\ \frac{1}{2}(f'(M) - f'(m))\left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2} \end{cases} \\ &\leq \frac{1}{4}(M-m)(f'(M) - f'(m)), \end{aligned} \tag{1.4}$$

for any $x \in H$ with $\|x\| = 1$.

We also have the inequality

$$\begin{aligned} &(0 \leq) \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \\ &\leq \frac{1}{4}(M-m)(f'(M) - f'(m)) \\ &- \begin{cases} [\langle Mx - Ax, Ax - mx\rangle \langle f'(M)x - f'(A)x, f'(A)x - f'(m)x\rangle]^{\frac{1}{2}}, \\ \\ \left|\langle Ax, x\rangle - \frac{M+m}{2}\right| \left|\langle f'(A)x, x\rangle - \frac{f'(M)+f'(m)}{2}\right| \end{cases} \\ &\leq \frac{1}{4}(M-m)(f'(M) - f'(m)), \end{aligned} \tag{1.5}$$

for any $x \in H$ with $\|x\| = 1$.

Moreover, if $m > 0$ and $f'(m) > 0$, then we also have

$$\begin{aligned} &(0 \leq) \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \\ &\leq \begin{cases} \frac{1}{4}\frac{(M-m)\left(f'(M)-f'(m)\right)}{\sqrt{Mmf'(M)f'(m)}} \langle Ax, x\rangle \langle f'(A)x, x\rangle, \\ \\ \left(\sqrt{M} - \sqrt{m}\right)\left(\sqrt{f'(M)} - \sqrt{f'(m)}\right)[\langle Ax, x\rangle \langle f'(A)x, x\rangle]^{\frac{1}{2}}, \end{cases} \end{aligned} \tag{1.6}$$

for any $x \in H$ with $\|x\| = 1$.

In [13] we obtained the following operator version for Slater's inequality as well as a reverse of it:

Theorem 6 (Dragomir, 2008, [13])**.** *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operator on the Hilbert space H with* $\operatorname{Sp}(A) \subseteq [m, M] \subset \mathring{I}$ *and $f'(A)$ is a positive invertible operator on H then*

$$0 \le f\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right) - \langle f(A)x, x\rangle \le f'\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right)\left[\frac{\langle Af'(A)x, x\rangle - \langle Ax, x\rangle\langle f'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right], \tag{1.7}$$

for any $x \in H$ with $\|x\| = 1$.

For other similar results, see [13].

In order to state other new results on Slater type trace inequalities we need some preliminary facts as follows.

2. Some Facts on Trace of Operators

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I}\|Ae_i\|^2 < \infty. \tag{2.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I}\|Ae_i\|^2 = \sum_{j\in I}\|Af_j\|^2 = \sum_{j\in I}\|A^*f_j\|^2 \tag{2.2}$$

showing that the definition (2.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of Hilbert-Schmidt operators in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I}\|Ae_i\|^2\right)^{1/2} \tag{2.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H. This definition does not depend on the choice of the orthonormal basis.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (2.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 7. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I}\langle Ae_i, Be_i\rangle = \sum_{i\in I}\langle B^*Ae_i, e_i\rangle \tag{2.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$*;*

(ii) We have the inequalities

$$\|A\| \leq \|A\|_2 \tag{2.5}$$

for any $A \in \mathcal{B}_2(H)$ *and*

$$\|AT\|_2, \ \ \|TA\|_2 \leq \|T\| \, \|A\|_2 \tag{2.6}$$

for any $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$*;*

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$*, i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_2(H)\,\mathcal{B}(H) \subseteq \mathcal{B}_2(H);$$

(iv) $\mathcal{B}_{fin}(H)$*, the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_2(H)$*;*

(v) $\mathcal{B}_2(H) \subseteq \mathcal{K}(H)$*, where* $\mathcal{K}(H)$ *denotes the algebra of compact operators on* H*.*

If $\{e_i\}_{i\in I}$ is an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A|\, e_i, e_i \rangle < \infty. \tag{2.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A \in \mathcal{B}(H)$*, then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$*;*

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$*;*

(ii) A *(or* $|A|$*) is the product of two elements of* $\mathcal{B}_2(H)$*.*

The following properties are also well known:

Theorem 8. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \ \text{ and } \ \|A\|_2 \leq \|A\|_1 \tag{2.8}$$

for any $A \in \mathcal{B}_1(H)$*;*

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$*, i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_1(H)\,\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\,\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\left\{ |\langle A, B\rangle_2| \ \mid B \in \mathcal{B}_2(H), \ \ \|B\| \leq 1 \right\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

(iv) We have the following isometric isomorphisms

$$\mathcal{B}_1(H) \cong K(H)^* \ \text{ and } \mathcal{B}_1(H)^* \cong \mathcal{B}(H),$$

where $K(H)^*$ *is the dual space of* $K(H)$ *and* $\mathcal{B}_1(H)^*$ *is the dual space of* $\mathcal{B}_1(H)$*.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle, \tag{2.9}$$

where $\{e_i\}_{i \in I}$ is an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (2.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 9. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{2.10}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{2.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;
(iv) If A, $B \in \mathcal{B}_2(H)$ *then* AB, $BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;
(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

Utilising the trace notation we obviously have that

$$\langle A, B \rangle_2 = \operatorname{tr}(B^*A) = \operatorname{tr}(AB^*) \text{ and } \|A\|_2^2 = \operatorname{tr}(A^*A) = \operatorname{tr}\left(|A|^2\right)$$

for any A, $B \in \mathcal{B}_2(H)$.

The following Hölder's type inequality has been obtained by Ruskai in [30]

$$|\operatorname{tr}(AB)| \le \operatorname{tr}(|AB|) \le \left[\operatorname{tr}\left(|A|^{1/\alpha}\right)\right]^{\alpha} \left[\operatorname{tr}\left(|B|^{1/(1-\alpha)}\right)\right]^{1-\alpha} \tag{2.12}$$

where $\alpha \in (0,1)$ and A, $B \in \mathcal{B}(H)$ with $|A|^{1/\alpha}$, $|B|^{1/(1-\alpha)} \in \mathcal{B}_1(H)$.

In particular, for $\alpha = \frac{1}{2}$ we get the Schwarz inequality

$$|\operatorname{tr}(AB)| \le \operatorname{tr}(|AB|) \le \left[\operatorname{tr}\left(|A|^2\right)\right]^{1/2} \left[\operatorname{tr}\left(|B|^2\right)\right]^{1/2} \tag{2.13}$$

with A, $B \in \mathcal{B}_2(H)$.

If A and B are selfadjoint operators with $A \le B$ and $P \in \mathcal{B}_1(H)$ with $P \ge 0$, then $P^{1/2}AP^{1/2} \le P^{1/2}BP^{1/2}$. Since tr is a positive linear functional and since $\operatorname{tr}(XY) = \operatorname{tr}(YX)$, it follows that $\operatorname{tr}(PA) = \operatorname{tr}\left(P^{1/2}AP^{1/2}\right) \le \operatorname{tr}\left(P^{1/2}BP^{1/2}\right) = \operatorname{tr}(PB)$. Therefore, if A and B are selfadjoint operators with $A \le B$ and $P \in \mathcal{B}_1(H)$ with $P \ge 0$, then

$$\operatorname{tr}(PA) \le \operatorname{tr}(PB). \tag{2.14}$$

If $A \ge 0$ and $P \in \mathcal{B}_1(H)$ with $P \ge 0$, then

$$0 \le \operatorname{tr}(PA) \le \|A\| \operatorname{tr}(P). \tag{2.15}$$

Indeed, since $A \le \|A\| 1_H$ for $A \ge 0$, then (2.15) follows by (2.14).

Moreover, for any selfadjoint A, $-|A| \le A \le |A|$. So it follows by (2.14) that

$$-\operatorname{tr}(P|A|) \le \operatorname{tr}(PA) \le \operatorname{tr}(P|A|)$$

i.e.,

$$|\operatorname{tr}(PA)| \le \operatorname{tr}(P|A|) \tag{2.16}$$

for any A a selfadjoint operator and $P \in \mathcal{B}_1(H)$ with $P \ge 0$.

For the theory of trace functionals and their applications the reader is referred to [33].

For some classical trace inequalities see [5], [7], [29] and [38], which are continuations of the work of Bellman [2]. For related works the reader can refer to [1], [3], [5], [18], [19], [20], [21], [31] and [35].

3. Slater Type Trace Inequalities

We denote by $\mathcal{B}_1^+(H) := \{P : P \in \mathcal{B}_1(H) \text{ and } P \geq 0\}$.

The following result holds:

Theorem 10. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operator on the Hilbert space H with $\operatorname{Sp}(A) \subseteq [m, M] \subset \mathring{I}$ and $f'(A)$ is a positive invertible operator on H, then*

$$0 \leq f\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) - \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)} \leq f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right)\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right), \tag{3.1}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Proof. Since f is convex and differentiable on $\mathring{I}$, then we have

$$f'(s)(t-s) \leq f(t) - f(s) \leq f'(t)(t-s) \tag{3.2}$$

for any $t, s \in [m, M]$.

Now, if we fix $t \in [m, M]$ and apply the property (P) for the operator A, then we have

$$tf'(A) - Af'(A) \leq f(t) \cdot 1_H - f(A) \leq f'(t)\, t \cdot 1_H - f'(t) A \tag{3.3}$$

for any $t \in [m, M]$.

If we apply the property (2.14) to the inequality (3.3) then we have

$$t \operatorname{tr}[Pf'(A)] - \operatorname{tr}[PAf'(A)] \leq f(t)\operatorname{tr}(P) - \operatorname{tr}[Pf(A)] \leq f'(t)\, t \operatorname{tr}(P) - f'(t)\operatorname{tr}(PA) \tag{3.4}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Now, since A is selfadjoint with $m1_H \leq A \leq M1_H$ and $f'(A)$ is positive, then

$$mf'(A) \leq Af'(A) \leq Mf'(A).$$

If we apply again the property (2.14), then we get

$$m \operatorname{tr}[Pf'(A)] \leq \operatorname{tr}[PAf'(A)] \leq M \operatorname{tr}[Pf'(A)],$$

which shows that

$$t_0 := \frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]} \in [m, M].$$

Observe that since $f'(A)$ is a positive invertible operator on H, then $\operatorname{tr}[Pf'(A)] > 0$ for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Finally, if we put $t=t_0$ in the equation (3.4), then we get

$$\begin{aligned}
&\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\operatorname{tr}[Pf'(A)]-\operatorname{tr}[PAf'(A)] \quad (3.5)\\
&\leq f\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right)\operatorname{tr}(P)-\operatorname{tr}[Pf(A)]\\
&\leq f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right)\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\operatorname{tr}(P)\\
&-f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right)\operatorname{tr}(PA),
\end{aligned}$$

which is equivalent to the desired result (3.1). □

Remark 1. *It is important to observe that, the condition that $f'(A)$ is a positive invertible operator on H can be replaced with the more general assumption that*

$$\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\in\mathring{I} \text{ and } \operatorname{tr}[Pf'(A)]\neq 0 \tag{3.6}$$

for any $P\in\mathcal{B}_1^+(H)\setminus\{0\}$, which may be easily verified for particular convex functions f in various examples as follows.

Also, as pointed out by the referee, if $\langle f'(A)x,x\rangle>0$ for any $x\in H$, $x\neq 0$, then $\operatorname{tr}[Pf'(A)]>0$ for any $P\in\mathcal{B}_1^+(H)\setminus\{0\}$ and the inequality (3.1) is valid as well.

Remark 2. *Now, if the function is concave on $\mathring{I}$ and the condition (3.6) holds, then we have the inequalities*

$$\begin{aligned}
0&\leq\frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)}-f\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) \quad (3.7)\\
&\leq f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right),
\end{aligned}$$

for any $P\in\mathcal{B}_1^+(H)\setminus\{0\}$.

Utilising the inequality (3.7) for the concave function $f:(0,\infty)\to\mathbb{R}$, $f(t)=\ln t$, then we can state that

$$0\leq\frac{\operatorname{tr}(P\ln A)}{\operatorname{tr}(P)}-\ln\left(\frac{\operatorname{tr}(P)}{\operatorname{tr}(PA^{-1})}\right)\leq\frac{\operatorname{tr}(PA^{-1})}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-1 \tag{3.8}$$

for any positive invertible operator A and P with $P\in\mathcal{B}_1^+(H)\setminus\{0\}$.

Utilising the inequality (3.1) for the convex function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t^{-1}$, then we can state that

$$0\leq\frac{\operatorname{tr}(PA^{-2})}{\operatorname{tr}(PA^{-1})}-\frac{\operatorname{tr}(PA^{-1})}{\operatorname{tr}(P)}\leq\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PA^{-2})}{\operatorname{tr}(PA^{-1})}-\frac{\operatorname{tr}(PA^{-1})}{\operatorname{tr}(PA^{-2})}, \tag{3.9}$$

for any positive invertible operator $A$and P with $P\in\mathcal{B}_1^+(H)\setminus\{0\}$.

If we take $B=A^{-1}$ in (3.9), then we get the equivalent inequality

$$0\leq\frac{\operatorname{tr}(PB^2)}{\operatorname{tr}(PB)}-\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}\leq\frac{\operatorname{tr}(PB^2)}{\operatorname{tr}(PB)}\frac{\operatorname{tr}(PB^{-1})}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PB)}{\operatorname{tr}(PB^2)}, \tag{3.10}$$

for any positive invertible operator B and P with $P\in\mathcal{B}_1(H)\setminus\{0\}$.

If we write the inequality (3.1) for the convex function $f(t) = \exp(\alpha t)$ with $\alpha \in \mathbb{R} \setminus \{0\}$, then we get

$$0 \leq \exp\left(\alpha \frac{\operatorname{tr}[PA\exp(\alpha A)]}{\operatorname{tr}[P\exp(\alpha A)]}\right) - \frac{\operatorname{tr}[P\exp(\alpha A)]}{\operatorname{tr}(P)} \tag{3.11}$$
$$\leq \alpha \exp\left(\alpha \frac{\operatorname{tr}[PA\exp(\alpha A)]}{\operatorname{tr}[P\exp(\alpha A)]}\right)\left(\frac{\operatorname{tr}[PA\exp(\alpha A)]}{\operatorname{tr}[P\exp(\alpha A)]} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right),$$

for any selfadjoint operator A and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

4. Further Reverses

We use the following Grüss' type inequalities [14]:

Lemma 1. *Let S be a selfadjoint operator with $m1_H \leq S \leq M1_H$ and $f : [m, M] \to \mathbb{C}$ a continuous function of bounded variation on $[m, M]$. For any $C \in \mathcal{B}(H)$ and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$ we have the inequality*

$$\left|\frac{\operatorname{tr}(Pf(S)C)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \tag{4.1}$$
$$\leq \frac{1}{2}\bigvee_m^M (f) \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)$$
$$\leq \frac{1}{2}\bigvee_m^M (f) \left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2},$$

where $\bigvee_m^M (f)$ is the total variation of f on the interval.

If the function $f : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$, i.e.

$$|f(t) - f(s)| \leq L|t - s|$$

for any $t, s \in [m, M]$, then

$$\left|\frac{\operatorname{tr}(Pf(S)C)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \tag{4.2}$$
$$\leq L\left\|S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)$$
$$\leq L\left\|S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right\| \left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2}$$

for any $C \in \mathcal{B}(H)$ and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Proof. For the sake of completeness we give here a simple proof.

We observe that, for any $\lambda \in \mathbb{C}$ we have

$$\begin{aligned}
(4.3)\quad & \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left[P\left(A-\lambda 1_H\right)\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right)\right] \\
& =\frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left[PA\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right)\right] \\
& -\frac{\lambda}{\operatorname{tr}(P)} \operatorname{tr}\left[P\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right)\right] \\
& =\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} .
\end{aligned}$$

Taking the modulus in (4.3) and utilising the properties of the trace, we have

$$\begin{aligned}
(4.4)\quad & \left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\
& =\frac{1}{\operatorname{tr}(P)}\left|\operatorname{tr}\left[P\left(A-\lambda 1_H\right)\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right)\right]\right| \\
& =\frac{1}{\operatorname{tr}(P)}\left|\operatorname{tr}\left[\left(A-\lambda 1_H\right)\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right) P\right]\right| \\
& \leq\left\|A-\lambda 1_H\right\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right) P\right|\right)
\end{aligned}$$

for any $\lambda \in \mathbb{C}$, where for the last inequality we used the inequality (2.11).

From the inequality (4.4) we have

$$\begin{aligned}
(4.5)\quad & \left|\frac{\operatorname{tr}(Pf(S) C)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\
& \leq\left\|f(S)-\lambda 1_H\right\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right) P\right|\right)
\end{aligned}$$

for any $\lambda \in \mathbb{C}$.

From (4.5) we get

$$\begin{aligned}
(4.6)\quad & \left|\frac{\operatorname{tr}(Pf(S) C)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\
& \leq\left\|f(S)-\frac{f(m)+f(M)}{2} 1_H\right\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right) P\right|\right) .
\end{aligned}$$

Since f is of bounded variation on $[m, M]$, then we have

$$\begin{aligned}
(4.7)\quad \left|f(t)-\frac{f(m)+f(M)}{2}\right| & =\left|\frac{f(t)-f(m)+f(t)-f(M)}{2}\right| \\
& \leq \frac{1}{2}[|f(t)-f(m)|+|f(M)-f(t)|] \leq \frac{1}{2} \bigvee_m^M(f)
\end{aligned}$$

for any $t \in[m, M]$.

From (4.7) we get in the order $\mathcal{B}(H)$ that

$$\left|f(S)-\frac{f(m)+f(M)}{2} 1_H\right| \leq \frac{1}{2} \bigvee_m^M(f) 1_H,$$

which implies that

$$\left\| f(S) - \frac{f(m)+f(M)}{2} 1_H \right\| \leq \frac{1}{2} \bigvee_m^M (f) . \tag{4.8}$$

Making use of (4.7) and (4.8) we get the first inequality in (4.1).

The second part is obvious by the Schwarz inequality for traces

$$\frac{\operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right) P\right|\right)}{\operatorname{tr}(P)} \leq \left(\frac{\operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right) P^{1/2}\right|^2\right)}{\operatorname{tr}(P)} \right)^{1/2},$$

and by noticing that

$$\frac{\operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right) P^{1/2}\right|^2\right)}{\operatorname{tr}(P)} = \frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2 \tag{4.9}$$

for any $C \in \mathcal{B}(H)$ and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

From (4.5) we also have

$$\begin{aligned} &\left| \frac{\operatorname{tr}(Pf(S)C)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} \right| \\ &\leq \left\| f(S) - f\left(\frac{\operatorname{tr}(SP)}{\operatorname{tr}(P)}\right) 1_H \right\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right) P\right|\right) \end{aligned} \tag{4.10}$$

any $C \in \mathcal{B}(H)$ and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Since

$$|f(t) - f(s)| \leq L|t-s|$$

for any $t,\ s \in [m, M]$, then we have in the order $\mathcal{B}(H)$ that

$$|f(S) - f(s) 1_H| \leq L |S - s1_H|$$

for any $s \in [m, M]$. In particular, we have

$$\left| f(S) - f\left(\frac{\operatorname{tr}(SP)}{\operatorname{tr}(P)}\right) 1_H \right| \leq L \left| S - \frac{\operatorname{tr}(SP)}{\operatorname{tr}(P)} 1_H \right|,$$

which implies that

$$\left\| f(S) - f\left(\frac{\operatorname{tr}(SP)}{\operatorname{tr}(P)}\right) 1_H \right\| \leq L \left\| S - \frac{\operatorname{tr}(SP)}{\operatorname{tr}(P)} 1_H \right\|$$

and by (4.10) we get the first inequality in (4.2).

The second part is obvious. □

We also have the following reverse of Schwarz inequality [14]:

Lemma 2. *If C is a selfadjoint operator with $k1_H \leq C \leq K1_H$ for some real numbers $k < K$, then*

$$0 \leq \frac{\operatorname{tr}\left(PC^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2 \leq \frac{1}{2}(K-k)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \leq \frac{1}{2}(K-k)\left[\frac{\operatorname{tr}\left(PC^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \leq \frac{1}{4}(K-k)^2, \tag{4.11}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Proof. If we take in (4.1) $f(t) = t$ and $S = C$ we get

$$\left|\frac{\operatorname{tr}\left(PC^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2\right| \leq \frac{1}{2}(K-k)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \leq \frac{1}{2}(K-k)\left[\frac{\operatorname{tr}\left(PC^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2\right]^{1/2}. \tag{4.12}$$

Since by (4.9) we have

$$\frac{\operatorname{tr}\left(PC^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2 \geq 0,$$

then by (4.12) we get

$$0 \leq \frac{\operatorname{tr}\left(PC^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2 \leq \frac{1}{2}(K-k)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \leq \frac{1}{2}(K-k)\left[\frac{\operatorname{tr}\left(PC^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2\right]^{1/2}. \tag{4.13}$$

Utilising the inequality between the first and last term in (4.13) we also have

$$\left[\frac{\operatorname{tr}\left(PC^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \leq \frac{1}{2}(K-k),$$

which proves the last part of (4.11). □

Theorem 11. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operator on the Hilbert space H with $\operatorname{Sp}(A) \subseteq [m, M] \subset \mathring{I}$ and $f'(A)$ is a positive invertible operator on H, or*

$$\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]} \in \mathring{I}, \quad \operatorname{tr}[Pf'(A)] \neq 0$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, *then*

$$0 \le f\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) - \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)} \le \frac{\operatorname{tr}(P)}{\operatorname{tr}[Pf'(A)]} f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) L(P,A,f'(A)), \tag{4.14}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, *where*

$$\begin{aligned} L(P,A,f'(A)) &:= \frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}[Pf'(A)]}{\operatorname{tr}(P)} \\ &\le \begin{cases} \frac{1}{2}(f'(M)-f'(m))\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ \\ \frac{1}{2}(M-m)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(f'(A) - \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}1_H\right)P\right|\right) \end{cases} \\ &\le \begin{cases} \frac{1}{2}(f'(M)-f'(m))\left[\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M-m)\left[\frac{\operatorname{tr}\left(P[f'(A)]^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \end{cases} \\ &\le \frac{1}{4}(f'(M)-f'(m))(M-m). \end{aligned}$$

Proof. Utilising Lemma 1 and Lemma 2 we have

$$\begin{aligned} 0 &\le \frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \\ &\le \frac{1}{2}(f'(M)-f'(m))\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ &\le \frac{1}{2}(f'(M)-f'(m))\left[\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ &\le \frac{1}{4}(f'(M)-f'(m))(M-m) \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} 0 &\le \frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \\ &\le \frac{1}{2}(M-m)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(f'(A) - \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ &\le \frac{1}{2}(M-m)\left[\frac{\operatorname{tr}\left(P[f'(A)]^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ &\le \frac{1}{4}(f'(M)-f'(m))(M-m) \end{aligned} \tag{4.16}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

The positivity of

$$\frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}$$

follows by Čebyšev's trace inequality for synchronous functions of selfadjoint operators, see [15]. □

The case of convex and monotonic functions is as follows:

Corollary 1. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operator on the Hilbert space H with $\operatorname{Sp}(A) \subseteq [m, M] \subset \mathring{I}$ and $f'(m) > 0$, then*

$$0 \le f\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) - \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)} \le \frac{f'(M)}{f'(m)} L(P, A, f'(A)), \tag{4.17}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

The proof follows by (4.14) observing that

$$0 \le \frac{\operatorname{tr}(P)}{\operatorname{tr}[Pf'(A)]} f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) \le \frac{f'(M)}{f'(m)}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

If we consider the monotonic nondecreasing convex function $f(t) = t^p$ with $p \ge 1$ and $t \ge 0$, then by (4.17) we have the sequence of inequalities

$$\begin{aligned} 0 &\le \left(\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(PA^{p-1})}\right)^p - \frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} \\ &\le p\left(\frac{M}{m}\right)^{p-1}\left(\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(PA^{p-1}\right)}{\operatorname{tr}(P)}\right) \\ &\le \frac{1}{2}p^2\left(\frac{M}{m}\right)^{p-1} \\ &\quad \times \begin{cases} \left(M^{p-1} - m^{p-1}\right)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ \\ (M-m)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(A^{p-1} - \frac{\operatorname{tr}\left(PA^{p-1}\right)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \end{cases} \\ &\le \frac{1}{2}p^2\left(\frac{M}{m}\right)^{p-1} \\ &\quad \times \begin{cases} \left(M^{p-1} - m^{p-1}\right)\left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ \\ (M-m)\left[\frac{\operatorname{tr}\left(PA^{2(p-1)}\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}\left(PA^{p-1}\right)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \end{cases} \\ &\le \frac{1}{4}p^2\left(\frac{M}{m}\right)^{p-1}\left(M^{p-1} - m^{p-1}\right)(M-m) \end{aligned} \tag{4.18}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$ and A with $\operatorname{Sp}(A) \subseteq [m, M] \subset (0, \infty)$.

Theorem 12. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and twice differentiable function on $\mathring{I}$ whose second derivative f'' is bounded on $\mathring{I}$, i.e. there is a positive constant K such that $0 \le f''(t) \le K$ for any $t \in \mathring{I}$. If A is a selfadjoint operator on the Hilbert space H with $\operatorname{Sp}(A) \subseteq [m, M] \subset \mathring{I}$ and $f'(A)$ is a positive invertible operator on H, or*

$$\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]} \in \mathring{I}, \quad \operatorname{tr}[Pf'(A)] \neq 0$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, then

$$\begin{aligned}
(4.19) \qquad 0 &\le f\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) - \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)} \\
&\le K\left\|A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\
&\times \frac{\operatorname{tr}(P)}{\operatorname{tr}[Pf'(A)]} f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) \\
&\le K\left\|A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right\| \left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\
&\times \frac{\operatorname{tr}(P)}{\operatorname{tr}[Pf'(A)]} f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) \\
&\le \frac{1}{2}(M-m)K\left\|A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right\| \frac{\operatorname{tr}(P)}{\operatorname{tr}[Pf'(A)]} f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right)
\end{aligned}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Proof. From (4.14) we have

$$\begin{aligned}
(4.20) \qquad 0 &\le f\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) - \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)} \\
&\le \frac{\operatorname{tr}(P)}{\operatorname{tr}[Pf'(A)]} f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) L(P, A, f'(A)),
\end{aligned}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

From (4.2) we also have

$$\begin{aligned}
(4.21) \qquad &(0 \le) L(P, A, f'(A)) \\
&\le K\left\|A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\
&\le K\left\|A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right\| \left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2}
\end{aligned}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Therefore, by (4.20) and (4.21) we get

$$\begin{aligned}
0 &\le f\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right)-\frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)} \\
&\le K\left\|A-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(A-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\
&\times\frac{\operatorname{tr}(P)}{\operatorname{tr}[Pf'(A)]}f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) \\
&\le K\left\|A-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right\|\left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\
&\times\frac{\operatorname{tr}(P)}{\operatorname{tr}[Pf'(A)]}f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right)
\end{aligned}$$

that proves the second and third inequalities in (4.19).

The last part follows by Lemma 2. □

The inequality (4.19) can be also written for the convex function $f(t)=t^p$ with $p\ge 1$ and $t\ge 0$, however the details are not presented here.

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