Received 15/01/14

GENERALIZATION OF HERMITE-HADAMARD TYPE INEQUALITIES FOR n-TIMES DIFFERENTIABLE FUNCTIONS WHICH ARE s-PREINVEX IN THE SECOND SENSE WITH APPLICATIONS

M. A. LATIF[1] AND S. S. DRAGOMIR[2,3]

ABSTRACT. In this paper, Hermite-Hadamard inequality for differentiable preinvex functions is generalized and refined for n-times differentiable functions which are s-preinvex in the second sense. Some recent results are also improved and applications to special means of positive numbers are given.

1. INTRODUCTION

The following definition for convex functions is well known in the mathematical literature:

Definition 1. *A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

holds for all $x, y \in I$ and $t \in [0,1]$.

A number of inequalities have been established for convex functions but the following double inequalites, know as Hermite-Hadmard inequalities, are famous in mathematical literature

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping and $a, b \in I \subseteq \mathbb{R}$ with $a < b$. The inequalities (1.1) hold in reversed direction if f is concave. A number of papers have been written on this inequality providing new proofs, noteworthy extensions, generalizations, refinements, counterparts and new Hermite-Hadamard-type inequalities and numerous applications, see for instance [7]-[11], [13], [15], [20]-[23], [29], [31], [32]-[34], [36], [37] and the references therein.

In recent years, the calssical convexity has been generalized and extended in a diverse manner. One of them is the preinvexity, introduced by Weir et al. [38] as a significant generalization of convex functions. Many researchers have studied the basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems, for example see the work of Mohn et al. [24],

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Date: November 15, 2012.

2000 *Mathematics Subject Classification.* 26D07, 26D10, 26D99.

Key words and phrases. Hermite-Hadamard's inequality, invex set, preinvex function, Hölder's inequality, s-preinvex function.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Noor [26] and Yang et al. [41]. It is well known that the preinvex functions and invex sets may not be convex functions and convex sets.

Let us recall some definitions and known results concerning invexity and preinvexity

Definition 2. [41] *A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to $\eta : K \times K \to \mathbb{R}^n$ if*

$$x + t\eta(y, x) \in K, \forall x, y \in K, t \in [0, 1].$$

The invex set K is also called a η-connected set.

Definition 3. [38] *Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}^n$. A function $f : K \to \mathbb{R}$ is said to be preinvex with respect to η, if for all $u, v \in K$ and $t \in [0, 1]$*

$$f(u + t\eta(v, u)) \leq (1 - t) f(u) + tf(v).$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

It is to be noted that every preinvex function is convex with respect to the map $\eta(x, y) = x - y$ but the converse is not true see for instance [38].

Noor [28], proved the following Hermite-Hadamard inequalities.

Theorem 1. [28] *Let $f : K = [a, a + \eta(b, a)] \to (0, \infty)$ be a preinvex function on the interval of the real numbers K° (the interior of K) and a, $b \in K^\circ$ with $\eta(b, a) > 0$. Then the following inequalities holds:*

$$f\left(\frac{2a + \eta(b, a)}{2}\right) \leq \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)\, dx \leq \frac{f(a) + f(b)}{2}. \tag{1.2}$$

Barani, Ghazanfari and Dragomir in [5], presented the follwoing estimates of the right-side of a Hermite-Hadamard type inequality in which some preinvex functions are involved.

Theorem 2. [5] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function .Assume $p \in \mathbb{R}$ with $p > 1$. If $\left|f'\right|^{\frac{p}{p-1}}$ is preinvex on K then, for every $a, b \in K$ with $\eta(b, a) \neq 0$, then the following inequality holds:*

$$\left|\frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{|\eta(b, a)|}{2(1 + p)^{\frac{1}{p}}} \left[\frac{\left|f'(a)\right|^{\frac{p}{p-1}} + \left|f'(b)\right|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}. \tag{1.3}$$

Theorem 3. [5]*Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $\left|f'\right|$ is preinvex on K then, for every $a, b \in K$ with $\eta(b, a) \neq 0$, then the following inequality holds:*

$$\left|\frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{|\eta(b, a)|}{8} \left(\left|f'(a)\right| + \left|f'(b)\right|\right). \tag{1.4}$$

More recently, Li [42] introduced the notion of s-preinvexity and established Hermite-Hadamard type inequalities for this class of functions.

Definition 4. [42] *Let $K \subseteq [0,\infty)$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}^n$. A function $f : K \to \mathbb{R}$ is said to be s-preinvex with respect to η, if for all $u, v \in K$, $t \in [0,1]$ and $s \in (0,1]$*

$$f(u + t\eta(v,u)) \leq (1-t)^s f(u) + t^s f(v).$$

The function f is said to be s-preconcave if and only if $-f$ is s-preinvex.

Theorem 4. [42] *Let $f : K = [a, a+\eta(b,a)] \subseteq [0,\infty) \to (0,\infty)$ be a s-preinvex function on the interval of the real numbers K° (the interior of K) and a, $b \in K^\circ$ with $\eta(b,a) > 0$. Then the following inequalities holds:*

$$2^{s-1} f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \leq \frac{f(a)+f(b)}{s+1}. \tag{1.5}$$

For more recent results on Hermite-Hadamard type and Simpson's type inequalities for preinvex, log-preinvex functions, s-preinvex functions, prequasiinvex functions and n-times differentiable preinvex functions, we refer the interested readers to [4, 16, 17, 18, 19, 35, 39, 40, 42].

The main purpose of the present paper is to establish new Hermite-Hadamard type inequalities in Section 2 that are connected with the right-side and left-side of Hermite-Hadamard inequality for n-times differentiable s-preinvex functions which generalize those results established for n-times differentiable preinvex functions and n-times differentiable convex functions.

2. Main Results

In order to prove our main results, we need the following lemmas:

Lemma 1. [16] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on K for $n \in \mathbb{N}$, $n \geq 1$. If $f^{(n)}$ is integrable on $[a, a+\eta(b,a)]$, then for every $a, b \in K$ with $\eta(b,a) > 0$, the following equality holds:*

$$\begin{aligned} &-\frac{f(a)+f(a+\eta(b,a))}{2} + \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \\ &\qquad + \sum_{k=2}^{n-1} \frac{(-1)^k (k-1) (\eta(b,a))^k}{2(k+1)!} f^{(k)}(a+\eta(b,a)) \\ &\qquad = \frac{(-1)^{n-1} (\eta(b,a))^n}{2n!} \int_0^1 t^{n-1} (n-2t) f^{(n)}(a+t\eta(b,a))dt, \end{aligned} \tag{2.1}$$

where the sum above takes 0 when $n=1$ and $n=2$.

Lemma 2. [16] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on K for $n \in \mathbb{N}$, $n \geq 1$. If $f^{(n)}$ is integrable on $[a, a+\eta(b,a)]$, then for every $a, b \in K$ with $\eta(b,a) > 0$, the* *following equality holds:*

$$\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!}f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx$$
$$=\frac{(-1)^{n+1}(\eta(b,a))^n}{n!}\int_0^1 K_n(t)f^{(n)}(a+t\eta(b,a))dt, \quad (2.2)$$

where

$$K_n(t):=\begin{cases} t^n, & t\in\left[0,\frac{1}{2}\right] \\ (t-1)^n, & t\in\left(\frac{1}{2},1\right]\end{cases}.$$

We are now ready to present our first result.

Theorem 5. *Let $K\subseteq[0,\infty)$ be an open invex subset with respect to $\eta: K\times K\to\mathbb{R}$. Suppose $f:K\to\mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on $[a,a+\eta(b,a)]$ for $n\in\mathbb{N}$, $n\geq 2$. If $\left|f^{(n)}\right|^q$ for $q\geq 1$, is s-preinvex on K for $n\in\mathbb{N}$, $n\geq 2$, then for every $a,b\in K$ with $\eta(b,a)>0$, we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right.$$
$$\left.-\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right|$$
$$\leq\frac{(\eta(b,a))^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{q}}\left(Q\left|f^{(n)}(a)\right|^q+P\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}, \quad (2.3)$$

where

$$P=\frac{n(n-1)+s(n-2)}{(n+s)(n+s+1)}, Q=nB(n,s+1)-2B(n+1,s+1)$$

and

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}$$

for $x>0$, $y>0$ is the Beta's function.

Proof. Suppose $n \geq 2$ and $q = 1$. By s-preinvexity of $\left|f^{(n)}\right|$ and lemma 1, we get

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx \right.\\
&\qquad\left. -\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right|\\
&\leq \frac{(\eta(b,a))^n}{2n!}\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(a+t\eta(b,a))\right|dt\\
&\leq \frac{(\eta(b,a))^n}{2n!}\int_0^1 t^{n-1}(n-2t)\left((1-t)^s\left|f^{(n)}(a)\right|+t^s\left|f^{(n)}(b)\right|\right)dt\\
&= \frac{(\eta(b,a))^n}{2n!}\left(\left|f^{(n)}(b)\right|\int_0^1 t^{n+s-1}(n-2t)\,dt+\left|f^{(n)}(a)\right|\int_0^1 t^{n-1}(n-2t)(1-t)^s\,dt\right).
\end{aligned} \tag{2.4}$$

Since

$$\int_0^1 t^{n+s-1}(n-2t)\,dt = \frac{n(n-1)+s(n-2)}{(n+s)(n+s+1)} = P$$

and

$$\begin{aligned}
\int_0^1 t^{n-1}(n-2t)(1-t)^s\,dt &= n\int_0^1 t^{n-1}(1-t)^s\,dt - 2\int_0^1 t^n(1-t)^s\,dt\\
&= nB(n,s+1)-2B(n+1,s+1) = Q.
\end{aligned}$$

Using the above observations in (2.4), we get (2.3). The proof for the case $q = 1$ is complete.

Assume now that $q > 1$, then by the s-preinvexity of $\left|f^{(n)}\right|^q$ on K, lemma 1 and the Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx \right.\\
&\qquad\left. -\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right|\\
&\leq \frac{(\eta(b,a))^n}{2n!}\left(\int_0^1 t^{n-1}(n-2t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\\
&\leq \frac{(\eta(b,a))^n}{2n!}\left(\int_0^1 t^{n-1}(n-2t)\,dt\right)^{1-\frac{1}{q}}\\
&\times\left(\left|f^{(n)}(b)\right|^q\int_0^1 t^{n+s-1}(n-2t)\,dt+\left|f^{(n)}(a)\right|^q\int_0^1 t^{n-1}(n-2t)(1-t)^s\right)^{\frac{1}{q}}
\end{aligned} \tag{2.5}$$

which is the inequality (2.3). Hence the proof of the theorem is completed. □

Remark 1. *If in Theorem 5, we take $s = 1$, we get Theorem 8 from* [16].

Corollary 1. *Suppose the assumptions of Theorem 5 are satisfied. Then for* $n = 2$, *we have*

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \\ &\qquad\le \frac{(\eta(b,a))^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{\left|f''(a)\right|^q+\left|f''(b)\right|^q}{(s+3)(s+2)}\right)^{\frac{1}{q}}.\end{aligned}\tag{2.6}$$

Remark 2. *If in Corollary 1* $s = 1$, *we get a result proved in Corollary 1 from* [16].

Remark 3. *If in Theorem 5, we take* $\eta(b,a) = b - a$. *Then one gets a result proved in Theorem 1.1 from* [11].

Theorem 6. *Let* $K \subseteq [0,\infty)$ *be an open invex subset with respect to* $\eta : K\times K \to \mathbb{R}$. *Suppose* $f : K \to \mathbb{R}$ *is a function such that* $f^{(n)}$ *exists on* K *and* $f^{(n)}$ *is integrable on* $[a, a+\eta(b,a)]$ *for* $n \in \mathbb{N}$, $n \ge 2$. *If* $\left|f^{(n)}\right|^q$ *for* $q \ge 1$, *is s-preinvex on* K *for* $n \in \mathbb{N}$, $n \ge 2$, *then for every* $a, b \in K$ *with* $\eta(b,a) > 0$, *we have the following inequality:*

$$\begin{aligned}&\Bigg|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx \\ &\qquad -\sum_{k=2}^{n-1}\frac{(-1)^k(k-1)(\eta(b,a))^k}{2(k+1)!}f^{(k)}(a+\eta(b,a))\Bigg| \\ &\qquad\qquad \le \frac{(\eta(b,a))^n(n-1)^{1-1/q}}{2n!} \\ &\times\Big[\{nB(nq-q+1,s+1)-2B(nq-q+2,s+1)\}\left|f^{(n)}(a)\right|^q \\ &\qquad \left\{+\frac{n}{nq-q+s+1}-\frac{2}{nq-q+s+2}\right\}\left|f^{(n)}(b)\right|^{q1/q}\Big]^{1/q},\end{aligned}\tag{2.7}$$

where

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}$$

for $x > 0$, $y > 0$ *is the Beta's function.*

Proof. The case when $q = 1$ is easy to prove so we assume that $q > 1$. By making use of Lemma 1, the Hölder inequality and the s-preinvexity of $\left|f^{(n)}\right|^{q}$, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(x)\,dx\right.$$
$$\left.-\sum_{k=2}^{n-1}\frac{(-1)^{k}(k-1)(\eta(b,a))^{k}}{2(k+1)!}f^{(k)}(a+\eta(b,a))\right|\le\frac{(\eta(b,a))^{n}}{2n!}\left(\int_{0}^{1}(n-2t)\,dt\right)^{1-1/q}$$
$$\times\int_{0}^{1}t^{q(n-1)}(n-2t)\left((1-t)^{s}\left|f^{(n)}(a)\right|^{q}+t^{s}\left|f^{(n)}(b)\right|^{q}\right)dt=\frac{(\eta(b,a))^{n}(n-1)^{1-1/q}}{2n!}$$
$$\times\left[\{nB(nq-q+1,s+1)-2B(nq-q+2,s+1)\}\left|f^{(n)}(a)\right|^{q}\right.$$
$$\left.+\left\{\frac{n}{nq-q+s+1}-\frac{2}{nq-q+s+2}\right\}\left|f^{(n)}(b)\right|^{q1/q}\right]^{1/q}. \tag{2.8}$$

This completes the proof of the theorem. □

Corollary 2. *Suppose the conditions of Theorem 6 are fulfilled and* $n = 2$. *Then*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(x)\,dx\right|$$
$$\le\frac{(\eta(b,a))^{2}}{2^{2-1/q}}\times\left[\{B(q+1,s+1)-B(q+2,s+1)\}\left|f''(a)\right|^{q}\right.$$
$$\left.+\frac{\left|f''(b)\right|^{q}}{(q+s+1)(q+s+2)}\right]^{1/q}, \tag{2.9}$$

where $B(x,y)$, $x,y > 0$ *is the Beta's function.*

Corollary 3. *If we take* $q = 1$ *and* $s = 1$ *in Corollary 3. Then*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(x)\,dx\right|$$
$$\le\frac{(\eta(b,a))^{2}}{24}\left[\left|f''(a)\right|^{q}+\left|f''(b)\right|\right]. \tag{2.10}$$

Remark 4. *For* $\eta(b,a) = b-a$, *we obtain new bounds of the difference between the middle and right side of Hermite-Hadamard inequalities (1.1) in terms of second order derivatives.*

Now we give some results related to left-side of Hermite-Hadamard's inequality for n-times differentiable s-preinvex functions.

Theorem 7. *Let* $K \subseteq [0,\infty)$ *be an open invex subset with respect to* $\eta : K\times K \to \mathbb{R}$. *Suppose* $f : K \to \mathbb{R}$ *is a function such that* $f^{(n)}$ *exists on* K *and* $f^{(n)}$ *is integrable on* $[a, a+\eta(b,a)]$ *for* $n \in \mathbb{N}$, $n \ge 1$. *If* $\left|f^{(n)}\right|^{q}$*for* $q > 1$, *is* s*-preinvex on* K *for* $n \in \mathbb{N}$, $n \ge 1$, $s \in (0,1]$, *then for every* $a, b \in K$ *with* $\eta(b,a) > 0$, *we have the* *following inequality:*

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!}f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right)\right.$$
$$\left.-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|\leq\frac{(\eta(b,a))^n}{2^n n!(np+1)^{\frac{1}{p}}}\left[\frac{\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(b)\right|^q}{s+1}\right]^{\frac{1}{q}}, \tag{2.11}$$

where $\frac{1}{p}+\frac{1}{q}=1$*.*

Proof. Suppose $n\geq 1$. By using lemma 2 and the s-preinvexity of $\left|f^{(n)}\right|^q$ on K for $n\in\mathbb{N}$, $n\geq 1$, $q>1$, we have

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!}f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|$$
$$\leq\frac{(\eta(b,a))^n}{n!}\int_0^1|K_n(t)|\left|f^{(n)}(a+t\eta(b,a))\right|dt$$
$$\leq\frac{(\eta(b,a))^n}{n!}\left(\int_0^1|K_n(t)|^p\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}. \tag{2.12}$$

Since

$$\int_0^1|K_n(t)|^p=\int_0^{\frac{1}{2}}t^{np}dt+\int_{\frac{1}{2}}^1(1-t)^{np}\,dt=\frac{1}{2^{np}(np+1)}$$

and

$$\int_0^1\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\leq\int_0^1(1-t)^s\left|f^{(n)}(a)\right|^q dt+\int_0^1 t^s\left|f^{(n)}(b)\right|^q dt$$
$$=\frac{\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(b)\right|^q}{s+1}$$

An application of the above observations (2.12), we get the desired inequality (2.11). This completes the proof of the theorem. □

Corollary 4. *Under the assuptions of Theorem 7, if* $n=2$*, then we obtain the following inequality:*

$$\left|f\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|$$
$$\leq\frac{(\eta(b,a))^2}{8(2p+1)^{\frac{1}{p}}}\left[\frac{\left|f''(a)\right|^q+\left|f''(b)\right|^q}{s+1}\right]^{\frac{1}{q}}, \tag{2.13}$$

where $\frac{1}{p}+\frac{1}{q}=1$*.*

Corollary 5. *In Corollary 4, if we take $s=1$, then one gets the following result:*

$$\left|f\left(a+\frac{1}{2}\eta\left(b,a\right)\right)-\frac{1}{\eta\left(b,a\right)}\int_{a}^{a+\eta(b,a)}f\left(x\right)dx\right| \leq \frac{\left(\eta\left(b,a\right)\right)^{2}}{8\left(2p+1\right)^{\frac{1}{p}}}\left[\frac{\left|f^{''}\left(a\right)\right|^{q}+\left|f^{''}\left(b\right)\right|^{q}}{2}\right]^{\frac{1}{q}}, \quad (2.14)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 6. *In Theorem 7, if $\eta\left(b,a\right)=b-a$, we have the following inequality:*

$$\left|\sum_{k=0}^{n-1}\frac{\left[\left(-1\right)^{k}+1\right]\left(b-a\right)^{k}}{2^{k+1}\left(k+1\right)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx\right| \leq \frac{\left(b-a\right)^{n}}{2^{n}n!\left(np+1\right)^{\frac{1}{p}}}\left[\frac{\left|f^{(n)}(a)\right|^{q}+\left|f^{(n)}(b)\right|^{q}}{s+1}\right]^{\frac{1}{q}}, \quad (2.15)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

A different approach leads us to the following result:

Theorem 8. *Let $K\subseteq[0,\infty)$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$. Suppose $f:K\to\mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on $\left[a,a+\eta\left(b,a\right)\right]$ for $n\in\mathbb{N}$, $n\geq1$. If $\left|f^{(n)}\right|^{q}$ s-is preinvex on K for $n\in\mathbb{N}$, $n\geq1$, $q\in\mathbb{R}$, $q>1$, then for every $a,b\in K$ with $\eta\left(b,a\right)>0$, we have the following inequality:*

$$\begin{aligned}&\left|\sum_{k=0}^{n-1}\frac{\left[\left(-1\right)^{k}+1\right]\left(\eta\left(b,a\right)\right)^{k}}{2^{k+1}\left(k+1\right)!}f^{(k)}\left(a+\frac{1}{2}\eta\left(b,a\right)\right)-\frac{1}{\eta\left(b,a\right)}\int_{a}^{a+\eta(b,a)}f\left(x\right)dx\right|\\ &\leq\frac{\left(\eta\left(b,a\right)\right)^{n}}{2^{n+\frac{1}{p}}\left(np+1\right)^{\frac{1}{p}}n!}\left[\left(\frac{\left(2^{s+1}-1\right)\left|f^{(n)}(a)\right|^{q}+\left|f^{(n)}(b)\right|^{q}}{2^{s+1}\left(s+1\right)}\right)^{\frac{1}{q}}\right.\\ &\qquad\left.+\left(\frac{\left|f^{(n)}(a)\right|^{q}+\left(2^{s+1}-1\right)\left|f^{(n)}(b)\right|^{q}}{2^{s+1}\left(s+1\right)}\right)^{\frac{1}{q}}\right], \quad (2.16)\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From lemma 2 and the power-mean integral inequality, we have

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!}f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|$$
$$\leq\frac{(\eta(b,a))^n}{n!}\left[\left(\int_0^{\frac{1}{2}}t^{np}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1(1-t)^{np}\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right]. \quad (2.17)$$

Since $\left|f^{(n)}\right|^q$ is s-preinvex on K in the second sense for $n\in\mathbb{N}$, $n\geq 1$, $q\in\mathbb{R}$, $q>1$ and $s\in(0,1]$. Hence for every $a,b\in K$ with $\eta(b,a)>0$, we have

$$\int_0^{\frac{1}{2}}t^n\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\leq\left|f^{(n)}(a)\right|^q\int_0^{\frac{1}{2}}(1-t)^s\,dt+\left|f^{(n)}(b)\right|^q\int_0^{\frac{1}{2}}t^s dt$$
$$=\frac{2^{s+1}-1}{2^{s+1}(s+1)}\left|f^{(n)}(a)\right|^q+\frac{1}{2^{s+1}(s+1)}\left|f^{(n)}(b)\right|^q \quad (2.18)$$

and

$$\int_0^{\frac{1}{2}}\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\leq\left|f^{(n)}(a)\right|^q\int_{\frac{1}{2}}^1(1-t)^s\,dt+\left|f^{(n)}(b)\right|^q\int_{\frac{1}{2}}^1 t^s dt$$
$$=\frac{1}{2^{s+1}(s+1)}\left|f^{(n)}(a)\right|^q+\frac{2^{s+1}-1}{2^{s+1}(s+1)}\left|f^{(n)}(b)\right|^q. \quad (2.19)$$

Also

$$\int_0^{\frac{1}{2}}t^{np}dt=\int_{\frac{1}{2}}^1(1-t)^{np}\,dt=\frac{1}{2^{np+1}(np+1)}. \quad (2.20)$$

Using (2.18), (2.19) and (2.20) in (2.17), we get the required inequality (2.16). This completes the proof of the theorem.oved in [26]. □

Remark 5. *For $s=1$, Theorem 8 becomes Theorem 11 from* [16].

Corollary 7. *For $s=1$ and $n=2$, we get the following inequality from* [16]*:*

$$\left|f\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|$$
$$\leq\frac{(\eta(b,a))^2}{8\cdot 2^{\frac{1}{p}}(2p+1)^{\frac{1}{p}}}\left[\left(\frac{3\left|f''(a)\right|^q+\left|f''(b)\right|^q}{8}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{8}\right)^{\frac{1}{q}}\right], \quad (2.21)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 9. *Let $K \subseteq [0,\infty)$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose $f : K \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on K and $f^{(n)}$ is integrable on $[a, a+\eta(b,a)]$ for $n \in \mathbb{N}$, $n \geq 1$. If $\left|f^{(n)}\right|^q$ s-is preinvex on K for $n \in \mathbb{N}$, $n \geq 1$, $q \in \mathbb{R}$, $q \geq 1$, then for every $a, b \in K$ with $\eta(b,a) > 0$, we have the following inequality:*

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!}f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|$$
$$\leq \frac{(\eta(b,a))^n(n+1)^{\frac{1}{q}}}{2^{(n+1)\left(1-\frac{1}{q}\right)}(n+1)!}\left[\left(L\left|f^{(n)}(a)\right|^q+M\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(M\left|f^{(n)}(a)\right|^q+N\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}\right], \quad (2.22)$$

where

$$L = B\left(\frac{1}{2};n+1,s+1\right), M = \frac{1}{2^{n+s+1}(n+s+1)},$$
$$N = B(s+1,n+1) - B\left(\frac{1}{2};s+1,n+1\right)$$

and

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}$$

for $x > 0$, $y > 0$ is the Beta's function, and

$$B(z;x,y) = \int_0^z t^{x-1}(1-t)^{y-1}$$

is the generalization of the Beta function $B(x,y)$.

Proof. It is not difficult to see that (2.22) holds true for $q = 1$. Suppose that $q > 1$. From lemma 2 and the Hölder's integral inequality, we have

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!}f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|$$
$$\leq \frac{(\eta(b,a))^n}{n!}\left[\left(\int_0^{\frac{1}{2}}t^n dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}t^n\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1(1-t)^n dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1(1-t)^n\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right]. \quad (2.23)$$

Since $\left|f^{(n)}\right|^q$ is s-preinvex on K in the second sense for $n \in \mathbb{N}$, $n \geq 1$, $q \in \mathbb{R}$, $q \geq 1$ and $s \in (0,1]$. Hence for every $a, b \in K$ with $\eta(b,a) > 0$, we have

$$\int_0^{\frac{1}{2}}t^n\left|f^{(n)}(a+t\eta(b,a))\right|^q dt \leq \left|f^{(n)}(a)\right|^q\int_0^{\frac{1}{2}}t^n(1-t)^s dt+\left|f^{(n)}(b)\right|^q\int_0^{\frac{1}{2}}t^{n+s}dt$$
$$= B\left(\frac{1}{2};n+1,s+1\right)\left|f^{(n)}(a)\right|^q+\frac{\left|f^{(n)}(b)\right|^q}{2^{n+s+1}(n+s+1)} \quad (2.24)$$

and

$$\begin{aligned}&\int_0^{\frac{1}{2}}(1-t)^n\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\\&\qquad\le\left|f^{(n)}(a)\right|^q\int_{\frac{1}{2}}^1(1-t)^{n+s}dt+\left|f^{(n)}(b)\right|^q\int_{\frac{1}{2}}^1 t^s(1-t)^n dt\\&\qquad=\frac{\left|f^{(n)}(a)\right|^q}{2^{n+s+1}(n+s+1)}+\left[B(s+1,n+1)-B\left(\frac{1}{2};s+1,n+1\right)\right]\left|f^{(n)}(b)\right|^q.\end{aligned}\tag{2.25}$$

Using (2.24), (2.25) and

$$\int_0^{\frac{1}{2}}t^{np}dt=\int_{\frac{1}{2}}^1(1-t)^{np}dt=\frac{1}{2^{np+1}(np+1)}.\tag{2.26}$$

we get the required inequality (2.22). This completes the proof of the theorem. □

Corollary 8. *If we choose* $n = 2$ *and* $s = 1$ *in the Theorem 9, we get the following inequality:*

$$\begin{aligned}&\left|f\left(a+\frac{1}{2}\eta(b,a)\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|\\&\qquad\le\frac{(\eta(b,a))^2\cdot 3^{\frac{1}{q}-1}}{2^{4-\frac{3}{q}}}\left[\left(\frac{5\left|f''(a)\right|^q+3\left|f''(b)\right|^q}{192}\right)^{\frac{1}{q}}\right.\\&\qquad\qquad\left.+\left(\frac{3\left|f''(a)\right|^q+5\left|f''(b)\right|^q}{192}\right)^{\frac{1}{q}}\right].\end{aligned}\tag{2.27}$$

Proof. Since for $n = 2$ and $s = 1$, $L = B\left(\frac{1}{2};3,2\right) = \frac{5}{192}$, $M = \frac{1}{64}$ and $N = B(2,3) - B\left(\frac{1}{2};2,3\right) = \frac{5}{192}$ and hence proof follows. □

3. Applications to Special Means

In the following we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 5. [3]*A function* $M : \mathbb{R}_+^2 \to \mathbb{R}_+$, *is called a Mean function if it has the following properties:*

1. *Homogeneity:* $M(ax, ay) = aM(x, y)$, *for all* $a > 0$,
2. *Symmetry :* $M(x, y) = M(y, x)$,
3. *Reflexivity :* $M(x, x) = x$,
4. *Monotonicity: If* $x \le x'$ *and* $y \le y'$, *then* $M(x, y) \le M(x', y')$,
5. *Internality:* $\min\{x, y\} \le M(x, y) \le \max\{x, y\}$.

We consider some means for arbitrary positive real numbers α, β (see for instance [3]).

(1) The arithmetic mean:
$$A:=A\left(\alpha,\beta\right)=\frac{\alpha+\beta}{2}$$
(2) The The geometric mean:
$$G:=G\left(\alpha,\beta\right)=\sqrt{\alpha\beta}$$
(3) The harmonic mean:
$$H:=H\left(\alpha,\beta\right)=\frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}$$
(4) The power mean:
$$P_r:=P_r\left(\alpha,\beta\right)=\left(\frac{\alpha^r+\beta^r}{2}\right)^{\frac{1}{r}},\ r\geq 1$$
(5) The identric mean:
$$I:=I\left(\alpha,\beta\right)=\begin{cases}\frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha\neq\beta\\ \alpha, & \alpha=\beta\end{cases}$$
(6) The logarithmic mean:
$$L:=L\left(\alpha,\beta\right)=\frac{\alpha-\beta}{\ln\left|\alpha\right|-\ln\left|\beta\right|},\ \ \left|\alpha\right|\neq\left|\beta\right|$$
(7) The generalized log-mean:
$$L_p:=L_p\left(\alpha,\beta\right)=\left[\frac{\beta^{p+1}-\alpha^{p+1}}{\left(p+1\right)\left(\beta-\alpha\right)}\right],\ \alpha\neq\beta,\ p\in\mathbb{R}\backslash\left\{-1,0\right\}.$$

It is well known that L_p is monotonic nondecreasing over $p\in\mathbb{R}$, with $L_{-1}:=L$ and $L_0:=I$. In particular, we have the following inequality $H\leq G\leq L\leq I\leq A$.

Now, let a and b be positive real numbers such that $a<b$. Consider the function $M:=M(a,b):[a,a+\eta(b,a)]\times[a,a+\eta(b,a)]\to\mathbb{R}^+$, which is one of the above mentioned means, therefore one can obtained variant inequalities for these means as follows:

Setting $\eta(b,a)=M(b,a)$ in (2.6), (2.13), one can obtain the following interesting inequalities involving means:

$$\left|\frac{f\left(a\right)+f\left(a+M(b,a)\right)}{2}-\frac{1}{M(b,a)}\int_a^{a+M(b,a)}f\left(x\right)dx\right|\\ \leq\frac{\left(M(b,a)\right)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{\left|f^{''}\left(a\right)\right|^q+\left|f^{''}\left(b\right)\right|^q}{\left(s+3\right)\left(s+2\right)}\right)^{\frac{1}{q}}. \quad (3.1)$$

$$\left|f\left(a+\frac{1}{2}M(b,a)\right)-\frac{1}{M(b,a)}\int_a^{a+M(b,a)}f\left(x\right)dx\right|\\ \leq\frac{\left(M(b,a)\right)^2}{8\left(2p+1\right)^{\frac{1}{p}}}\left[\frac{\left|f^{''}\left(a\right)\right|^q+\left|f^{''}\left(b\right)\right|^q}{s+1}\right]^{\frac{1}{q}}, \quad (3.2)$$

Letting $M = A$, G, H, P_r, I, L, L_p in (3.1) and in (3.2), we get the inequalities involving means for a particular choice of a twice differentiable s-preinvex function f, and the details are left to the interested reader.

References

[1] T. Antczak, Mean value in invexity analysis, Nonl. Anal., 60 (2005), 1473-1484.

[2] Merve Avci, Havva Kavurmaci and M. Emin Özdemir, New inequalities of Hermite-Hadamard type via s-convex functions in the second sense with applications, Applied Mathematics and Computation, 217 (2011) 5171-5176.

[3] P.S. Bullen, Handbook of Means and Their Inequalities, Kluwer Academic Publishers, Dordrecht, 2003.

[4] A. Barani, A.G. Ghazanfari, S.S. Dragomir, Hermite-Hadamard inequality through prequsi-invex functions, RGMIA Research Report Collection, 14(2011), Article 48, 7 pp.

[5] A. Barani, A.G. Ghazanfari, S.S. Dragomir, Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex, RGMIA Research Report Collection, 14(2011), Article 64, 11 pp.

[6] A. Ben-Israel and B. Mond, What is invexity?, J. Austral. Math. Soc., Ser. B, 28(1986), No. 1, 1-9.

[7] S.S. Dragomir and R.P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and trapezoidal formula, Appl. Math. Lett., 11(5) (1998), 91-95.

[8] S. S. Dragomir and C. E. M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000.

[9] S.S. Dragomir and S. Fitzpatrick, The Hadamard's inequality for s-convex functions in the second sense, Demonstratio Math., 32 (4) (1999), 687-696.

[10] H. Hudzik, L. Maligranda, Some remraks on s-convex functions, Aequationes Math. 48 (1994), 100-111.

[11] Wei-Dong Jiang, Da-Wei Niu, Yun Hua, and Feng Qi, Generalizations of Hermite-Hadamard inequality to n-time differentiable functions which are s-convex in the second sense, Analysis (Munich) 32 (2012), 1001–1012; Available online at http://dx.doi.org/10.1524/anly.2012.1161.

[12] M. A. Hanson, On sufficiency of the Kuhn-Tucker conditions, J. Math. Anal. Appl. 80 (1981) 545-550.

[13] Shu-Hong, Bo-Yan Xi and Feng Qi, Some new inequalities of Hermite-Hadamard type for n-times differentiable functions which are m-convex, Analysis (Munich) 32 (2012), no. 3, 247-262; Available online at http://dx.doi.org/10.1524/anly.2012.1167.

[14] J. Hadamard, Étude sur les propriétés des fonctions entières et en particulier d'une fonction considerée par Riemann, J. Math Pures Appl., 58 (1893), 171–215.

[15] Dah-Yang Hwang, Some inequalities for n-time differentiable mappings and applications, Kyugpook Math. J. 43(2003), 335-343.

[16] M. A. Latif, On Hermite-Hadamard type integral inequalities for n-times differentiable preinvex functions with applications, Stud. Univ. Babeş-Bolyai Math. 58(2013), No. 3, 325–343.

[17] M. A. Latif and S. S. Dragomir, Some weighted integral inequalities for differentiable preinvex and prequasiinvex functions with applications, Journal of Inequalities and Applications 2013, 2013:575.

[18] M. A. Latif, Some inequalities for prequasiinvex functions with applications, Konurlap Journal of Math.Vol. 1, no 2, 17-29.

[19] I. Iscan, Ostrowski type inequalites for functions whose derivatives are preinvex, arXiv:1204.2010v1.

[20] U.S. Kırmacı, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp., 147 (2004), 137-146.

[21] U.S. Kırmacı and M.E. Özdemir, On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp., 153 (2004), 361-368.

[22] U.S. Kırmacı, Improvement and further generalization of inequalities for differentiable mappings and applications, Computers and Math. with Appl., 55 (2008), 485-493.

[23] U. S. Kirmaci, M. Klaričić Bakula, M. E. Özdemir, J. Pečarić, Hadamard-type inequalities for s-convex functions, Appl. Math. and Comput., Volume 193, Issue 1(2007), Pages 26-35.

[24] S. R. Mohan and S. K. Neogy, On invex sets and preinvex functions, J. Math. Anal. Appl. 189 (1995), 901–908.

[25] M. A. Noor, Hermite-Hadamard integral inequalities for log-preinvex functions, J. Math. Anal. Approx. Theory, 2(2007), 126-131.

[26] M. A. Noor, Variational like inequalities, Optimization, 30(1994), 323-330.

[27] M. A. Noor, On Hadamard integral inequalities involving two log-preinvex functions, J. Inequal. Pure Appl. Math., 8(2007), No. 3, 1-6, Article 75.

[28] M. Aslam Noor, Hadamard integral inequalities for product of two preinvex function, Nonl. Anal. Forum, 14 (2009), 167-173.

[29] C.E.M. Pearce and J. Pečarić, Inequalities for differentiable mappings with application to special means and quadrature formulae, Appl. Math. Lett., 13(2) (2000), 51–55.

[30] R. Pini, Invexity and generalized Convexity, Optimization 22 (1991) 513-525.

[31] M. Z. Sarikaya, A. Saglam and H. Yıldırım, New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are convex and quasi-convex, International Journal of Open Problems in Computer Science and Mathematics (IJOPCM), 5(3), 2012.

[32] M. Z. Sarikaya, A. Saglam and H. Yıldırım, On some Hadamard-type inequalities for h-convex functions, Journal of Mathematical Inequalities, Volume 2, Number 3 (2008), 335-341.

[33] M. Z. Sarikaya, M. Avci and H. Kavurmaci, On some inequalities of Hermite-Hadamard type for convex functions, ICMS Iternational Conference on Mathematical Science. AIP Conference Proceedings 1309, 852 (2010).

[34] M. Z. Sarikaya and N. Aktan, On the generalization some integral inequalities and their applications Mathematical and Computer Modelling, Volume 54, Issues 9-10, November 2011, Pages 2175-2182.

[35] M. Z. Sarikaya, H. Bozkurt and N. Alp, On Haermite-Hadamard type integral inequalities for preinvex and log-preinvex functions, Contemporary Analysis and Applied Mathematics Vol.1, No.2, 237-252, 2013.

[36] Mehmet Zeki Sarikaya, Erhan Set, M. Emin Özdemir, On new inequalities of Simpson's type for s-convex functions, Computers and Mathematics with Applications, 60(2010), 2191-2199.

[37] A. Saglam, M. Z. Sarikaya and H. Yildirim, Some new inequalities of Hermite-Hadamard's type, Kyungpook Mathematical Journal, 50(2010), 399-410.

[38] T. Weir, and B. Mond, Preinvex functions in multiobjective optimization, Journal of Mathematical Analysis and Applications, 136 (1998) 29-38.

[39] Y. Wang, S. -H. Wang, and F. Qi, Simpson type integral inequalities in which the power of the absolute value of the first derivative of the integrand is s-preinvex, Facta Univ. (NIŠ), Ser. Math. Inform. Vol. 28, No 2 (2013), 151–159.

[40] Y. Wang - B.-Y. Xi - Feng Qi, Hermite-Hadamard type integral inequalities when the power of the absolute value of the first derivative of the integrand is preinvex, Le Matematiche. (to appear)

[41] X. M. Yang and D. Li, On properties of preinvex functions, J. Math. Anal. Appl. 256 (2001) 229-241.

[42] Li Jue-You, On Hadamard-type inequalities for s-preinvex functions, Journal of Chongqing Normal University (Natural Science) 27(2010). [http://en.cnki.com.cn/Article_en/CJFDTotal-CQSF201004003.htm]

[1]School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa

E-mail address: `m_amer_latif@hotmail.com`

[2]School of Engineering and Science, Victoria University, PO Box 14428 Melbourne City, MC 8001, Australia, [3]School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa

E-mail address: `sever.dragomir@vu.edu.au`