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TRACE INEQUALITIES OF SHISHA-MOND TYPE FOR OPERATORS IN HILBERT SPACES

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Abstract. Some trace inequalities of Shisha-Mond type for operators in Hilbert spaces are provided. Applications in connection to Grüss inequality and for convex functions of selfadjoint operators are also given.

1. Introduction

In 1967, Shisha and Mond [55, p. 301] proved the following reverse of Cauchy-Bunyakovsky-Schwarz inequality:

Theorem 1. *Let* $\overline{\mathbf{a}}=(a_1,\dots,a_n)$ *and* $\overline{\mathbf{b}}=(b_1,\dots,b_n)$ *be two positive* n*-tuples with*

$$0<m\le\frac{a_k}{b_k}\le M<\infty \text{ for each } k\in\{1,\dots,n\}, \tag{1.1}$$

then

$$0\le\left(\sum_{k=1}^n a_k^2\sum_{k=1}^n b_k^2\right)^{1/2}-\sum_{k=1}^n a_kb_k\le\frac{(M-m)^2}{4(M+m)}\sum_{k=1}^n b_k^2. \tag{1.2}$$

The equality holds in (1.2) if and only if there exists a subsequence $(k_1,...,k_p)$ *of* $\{1,\dots,n\}$ *such that*

$$\sum_{m=1}^p b_{k_m}^2=\frac{M+3m}{4(M+m)}\sum_{k=1}^n b_k^2,$$

$\frac{a_{k_m}}{b_{k_m}}=M$ *for every* $m=1,...,p$ *and* $\frac{a_k}{b_k}=m$ *for every* k *distinct from all* k_m.

Recall some other classical reverses of Cauchy-Bunyakovsky-Schwarz inequality when bounds for each n-tuple are available.

Let $\overline{\mathbf{a}}=(a_1,\dots,a_n)$ and $\overline{\mathbf{b}}=(b_1,\dots,b_n)$ be two positive *n-tuples* with

$$0<m_1\le a_i\le M_1<\infty \text{ and } 0<m_2\le b_i\le M_2<\infty; \tag{1.3}$$

for each $i\in\{1,\dots,n\}$, and some constants m_1, m_2, M_1, M_2.

The following reverses of the Cauchy-Bunyakovsky-Schwarz inequality for positive sequences of real numbers are well known:

a) *Pólya-Szegö's inequality* [51]:

$$\frac{\sum_{k=1}^n a_k^2\sum_{k=1}^n b_k^2}{\left(\sum_{k=1}^n a_kb_k\right)^2}\le\frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}}+\sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2.$$

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b) *Shisha-Mond's inequality* [55]:

$$\frac{\sum_{k=1}^{n} a_k^2}{\sum_{k=1}^{n} a_k b_k} - \frac{\sum_{k=1}^{n} a_k b_k}{\sum_{k=1}^{n} b_k^2} \le \left[\left(\frac{M_1}{m_2}\right)^{\frac{1}{2}} - \left(\frac{m_1}{M_2}\right)^{\frac{1}{2}}\right]^2.$$

c) *Ozeki's inequality* [48]:

$$\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 - \left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \frac{n^2}{4}\left(M_1 M_2 - m_1 m_2\right)^2.$$

d) *Diaz-Metcalf's inequality* [17]:

$$\sum_{k=1}^{n} b_k^2 + \frac{m_2 M_2}{m_1 M_1} \sum_{k=1}^{n} a_k^2 \le \left(\frac{M_2}{m_1} + \frac{m_2}{M_1}\right) \sum_{k=1}^{n} a_k b_k.$$

If $\overline{\mathbf{w}} = (w_1, \dots, w_n)$ is a positive sequence, then the following weighted inequalities also hold:

e) *Cassels' inequality* [58]. If the positive real sequences $\overline{\mathbf{a}} = (a_1, \dots, a_n)$ and $\overline{\mathbf{b}} = (b_1, \dots, b_n)$ satisfy the condition (1.1), then

$$\frac{\left(\sum_{k=1}^{n} w_k a_k^2\right)\left(\sum_{k=1}^{n} w_k b_k^2\right)}{\left(\sum_{k=1}^{n} w_k a_k b_k\right)^2} \le \frac{(M+m)^2}{4mM}.$$

f) *Greub-Reinboldt's inequality* [38]. We have

$$\left(\sum_{k=1}^{n} w_k a_k^2\right)\left(\sum_{k=1}^{n} w_k b_k^2\right) \le \frac{(M_1 M_2 + m_1 m_2)^2}{4 m_1 m_2 M_1 M_2} \left(\sum_{k=1}^{n} w_k a_k b_k\right)^2,$$

provided $\overline{\mathbf{a}} = (a_1, \dots, a_n)$ and $\overline{\mathbf{b}} = (b_1, \dots, b_n)$ satisfy the condition (1.3).

g) *Generalized Diaz-Metcalf's inequality* [17], see also [46, p. 123]. If $u, v \in [0,1]$ and $v \le u$, $u + v = 1$ and (1.1) holds, then one has the inequality

$$u \sum_{k=1}^{n} w_k b_k^2 + vMm \sum_{k=1}^{n} w_k a_k^2 \le (vm + uM) \sum_{k=1}^{n} w_k a_k b_k.$$

h) *Klamkin-McLenaghan's inequality* [40]. If $\overline{\mathbf{a}}$, $\overline{\mathbf{b}}$ satisfy (1.1), then

$$\left(\sum_{i=1}^{n} w_i a_i^2\right)\left(\sum_{i=1}^{n} w_i b_i^2\right) - \left(\sum_{i=1}^{n} w_i a_i b_i\right)^2 \le \left(M^{\frac{1}{2}} - m^{\frac{1}{2}}\right)^2 \sum_{i=1}^{n} w_i a_i b_i \sum_{i=1}^{n} w_i a_i^2. \tag{1.4}$$

For other recent results providing discrete reverse inequalities, see the monograph online [19].

The following reverse of Schwarz's inequality in inner product spaces holds [20].

Theorem 2 (Dragomir, 2003, [20])**.** *Let $A, a \in \mathbb{C}$ and $x, y \in H$, where H is a complex inner product space with the inner product $\langle \cdot, \cdot \rangle$. If*

$$\operatorname{Re}\langle Ay - x, x - ay \rangle \ge 0, \tag{1.5}$$

or equivalently,

$$\left\| x - \frac{a + A}{2} \cdot y \right\| \le \frac{1}{2} |A - a| \|y\|, \tag{1.6}$$

holds, then we have the inequality

$$0 \leq \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \leq \frac{1}{4} |A - a|^2 \|y\|^4 . \tag{1.7}$$

The constant $\frac{1}{4}$ *is sharp in (1.7).*

In 1935, G. Grüss [39] proved the following integral inequality which gives an approximation of the integral mean of the product in terms of the product of the integrals means as follows:

$$\left|\frac{1}{b-a}\int_a^b f(x) g(x) dx - \frac{1}{b-a}\int_a^b f(x) dx \cdot \frac{1}{b-a}\int_a^b g(x) dx\right| \leq \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma), \tag{1.8}$$

where $f,\ g : [a, b] \to \mathbb{R}$ are integrable on $[a, b]$ and satisfy the condition

$$\phi \leq f(x) \leq \Phi,\ \gamma \leq g(x) \leq \Gamma \tag{1.9}$$

for each $x \in [a, b]$, where $\phi,\ \Phi,\ \gamma,\ \Gamma$ are given real constants.

Moreover, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

In [22], in order to generalize the Grüss integral inequality in abstract structures the author has proved the following inequality in inner product spaces.

Theorem 3 (Dragomir, 1999, [22])**.** *Let* $(H, \langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}$ $(\mathbb{K} = \mathbb{R}, \mathbb{C})$ *and* $e \in H$, $\|e\| = 1$. *If* $\varphi,\ \gamma,\ \Phi,\ \Gamma$ *are real or complex numbers and* $x,\ y$ *are vectors in* H *such that the conditions*

$$\operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle \geq 0 \text{ and } \operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle \geq 0 \tag{1.10}$$

hold, then we have the inequality

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{4} |\Phi - \varphi| |\Gamma - \gamma| . \tag{1.11}$$

The constant $\frac{1}{4}$ *is best possible in the sense that it can not be replaced by a smaller constant.*

For other results of this type, see the recent monograph [25] and the references therein.

For other Grüss type results for integral and sums see the papers [1]-[3], [8]-[10], [11]-[13], [21]-[28], [35], [49], [62] and the references therein.

In order to state some reverses of Schwarz and Grüss type inequalities for trace operators on complex Hilbert spaces we need some preparations as follows.

2. Some Facts on Trace of Operators

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{2.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^* f_j\|^2 \tag{2.2}$$

showing that the definition (2.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of Hilbert-Schmidt operators in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{2.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H. This definition does not depend on the choice of the orthonormal basis.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm*.

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (2.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 4. *We have*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^*Ae_i, e_i\rangle \tag{2.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$*;*

(ii) We have the inequalities

$$\|A\| \le \|A\|_2 \tag{2.5}$$

for any $A \in \mathcal{B}_2(H)$ *and*

$$\|AT\|_2, \|TA\|_2 \le \|T\| \|A\|_2 \tag{2.6}$$

for any $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$*;*

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$*, i.e.*

$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H) \subseteq \mathcal{B}_2(H);$$

(iv) $\mathcal{B}_{fin}(H)$*, the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_2(H)$*;*

(v) $\mathcal{B}_2(H) \subseteq \mathcal{K}(H)$*, where* $\mathcal{K}(H)$ *denotes the algebra of compact operators on* H*.*

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{2.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A \in \mathcal{B}(H)$*, then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$*;*

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$*;*

(ii) A *(or* $|A|$*) is the product of two elements of* $\mathcal{B}_2(H)$*.*

The following properties are also well known:

Theorem 5. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \text{ and } \|A\|_2 \le \|A\|_1 \tag{2.8}$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_1(H)\,\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\,\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\left\{\langle A, B\rangle_2 \mid B \in \mathcal{B}_2(H),\ \|B\| \le 1\right\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

(iv) We have the following isometric isomorphisms

$$\mathcal{B}_1(H) \cong K(H)^* \text{ and } \mathcal{B}_1(H)^* \cong \mathcal{B}(H),$$

where $K(H)^*$ *is the dual space of* $K(H)$ *and* $\mathcal{B}_1(H)^*$ *is the dual space of* $\mathcal{B}_1(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{2.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (2.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 6. *We have*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{2.10}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT, TA \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{2.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If $A, B \in \mathcal{B}_2(H)$ *then* $AB, BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

Utilising the trace notation we obviously have that

$$\langle A, B\rangle_2 = \operatorname{tr}(B^*A) = \operatorname{tr}(AB^*) \text{ and } \|A\|_2^2 = \operatorname{tr}(A^*A) = \operatorname{tr}\left(|A|^2\right)$$

for any $A, B \in \mathcal{B}_2(H)$.

The following Hölder's type inequality has been obtained by Ruskai in [52]

$$|\operatorname{tr}(AB)| \le \operatorname{tr}(|AB|) \le \left[\operatorname{tr}\left(|A|^{1/\alpha}\right)\right]^{\alpha}\left[\operatorname{tr}\left(|B|^{1/(1-\alpha)}\right)\right]^{1-\alpha} \tag{2.12}$$

where $\alpha \in (0,1)$ and $A, B \in \mathcal{B}(H)$ with $|A|^{1/\alpha}, |B|^{1/(1-\alpha)} \in \mathcal{B}_1(H)$.

In particular, for $\alpha = \frac{1}{2}$ we get the Schwarz inequality

$$|\operatorname{tr}(AB)| \le \operatorname{tr}(|AB|) \le \left[\operatorname{tr}\left(|A|^2\right)\right]^{1/2}\left[\operatorname{tr}\left(|B|^2\right)\right]^{1/2} \tag{2.13}$$

with $A,\ B\in\mathcal{B}_2(H)$.

For the theory of trace functionals and their applications the reader is referred to [56].

For some classical trace inequalities see [14], [16], [47] and [61], which are continuations of the work of Bellman [5]. For related works the reader can refer to [4], [6], [14], [36], [41], [42], [44], [53] and [57].

We denote by

$$\mathcal{B}_1^+(H):=\{P:\ P\in\mathcal{B}_1(H),\ P \text{ and is selfadjoint and } P\geq 0\}.$$

We obtained recently the following result [33]:

Theorem 7. *For any $A,\ C\in\mathcal{B}(H)$ and $P\in\mathcal{B}_1^+(H)\setminus\{0\}$ we have the inequality*

$$\begin{aligned}\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| &\leq \inf_{\lambda\in\mathbb{C}}\|A-\lambda\cdot 1_H\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\\ &\leq \inf_{\lambda\in\mathbb{C}}\|A-\lambda\cdot 1_H\|\left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2},\end{aligned}\tag{2.14}$$

where $\|\cdot\|$ is the operator norm.

We also have [33]:

Corollary 1. *Let $\alpha,\ \beta\in\mathbb{C}$ and $A\in B(H)$ such that*

$$\left\|A-\frac{\alpha+\beta}{2}\cdot 1_H\right\|\leq\frac{1}{2}|\beta-\alpha|.$$

For any $C\in\mathcal{B}(H)$ and $P\in\mathcal{B}_1^+(H)\setminus\{0\}$ we have the inequality

$$\begin{aligned}\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| &\leq \frac{1}{2}|\beta-\alpha|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\\ &\leq \frac{1}{2}|\beta-\alpha|\left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2}.\end{aligned}\tag{2.15}$$

In particular, if $C\in\mathcal{B}(H)$ is such that

$$\left\|C-\frac{\alpha+\beta}{2}\cdot 1_H\right\|\leq\frac{1}{2}|\beta-\alpha|,$$

then

$$0\le\frac{\operatorname{tr}\left(P\left|C\right|^{2}\right)}{\operatorname{tr}\left(P\right)}-\left|\frac{\operatorname{tr}\left(PC\right)}{\operatorname{tr}\left(P\right)}\right|^{2}\le\frac{1}{2}\left|\beta-\alpha\right|\frac{1}{\operatorname{tr}\left(P\right)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}\left(PC\right)}{\operatorname{tr}\left(P\right)}1_{H}\right)P\right|\right)\le\frac{1}{2}\left|\beta-\alpha\right|\left[\frac{\operatorname{tr}\left(P\left|C\right|^{2}\right)}{\operatorname{tr}\left(P\right)}-\left|\frac{\operatorname{tr}\left(PC\right)}{\operatorname{tr}\left(P\right)}\right|^{2}\right]^{1/2}\le\frac{1}{4}\left|\beta-\alpha\right|^{2}. \tag{2.16}$$

Also

$$\left|\frac{\operatorname{tr}\left(PC^{2}\right)}{\operatorname{tr}\left(P\right)}-\left(\frac{\operatorname{tr}\left(PC\right)}{\operatorname{tr}\left(P\right)}\right)^{2}\right|\le\frac{1}{2}\left|\beta-\alpha\right|\frac{1}{\operatorname{tr}\left(P\right)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}\left(PC\right)}{\operatorname{tr}\left(P\right)}1_{H}\right)P\right|\right)\le\frac{1}{2}\left|\beta-\alpha\right|\left[\frac{\operatorname{tr}\left(P\left|C\right|^{2}\right)}{\operatorname{tr}\left(P\right)}-\left|\frac{\operatorname{tr}\left(PC\right)}{\operatorname{tr}\left(P\right)}\right|^{2}\right]^{1/2}\le\frac{1}{4}\left|\beta-\alpha\right|^{2}. \tag{2.17}$$

For other related results see [33].

3. Shisha-Mond Type Trace Inequalities

For two given operators T, $U\in B(H)$ and two given scalars α, $\beta\in\mathbb{C}$ consider the transform

$$\mathcal{C}_{\alpha,\beta}\left(T,U\right)=\left(T^{*}-\bar{\alpha}U^{*}\right)\left(\beta U-T\right).$$

This transform generalizes the transform

$$\mathcal{C}_{\alpha,\beta}\left(T\right):=\left(T^{*}-\bar{\alpha}1_{H}\right)\left(\beta 1_{H}-T\right)=\mathcal{C}_{\alpha,\beta}\left(T,1_{H}\right),$$

where 1_H is the identity operator, which has been introduced in [31] in order to provide some generalizations of the well known Kantorovich inequality for operators in Hilbert spaces.

We recall that a bounded linear operator T on the complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$ is called *accretive* if $\operatorname{Re}\langle Ty,y\rangle\ge 0$ for any $y\in H$.

Utilizing the following identity

$$\operatorname{Re}\left\langle\mathcal{C}_{\alpha,\beta}\left(T,U\right)x,x\right\rangle=\operatorname{Re}\left\langle\mathcal{C}_{\beta,\alpha}\left(T,U\right)x,x\right\rangle=\frac{1}{4}\left|\beta-\alpha\right|^{2}\left\|Ux\right\|^{2}-\left\|Tx-\frac{\alpha+\beta}{2}\cdot Ux\right\|^{2}=\frac{1}{4}\left|\beta-\alpha\right|^{2}\left\langle\left|U\right|^{2}x,x\right\rangle-\left\langle\left|T-\frac{\alpha+\beta}{2}\cdot U\right|^{2}x,x\right\rangle \tag{3.1}$$

that holds for any scalars α, β and any vector $x\in H$, we can give a simple characterization result that is useful in the following:

Lemma 1. *For α, $\beta\in\mathbb{C}$ and T, $U\in B(H)$ the following statements are equivalent:*

(i) *The transform $\mathcal{C}_{\alpha,\beta}\left(T,U\right)$ (or, equivalently, $\mathcal{C}_{\beta,\alpha}\left(T,U\right)$) is accretive;*

(ii) *We have the norm inequality*

$$\left\| Tx - \frac{\alpha+\beta}{2} \cdot Ux \right\| \leq \frac{1}{2} |\beta - \alpha| \|Ux\| \tag{3.2}$$

for any $x \in H$;

(iii) *We have the following inequality in the operator order*

$$\left| T - \frac{\alpha+\beta}{2} \cdot U \right|^2 \leq \frac{1}{4} |\beta - \alpha|^2 |U|^2 .$$

As a consequence of the above lemma we can state:

Corollary 2. *Let* α, $\beta \in \mathbb{C}$ *and* T, $U \in B(H)$. *If* $\mathcal{C}_{\alpha,\beta}(T,U)$ *is accretive, then*

$$\left\| T - \frac{\alpha+\beta}{2} \cdot U \right\| \leq \frac{1}{2} |\beta - \alpha| \|U\| . \tag{3.3}$$

Remark 1. *In order to give examples of linear operators* $T, U \in B(H)$ *and numbers* α, $\beta \in \mathbb{C}$ *such that the transform* $\mathcal{C}_{\alpha,\beta}(T,U)$ *is accretive, it suffices to select two bounded linear operator* S *and* V *and the complex numbers* z, w $(w \neq 0)$ *with the property that* $\|Sx - zVx\| \leq |w| \|Vx\|$ *for any* $x \in H$, *and, by choosing* $T = S$, $U = V$, $\alpha = \frac{1}{2}(z + w)$ *and* $\beta = \frac{1}{2}(z - w)$ *we observe that* T *and* U *satisfy (3.2), i.e.,* $\mathcal{C}_{\alpha,\beta}(T,U)$ *is accretive.*

The following result is useful in the sequel:

Lemma 2. *Let, either* $P \in \mathcal{B}_+(H)$, A, $B \in \mathcal{B}_2(H)$ *or* $P \in \mathcal{B}_1^+(H)$, A, $B \in \mathcal{B}(H)$ *and* γ, $\Gamma \in \mathbb{C}$. *Then*

$$\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^* - \overline{\gamma} B^*\right)(\Gamma B - A)\right]\right) \geq 0 \tag{3.4}$$

if and only if

$$\operatorname{tr}\left(P \left| A - \frac{\gamma+\Gamma}{2} B \right|^2\right) \leq \frac{1}{4} |\Gamma - \gamma|^2 \operatorname{tr}\left(P |B|^2\right). \tag{3.5}$$

To simplify the writing, we the say that (A, B) *satisfies the* P-(γ, Γ)*-trace property.*

Proof. Doing the calculation, we have the equality

$$\begin{aligned} &\frac{1}{4} |\Gamma - \gamma|^2 P |B|^2 - P \left| A - \frac{\gamma+\Gamma}{2} B \right|^2 \\ &= P \left[-|A|^2 + \frac{\overline{\gamma+\Gamma}}{2} B^* A + \frac{\gamma+\Gamma}{2} A^* B - \operatorname{Re}(\Gamma \overline{\gamma}) |B|^2 \right] \end{aligned} \tag{3.6}$$

for any bounded operators A, B, P and the complex numbers γ, $\Gamma \in \mathbb{C}$.

Taking the trace in (3.6) we get after some simple manipulation

$$\begin{aligned} &\frac{1}{4} |\Gamma - \gamma|^2 \operatorname{tr}\left(P |B|^2\right) - \operatorname{tr}\left(P \left| A - \frac{\gamma+\Gamma}{2} B \right|^2\right) \\ &= -\operatorname{tr}\left(P |A|^2\right) - \operatorname{Re}(\Gamma \overline{\gamma}) \operatorname{tr}\left(P |B|^2\right) \\ &+ \operatorname{Re}\left[\overline{\gamma} \operatorname{tr}(PB^*A)\right] + \operatorname{Re}\left[\Gamma \overline{\operatorname{tr}(PB^*A)}\right]. \end{aligned} \tag{3.7}$$

Since

$$\begin{aligned}&\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^{*}-\overline{\gamma}B^{*}\right)\left(\Gamma B-A\right)\right]\right)\\&=\operatorname{Re}\left[\Gamma\overline{\operatorname{tr}\left(PB^{*}A\right)}+\overline{\gamma}\operatorname{tr}\left(PB^{*}A\right)\right]-\operatorname{tr}\left(P\left|B\right|^{2}\right)\operatorname{Re}\left(\overline{\gamma}\Gamma\right)-\operatorname{tr}\left(P\left|A\right|^{2}\right),\end{aligned}$$

then we get

$$\begin{aligned}&\frac{1}{4}\left|\Gamma-\gamma\right|^{2}\operatorname{tr}\left(P\left|B\right|^{2}\right)-\operatorname{tr}\left(P\left|A-\frac{\gamma+\Gamma}{2}B\right|^{2}\right)\\&=\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^{*}-\overline{\gamma}B^{*}\right)\left(\Gamma B-A\right)\right]\right),\end{aligned}\tag{3.8}$$

which proves the desired equivalence. □

Corollary 3. *Let, either $P\in\mathcal{B}_{+}(H)$, A, $B\in\mathcal{B}_{2}(H)$ or $P\in\mathcal{B}_{1}^{+}(H)$, A, $B\in\mathcal{B}(H)$ and γ, $\Gamma\in\mathbb{C}$. If the transform $\mathcal{C}_{\gamma,\Gamma}(A,B)$ is accretive, then (A,B) satisfies the P-(γ,Γ)-trace property.*

We have the following result:

Theorem 8. *Let, either $P\in\mathcal{B}_{+}(H)$, A, $B\in\mathcal{B}_{2}(H)$ or $P\in\mathcal{B}_{1}^{+}(H)$, A, $B\in\mathcal{B}(H)$ and γ, $\Gamma\in\mathbb{C}$ with $\Gamma+\gamma\neq0$.*

(i) If (A,B) satisfies the P-(γ,Γ)-trace property, then we have

$$\begin{aligned}&\sqrt{\operatorname{tr}\left(P\left|A\right|^{2}\right)\operatorname{tr}\left(P\left|B\right|^{2}\right)}\\&\leq\frac{\operatorname{Re}\left(\gamma+\Gamma\right)\operatorname{Re}\operatorname{tr}\left(PB^{*}A\right)+\operatorname{Im}\left(\gamma+\Gamma\right)\operatorname{Im}\operatorname{tr}\left(PB^{*}A\right)}{\left|\Gamma+\gamma\right|}\\&+\frac{1}{4}\frac{\left|\Gamma-\gamma\right|^{2}}{\left|\Gamma+\gamma\right|}\operatorname{tr}\left(P\left|B\right|^{2}\right)\\&\leq\left|\operatorname{tr}\left(PB^{*}A\right)\right|+\frac{1}{4}\frac{\left|\Gamma-\gamma\right|^{2}}{\left|\Gamma+\gamma\right|}\operatorname{tr}\left(P\left|B\right|^{2}\right).\end{aligned}\tag{3.9}$$

(ii) If the transform $\mathcal{C}_{\gamma,\Gamma}(A,B)$ is accretive, then the inequality (3.9) also holds.

Proof. (i) If (A,B) satisfies the P-(γ,Γ)-trace property, then

$$\operatorname{tr}\left(P\left|A-\frac{\gamma+\Gamma}{2}B\right|^{2}\right)\leq\frac{1}{4}\left|\Gamma-\gamma\right|^{2}\operatorname{tr}\left(P\left|B\right|^{2}\right)$$

that is equivalent to

$$\operatorname{tr}\left(P\left|A\right|^{2}\right)-\operatorname{Re}\left[\left(\overline{\gamma}+\overline{\Gamma}\right)\operatorname{tr}\left(PB^{*}A\right)\right]+\frac{1}{4}\left|\Gamma+\gamma\right|^{2}\operatorname{tr}\left(P\left|B\right|^{2}\right)\leq\frac{1}{4}\left|\Gamma-\gamma\right|^{2}\operatorname{tr}\left(P\left|B\right|^{2}\right),$$

which implies that

$$\begin{aligned}&\operatorname{tr}\left(P\left|A\right|^{2}\right)+\frac{1}{4}\left|\Gamma+\gamma\right|^{2}\operatorname{tr}\left(P\left|B\right|^{2}\right)\\&\leq\operatorname{Re}\left[\left(\overline{\gamma}+\overline{\Gamma}\right)\operatorname{tr}\left(PB^{*}A\right)\right]+\frac{1}{4}\left|\Gamma-\gamma\right|^{2}\operatorname{tr}\left(P\left|B\right|^{2}\right).\end{aligned}\tag{3.10}$$

Making use of the elementary inequality

$$2\sqrt{pq}\leq p+q,\ \ p,q\geq0,$$

we also have

$$(3.11)\quad |\Gamma+\gamma|\left[\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\leq \operatorname{tr}\left(P|A|^2\right)+\frac{1}{4}|\Gamma+\gamma|^2\operatorname{tr}\left(P|B|^2\right).$$

Utilising (3.10) and (3.11) we get

$$(3.12)\quad \begin{aligned}&|\Gamma+\gamma|\left[\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\\ &\leq \operatorname{Re}\left[\left(\overline{\gamma}+\overline{\Gamma}\right)\operatorname{tr}\left(PB^*A\right)\right]+\frac{1}{4}|\Gamma-\gamma|^2\operatorname{tr}\left(P|B|^2\right).\end{aligned}$$

Dividing by $|\Gamma+\gamma|>0$ and observing that

$$\operatorname{Re}\left[\left(\overline{\gamma}+\overline{\Gamma}\right)\operatorname{tr}\left(PB^*A\right)\right]=\operatorname{Re}\left(\gamma+\Gamma\right)\operatorname{Re}\operatorname{tr}\left(PB^*A\right)+\operatorname{Im}\left(\gamma+\Gamma\right)\operatorname{Im}\operatorname{tr}\left(PB^*A\right)$$

we get the first inequality in (3.9).

The second inequality in (3.9) is obvious by Schwarz inequality

$$(ab+cd)^2\leq\left(a^2+c^2\right)\left(b^2+d^2\right),\ a,b,c,d\in\mathbb{R}.$$

The (ii) is obvious from (i). □

Remark 2. *We observe that the inequality between the first and last term in (3.9) is equivalent to*

$$(3.13)\quad 0\leq\sqrt{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right)}-\left|\operatorname{tr}\left(PB^*A\right)\right|\leq\frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\operatorname{tr}\left(P|B|^2\right).$$

Corollary 4. *Let, either* $P\in\mathcal{B}_+(H)$, $A\in\mathcal{B}_2(H)$ *or* $P\in\mathcal{B}_1^+(H)$, $A\in\mathcal{B}(H)$ *and* $\gamma,\ \Gamma\in\mathbb{C}$ *with* $\gamma+\Gamma\neq0$.

(i) If A *satisfies the* P-(γ,Γ)*-trace property, namely*

$$(3.14)\quad \operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}1_H\right)\left(\Gamma1_H-A\right)\right]\right)\geq0$$

or, equivalently

$$(3.15)\quad \operatorname{tr}\left(P\left|A-\frac{\gamma+\Gamma}{2}1_H\right|^2\right)\leq\frac{1}{4}|\Gamma-\gamma|^2\operatorname{tr}(P),$$

then we have

$$(3.16)\quad \begin{aligned}&\sqrt{\frac{\operatorname{tr}\left(P|A|^2\right)}{\operatorname{tr}(P)}}\\ &\leq\frac{\operatorname{Re}\left(\gamma+\Gamma\right)\frac{\operatorname{Re}\operatorname{tr}(PA)}{\operatorname{tr}(P)}+\operatorname{Im}\left(\gamma+\Gamma\right)\frac{\operatorname{Im}\operatorname{tr}(PA)}{\operatorname{tr}(P)}}{|\Gamma+\gamma|}+\frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\\ &\leq\left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|+\frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}.\end{aligned}$$

(ii) If the transform $\mathcal{C}_{\gamma,\Gamma}(A)$ *is accretive, then the inequality (3.9) also holds.*

(iii) We have

$$(3.17)\quad 0\leq\sqrt{\frac{\operatorname{tr}\left(P|A|^2\right)}{\operatorname{tr}(P)}}-\left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|\leq\frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}.$$

Remark 3. *The case of selfadjoint operators is as follows.*

Let A, B be selfadjoint operators and either $P \in \mathcal{B}_+(H)$, A, $B \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, A, $B \in \mathcal{B}(H)$ and m, $M \in \mathbb{R}$ with $m + M \neq 0$.

(i) If (A, B) satisfies the P-(m, M)-trace property, then we have

$$\sqrt{\operatorname{tr}(PA^2)\operatorname{tr}(PB^2)} \leq \operatorname{Re}\operatorname{tr}(PBA) + \frac{(M-m)^2}{4|M+m|}\operatorname{tr}\left(PB^2\right) \leq |\operatorname{tr}(PBA)| + \frac{(M-m)^2}{4|M+m|}\operatorname{tr}\left(PB^2\right) \tag{3.18}$$

and

$$0 \leq \sqrt{\operatorname{tr}(PA^2)\operatorname{tr}(PB^2)} - \operatorname{Re}\operatorname{tr}(PBA) \leq \frac{(M-m)^2}{4|M+m|}\operatorname{tr}\left(PB^2\right).$$

(ii) If the transform $\mathcal{C}_{m,M}(A, B)$ is accretive, then the inequality (3.18) also holds.

(iii) If $(A - mB)(MB - A) \geq 0$, then (3.18) is valid.

We observe that the inequality (3.18) in the case when $M > m > 0$ is the operator trace inequality version of Shisha-Mond inequality (1.1) from Introduction.

Corollary 5. *Let A, B be selfadjoint operators and either $P \in \mathcal{B}_+(H)$, A, $B \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, A, $B \in \mathcal{B}(H)$ and m, $M \in \mathbb{R}$ with $m + M \neq 0$.*

(i) If (A, B) satisfies the P-(m, M)-trace property, then we have

$$\left(\sqrt{\operatorname{tr}(PA^2)} + \sqrt{\operatorname{tr}(PB^2)}\right)^2 - \operatorname{tr}\left(P(A+B)^2\right) \leq \frac{(M-m)^2}{4|M+m|}\operatorname{tr}\left(PB^2\right) \tag{3.19}$$

and

$$\sqrt{\operatorname{tr}(PA^2)} + \sqrt{\operatorname{tr}(PB^2)} - \sqrt{\operatorname{tr}\left(P(A+B)^2\right)} \leq \frac{\sqrt{2}}{2}\frac{M-m}{\sqrt{|M+m|}}\sqrt{\operatorname{tr}(PB^2)}. \tag{3.20}$$

Proof. Observe that

$$\left(\sqrt{\operatorname{tr}(PA^2)} + \sqrt{\operatorname{tr}(PB^2)}\right)^2 - \operatorname{tr}\left(P(A+B)^2\right) = 2\left(\sqrt{\operatorname{tr}(PA^2)\operatorname{tr}(PB^2)} - \operatorname{Re}\operatorname{tr}(PBA)\right).$$

Utilising (3.18) we deduce (3.19).

The inequality (3.20) follows from (3.19). □

4. Trace Inequalities of Grüss Type

Let P be a selfadjoint operator with $P \geq 0$. The functional $\langle\cdot,\cdot\rangle_{2,P}$ defined by

$$\langle A, B\rangle_{2,P} := \operatorname{tr}(PB^*A) = \operatorname{tr}(APB^*) = \operatorname{tr}(B^*AP)$$

is a *nonnegative Hermitian form* on $\mathcal{B}_2(H)$, i.e. $\langle\cdot,\cdot\rangle_{2,P}$ satisfies the properties:

(h) $\langle A, A\rangle_{2,P} \geq 0$ for any $A \in \mathcal{B}_2(H)$;

(hh) $\langle\cdot,\cdot\rangle_{2,P}$ is linear in the first variable;

(hhh) $\langle B, A\rangle_{2,P} = \overline{\langle A, B\rangle}_{2,P}$ for any A, $B \in \mathcal{B}_2(H)$.

Using the properties of the trace we also have the following representations

$$\|A\|_{2,P}^2 := \operatorname{tr}\left(P|A|^2\right) = \operatorname{tr}(APA^*) = \operatorname{tr}\left(|A|^2 P\right)$$

and

$$\langle A, B\rangle_{2,P} = \operatorname{tr}(APB^*) = \operatorname{tr}(B^*AP)$$

for any $A,\ B\in\mathcal{B}_2(H)$.

The same definitions can be considered if $P\in\mathcal{B}_1^+(H)$ and $A,\ B\in\mathcal{B}(H)$.

We have the following Grüss type inequality:

Theorem 9. *Let, either* $P\in\mathcal{B}_+(H)$, $A,\ B,\ C\in\mathcal{B}_2(H)$ *or* $P\in\mathcal{B}_1^+(H)$, $A,\ B,\ C\in\mathcal{B}(H)$ *with* $P|A|^2,\ P|B|^2,\ P|C|^2\neq 0$ *and* $\lambda,\ \Gamma,\ \delta,\ \Delta\in\mathbb{C}$ *with* $\gamma+\Gamma\neq 0$, $\delta+\Delta\neq 0$. *If* (A,C) *has the trace* P-(λ,Γ)*-property and* (B,C) *has the trace* P-(δ,Δ)*-property, then*

$$\left|\frac{\operatorname{tr}(PB^*A)}{\operatorname{tr}\left(P|C|^2\right)}-\frac{\operatorname{tr}(PC^*A)}{\operatorname{tr}\left(P|C|^2\right)}\frac{\operatorname{tr}(PB^*C)}{\operatorname{tr}\left(P|C|^2\right)}\right|^2 \leq\frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\frac{|\Delta-\delta|^2}{|\Delta+\delta|}\sqrt{\frac{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right)}{\left[\operatorname{tr}\left(P|C|^2\right)\right]^2}}. \tag{4.1}$$

Proof. We prove in the case that $P\in\mathcal{B}_+(H)$ and $A,\ B,\ C\in\mathcal{B}_2(H)$.

Making use of the Schwarz inequality for the nonnegative hermitian form $\langle\cdot,\cdot\rangle_{2,P}$ we have

$$\left|\langle A,B\rangle_{2,P}\right|^2\leq\langle A,A\rangle_{2,P}\langle B,B\rangle_{2,P}$$

for any $A,\ B\in\mathcal{B}_2(H)$.

Let $C\in\mathcal{B}_2(H)$, $C\neq 0$. Define the mapping $[\cdot,\cdot]_{2,P,C}:\mathcal{B}_2(H)\times\mathcal{B}_2(H)\to\mathbb{C}$ by

$$[A,B]_{2,P,C}:=\langle A,B\rangle_{2,P}\|C\|_{2,P}^2-\langle A,C\rangle_{2,P}\langle C,B\rangle_{2,P}.$$

Observe that $[\cdot,\cdot]_{2,P,C}$ is a nonnegative Hermitian form on $\mathcal{B}_2(H)$ and by Schwarz inequality we also have

$$\left|\langle A,B\rangle_{2,P}\|C\|_{2,P}^2-\langle A,C\rangle_{2,P}\langle C,B\rangle_{2,P}\right|^2 \leq\left[\|A\|_{2,P}^2\|C\|_{2,P}^2-\left|\langle A,C\rangle_{2,P}\right|^2\right]\left[\|B\|_{2,P}^2\|C\|_{2,P}^2-\left|\langle B,C\rangle_{2,P}\right|^2\right]$$

for any $A,\ B\in\mathcal{B}_2(H)$, namely

$$\begin{aligned}&\left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right)-\operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right|^2\\&\leq\left[\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right)-|\operatorname{tr}(PC^*A)|^2\right]\\&\times\left[\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right)-|\operatorname{tr}(PB^*C)|^2\right],\end{aligned} \tag{4.2}$$

where for the last term we used the equality $\left|\langle B,C\rangle_{2,P}\right|^2=\left|\langle C,B\rangle_{2,P}\right|^2$.

Since (A,C) has the trace P-(λ,Γ)-property and (B,C) has the trace P-(δ,Δ) -property, then by (3.13) we have

$$0\leq\sqrt{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right)}-|\operatorname{tr}(PC^*A)|\leq\frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\operatorname{tr}\left(P|C|^2\right)$$

and

$$0\leq\sqrt{\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right)}-|\operatorname{tr}(PC^*B)|\leq\frac{1}{4}\frac{|\Delta-\delta|^2}{|\Delta+\delta|}\operatorname{tr}\left(P|C|^2\right)$$

which imply

$$0 \le \operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PC^*A)|^2 \tag{4.3}$$
$$\le \frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\operatorname{tr}\left(P|C|^2\right)\left(\sqrt{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right)} + |\operatorname{tr}(PC^*A)|\right)$$
$$\le \frac{1}{2}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\operatorname{tr}\left(P|C|^2\right)\sqrt{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right)}$$

and

$$0 \le \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PB^*C)|^2 \tag{4.4}$$
$$\le \frac{1}{4}\frac{|\Delta-\delta|^2}{|\Delta+\delta|}\operatorname{tr}\left(P|C|^2\right)\left(\sqrt{\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right)} + |\operatorname{tr}(PC^*B)|\right)$$
$$\le \frac{1}{2}\frac{|\Delta-\delta|^2}{|\Delta+\delta|}\operatorname{tr}\left(P|C|^2\right)\sqrt{\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right)}.$$

If we multiply the inequalities (4.3) and (4.4) we get

$$\left[\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PC^*A)|^2\right] \tag{4.5}$$
$$\times\left[\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right) - |\operatorname{tr}(PB^*C)|^2\right]$$
$$\le \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\frac{|\Delta-\delta|^2}{|\Delta+\delta|}\operatorname{tr}\left(P|C|^2\right)\sqrt{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right)}$$
$$\times\operatorname{tr}\left(P|C|^2\right)\sqrt{\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right)}.$$

If we use (4.2) and (4.5) we get

$$\left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right) - \operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right|^2 \tag{4.6}$$
$$\le \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\frac{|\Delta-\delta|^2}{|\Delta+\delta|}\operatorname{tr}\left(P|C|^2\right)\sqrt{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right)}$$
$$\times\operatorname{tr}\left(P|C|^2\right)\sqrt{\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right)}.$$

Since $P|C|^2 \ne 0$ then by (4.6) we get the desired result (4.1). □

Corollary 6. *Let, either $P \in \mathcal{B}_+(H)$, $A, B \in \mathcal{B}_2$ or $P \in \mathcal{B}_1^+(H)$, $A, B \in \mathcal{B}(H)$ with $P|A|^2$, $P|B|^2 \ne 0$ and $\lambda, \Gamma, \delta, \Delta \in \mathbb{C}$ with $\gamma + \Gamma \ne 0$, $\delta + \Delta \ne 0$. If A has the trace P-(λ, Γ)-property and B has the trace P-(δ, Δ)-property, then*

$$\left|\frac{\operatorname{tr}(PB^*A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB^*)}{\operatorname{tr}(P)}\right|^2 \tag{4.7}$$
$$\le \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\frac{|\Delta-\delta|^2}{|\Delta+\delta|}\sqrt{\frac{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right)}{[\operatorname{tr}(P)]^2}}.$$

The case of selfadjoint operators is useful for applications.

Remark 4. *Assume that A, B, C are selfadjoint operators. If, either $P \in \mathcal{B}_+(H)$, A, B, $C \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, A, B, $C \in \mathcal{B}(H)$ with PA^2, PB^2, $PC^2 \neq 0$ and m, M, n, $N \in \mathbb{R}$ with $m+M$, $n+N \neq 0$. If (A,C) has the trace P-(m,M)-property and (B,C) has the trace P-(n,N)-property, then*

$$\left|\frac{\operatorname{tr}(PBA)}{\operatorname{tr}(PC^2)} - \frac{\operatorname{tr}(PCA)}{\operatorname{tr}(PC^2)}\frac{\operatorname{tr}(PBC)}{\operatorname{tr}(PC^2)}\right|^2 \leq \frac{1}{4}\cdot\frac{(M-m)^2}{|M+m|}\frac{(N-n)^2}{|N+n|}\sqrt{\frac{\operatorname{tr}(PA^2)\operatorname{tr}(PB^2)}{[\operatorname{tr}(PC^2)]^2}}. \tag{4.8}$$

If A has the trace P-(k,K)-property and B has the trace P-(l,L)-property, then

$$\left|\frac{\operatorname{tr}(PBA)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}\right|^2 \leq \frac{1}{4}\cdot\frac{(K-k)^2}{|K+k|}\frac{(L-l)^2}{|L+l|}\sqrt{\frac{\operatorname{tr}(PA^2)\operatorname{tr}(PB^2)}{[\operatorname{tr}(P)]^2}}, \tag{4.9}$$

where $k+K$, $l+L \neq 0$.

5. Applications for Convex Functions

In the paper [34] we obtained amongst other the following reverse of the Jensen trace inequality:

Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m,M]$ for some scalars m, M with $m < M$. If f is a continuously differentiable convex function on $[m,M]$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$, then we have

$$\begin{aligned} 0 &\leq \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \\ &\leq \frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\cdot\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} \\ &\leq \begin{cases} \frac{1}{2}[f'(M)-f'(m)]\frac{\operatorname{tr}\left(P\left|A-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)} \\ \\ \frac{1}{2}(M-m)\frac{\operatorname{tr}\left(P\left|f'(A)-\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)} \end{cases} \\ &\leq \begin{cases} \frac{1}{2}[f'(M)-f'(m)]\left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M-m)\left[\frac{\operatorname{tr}\left(P[f'(A)]^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}\left(Pf'(A)\right)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \end{cases} \\ &\leq \frac{1}{4}[f'(M)-f'(m)](M-m). \end{aligned} \tag{5.1}$$

Let $\mathcal{M}_n(\mathbb{C})$ be the space of all square matrices of order n with complex elements and $A \in \mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix such that $Sp(A) \subseteq [m,M]$ for some scalars m, M with $m < M$. If f is a continuously differentiable convex function on $[m,M]$, then by taking $P=I_n$ in (5.1) we get

$$
\begin{aligned}
0 &\le \frac{\operatorname{tr}(f(A))}{n} - f\left(\frac{\operatorname{tr}(A)}{n}\right) \\
&\le \frac{\operatorname{tr}(f'(A)A)}{n} - \frac{\operatorname{tr}(A)}{n}\cdot\frac{\operatorname{tr}(f'(A))}{n} \\
&\le \begin{cases} \frac{1}{2}\left[f'(M)-f'(m)\right]\frac{\operatorname{tr}\left(\left|A-\frac{\operatorname{tr}(A)}{n}1_H\right|\right)}{n} \\ \\ \frac{1}{2}(M-m)\frac{\operatorname{tr}\left(\left|f'(A)-\frac{\operatorname{tr}(f'(A))}{n}1_H\right|\right)}{n} \end{cases} \\
&\le \begin{cases} \frac{1}{2}\left[f'(M)-f'(m)\right]\left[\frac{\operatorname{tr}(A^2)}{n}-\left(\frac{\operatorname{tr}(A)}{n}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M-m)\left[\frac{\operatorname{tr}\left(\left[f'(A)\right]^2\right)}{n}-\left(\frac{\operatorname{tr}(f'(A))}{n}\right)^2\right]^{1/2} \end{cases} \\
&\le \frac{1}{4}\left[f'(M)-f'(m)\right](M-m).
\end{aligned} \tag{5.2}
$$

The following reverse inequality also holds:

Proposition 2. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq[m,M]$ for some scalars m, M with $m+M\neq 0$. If f is a continuously differentiable convex function on $[m,M]$ with $f'(m)+f'(M)\neq 0$ and $P\in\mathcal{B}_1(H)\setminus\{0\}$, $P\ge 0$, then we have*

$$
\begin{aligned}
0 &\le \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \\
&\le \frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\cdot\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} \\
&\le \frac{1}{2}\cdot\frac{|M-m|\,|f'(M)-f'(m)|}{\sqrt{|m+M|}\sqrt{|f'(m)+f'(M)|}}\sqrt[4]{\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(P\left[f'(A)\right]^2\right)}{\operatorname{tr}(P)}}.
\end{aligned} \tag{5.3}
$$

The proof follows by the inequality (4.9) and the details are omitted,

Let $A\in\mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix such that $Sp(A)\subseteq[m,M]$ for some scalars m, M with $m+M\neq 0$. If f is a continuously differentiable convex function on $[m,M]$ with $f'(m)+f'(M)\neq 0$ then by taking $P=I_n$ in (5.3) we get

$$
\begin{aligned}
0 &\le \frac{\operatorname{tr}(f(A))}{n} - f\left(\frac{\operatorname{tr}(A)}{n}\right) \\
&\le \frac{\operatorname{tr}(f'(A)A)}{n} - \frac{\operatorname{tr}(A)}{n}\cdot\frac{\operatorname{tr}(f'(A))}{n} \\
&\le \frac{1}{2}\cdot\frac{|M-m|\,|f'(M)-f'(m)|}{\sqrt{|m+M|}\sqrt{|f'(m)+f'(M)|}}\sqrt[4]{\frac{\operatorname{tr}(A^2)}{n}\frac{\operatorname{tr}\left(\left[f'(A)\right]^2\right)}{n}}.
\end{aligned} \tag{5.4}
$$

We consider the power function $f:(0,\infty)\to(0,\infty)$, $f(t)=t^r$ with $t\in\mathbb{R}\setminus\{0\}$. For $r\in(-\infty,0)\cup[1,\infty)$, f is convex while for $r\in(0,1)$, f is concave.

Let $r\geq 1$ and A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq[m,M]$ for some scalars m, M with $0<m<M$. If $P\in\mathcal{B}_1^+(H)\setminus\{0\}$, then

$$0\leq\frac{\operatorname{tr}(PA^r)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^r \tag{5.5}$$
$$\leq r\left[\frac{\operatorname{tr}(PA^r)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\cdot\frac{\operatorname{tr}\left(PA^{r-1}\right)}{\operatorname{tr}(P)}\right]$$
$$\leq\frac{1}{2}p\frac{(M-m)\left(M^{p-1}-m^{p-1}\right)}{(m+M)^{1/2}\left(m^{p-1}+M^{p-1}\right)^{1/2}}\sqrt[4]{\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(PA^{2(p-1)}\right)}{\operatorname{tr}(P)}}.$$

Consider the convex function $f:\mathbb{R}\to(0,\infty)$, $f(t)=\exp t$ and let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq[m,M]$ for some scalars m, M with $m<M$. If $P\in\mathcal{B}_1^+(H)\setminus\{0\}$, then using (5.3) we have

$$0\leq\frac{\operatorname{tr}(P\exp A)}{\operatorname{tr}(P)}-\exp\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \tag{5.6}$$
$$\leq\frac{\operatorname{tr}(PA\exp A)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\cdot\frac{\operatorname{tr}(P\exp A)}{\operatorname{tr}(P)}$$
$$\leq\frac{1}{2}\frac{|M-m|\left(\exp(M)-\exp(m)\right)}{\sqrt{|m+M|}\sqrt{\exp m+\exp M}}\sqrt[4]{\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(P\exp(2A))}{\operatorname{tr}(P)}}.$$

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