Received 26/09/14

# NEW HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR GA-CONVEX WITH APPLICATIONS

M. A. LATIF

Abstract. In this paper, some refinements of Hermite-Hadamard type inequalities for GA-convex functions are obtained. Applications of the obtained results to special means are given.

## 1. Introduction

The classical or the usual convexity is defined as follows:

A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on $I$ if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

A number of papers have been written on inequalities using the classical convexity and one of the most fascinating inequalities in mathematical analysis is stated as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Both the inequalities hold in reversed direction if $f$ is concave. The inequalities stated in (1.1) are known as Hermite-Hadamard inequalities.

A number of papers have been written on the inequality (1.1), which provide new proofs, noteworthy extensions, generalizations, refinements, counterparts, new Hermite-Hadamard-type inequalities and numerous applications, see for instance [2]-[8], [12]-[19] and the references therein.

The usual notion of convex functions have been generalized in diverse manners. One of them is the so called GA-convex functions and is stated in the definition below.

**Definition 1.** [10, 11] *A function $f : I \subseteq \mathbb{R}_0 = [0,\infty)$ is said to be GA-convex function on $I$ if*

$$f\left(x^{\lambda}y^{1-\lambda}\right) \leq \lambda f(x) + (1-\lambda)f(y)$$

*holds for all $x$, $y \in I$ and $\lambda \in [0,1]$, where $x^{\lambda}y^{1-\lambda}$ and $\lambda f(x) + (1-\lambda)f(y)$ are respectively the weighted geometric mean of two positive numbers $x$ and $y$ and the weighted arithmetic mean of $f(x)$ and $f(y)$.*

---

*Date*: November 15, 2012.

2000 *Mathematics Subject Classification.* Primary 26D15, Secondary 26A51, 26E60, 41A55.

*Key words and phrases.* Hermite-Hadamard's inequality, convex function, GA-convex function, Hölder's inequality.

This paper is in final form and no version of it will be submitted for publication elsewhere.

In what follows we will use the following notations of means:
For positive numbers $a>0$ and $b>0$ with $a\neq b$

$$A(a,b)=\frac{a+b}{2},\qquad L(a,b)=\frac{b-a}{\ln b-\ln a}$$

and

$$L_p(a,b)=\begin{cases}\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & p\neq -1,0\\ L(a,b), & p=-1\\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & p=0\end{cases}$$

are the arithmetic mean, the logarithmic mean and the generalized logarithmic mean of order $p\in\mathbb{R}$ respectively. For further information on means, we refer the readers to [1, 14, 15] and the references therein.

Most recently, Zhang et al. in [19] established the following Hermite-Hadamard type integral inequalities for GA-convex function.

**Theorem 1.** [19] *Let $f:I\subseteq\mathbb{R}_+=(0,\infty)\to\mathbb{R}$ be a differentiable function on $I^\circ$ and $a,\,b\in I$ with $a<b$ and $f'\in L([a,b])$. If $\left|f'\right|^q$ is GA-convex on $[a,b]$ for $q\geq 1$, we have the following inequality:*

$$\left|bf(b)-af(a)-\int_a^b f(x)\,dx\right|\leq\frac{[(b-a)A(a,b)]^{1-\frac{1}{q}}}{2^{\frac{1}{q}}}\times\left\{\left[L\left(a^2,b^2\right)-a^2\right]\left|f'(a)\right|^q+\left[b^2-L\left(a^2,b^2\right)\right]\left|f'(b)\right|^q\right\}^{\frac{1}{q}}. \quad (1.2)$$

**Theorem 2.** [19] *Let $f:I\subseteq\mathbb{R}_+=(0,\infty)\to\mathbb{R}$ be a function differentiable function on $I^\circ$ and $a,\,b\in I$ with $a<b$ and $f'\in L([a,b])$. If $\left|f'\right|^q$ is GA-convex on $[a,b]$ for $q>1$, we have the following inequality:*

$$\left|bf(b)-af(a)-\int_a^b f(x)\,dx\right|\leq(\ln b-\ln a)\left[L\left(a^{\frac{2q}{q-1}},b^{\frac{2q}{q-1}}\right)\right]^{1-\frac{1}{q}}\left[A\left(\left|f'(a)\right|^q\left|f'(b)\right|^q\right)\right]^{\frac{1}{q}}. \quad (1.3)$$

**Theorem 3.** [19] *Let $f:I\subseteq\mathbb{R}_+=(0,\infty)\to\mathbb{R}$ be a differentiable function on $I^\circ$ and $a,\,b\in I$ with $a<b$ and $f'\in L([a,b])$. If $\left|f'\right|^q$ is GA-convex on $[a,b]$ for $q\geq 1$, we have the following inequality:*

$$\left|bf(b)-af(a)-\int_a^b f(x)\,dx\right|\leq\frac{(\ln b-\ln a)^{1-\frac{1}{q}}}{(2q)^{\frac{1}{q}}}\left[L\left(a^{\frac{2q}{q-1}},b^{\frac{2q}{q-1}}\right)\right]^{1-\frac{1}{q}}\times\left\{\left[L\left(a^{2q},b^{2q}\right)-a^{2q}\right]\left|f'(a)\right|^q+\left[b^{2q}-L\left(a^{2q},b^{2q}\right)\right]\left|f'(b)\right|^q\right\}^{\frac{1}{q}}. \quad (1.4)$$

**Theorem 4.** [19] *Let $f:I\subseteq\mathbb{R}_+=(0,\infty)\to\mathbb{R}$ be a differentiable function on $I^\circ$ and $a,\,b\in I$ with $a<b$ and $f'\in L([a,b])$. If $\left|f'\right|^q$ is GA-convex on $[a,b]$ for $q>1$*

*and* $2q > p > 0$. *Then*

$$\left| bf(b) - af(a) - \int_a^b f(x)\,dx \right| \le \frac{(\ln b - \ln a)^{1-\frac{1}{q}}}{p^{\frac{1}{q}}} \left[ L\left(a^{\frac{2q-p}{q-1}}, b^{\frac{2q-p}{q-1}}\right)\right]^{1-\frac{1}{q}}$$

$$\times \left\{ \left[L(a^p, b^p) - a^p\right] \left|f'(a)\right|^q + \left[b^p - L(a^p, b^p)\right] \left|f'(b)\right|^q \right\}^{\frac{1}{q}}. \quad (1.5)$$

For the applications of the above results to special means we refer the readers to [19].

Motivated by these results, the main purpose of the present paper is to establish Hermite-Hadamard type integral inequalities for GA-convex functions, which we believe are better than above presented results.

## 2. Main Results

In order to prove our main results, we need the following lemma:

**Lemma 1.** *Let* $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ *be a differentiable function on* $I^\circ$ *and* $a$, $b \in I$ *with* $a < b$. *If* $f' \in L([a, b])$, *then the following equality holds:*

$$bf(b) - af(a) - \int_a^b f(x)\,dx$$
$$= \frac{\ln b - \ln a}{2} \left[ \int_0^1 b^{1+t} a^{1-t} f'\left(b^{\frac{1+t}{2}} a^{\frac{1-t}{2}}\right) dt + \int_0^1 b^{1-t} a^{1+t} f'\left(b^{\frac{1-t}{2}} a^{\frac{1+t}{2}}\right) dt \right] \quad (2.1)$$

*Proof.* Let

$$I_1 = \int_0^1 b^{1+t} a^{1-t} f'\left(b^{\frac{1+t}{2}} a^{\frac{1-t}{2}}\right) dt$$

and

$$I_2 = \int_0^1 b^{1-t} a^{1+t} f'\left(b^{\frac{1-t}{2}} a^{\frac{1+t}{2}}\right) dt.$$

Now we observe that

$$\begin{aligned} I_1 &= \int_0^1 b^{1+t} a^{1-t} f'\left(b^{\frac{1+t}{2}} a^{\frac{1-t}{2}}\right) dt \\ &= \frac{2}{\ln b - \ln a} \int_0^1 b^{\frac{1+t}{2}} a^{\frac{1-t}{2}} f'\left(b^{\frac{1+t}{2}} a^{\frac{1-t}{2}}\right) d\left(b^{\frac{1+t}{2}} a^{\frac{1-t}{2}}\right). \end{aligned}$$

By the change of the variable $x = b^{\frac{1+t}{2}} a^{\frac{1-t}{2}}$ and by integrating by parts, we have

$$\begin{aligned} I_1 &= \frac{2}{\ln b - \ln a} \int_{\sqrt{ab}}^b x f'(x)\,dx \qquad (2.2) \\ &= \frac{2\left[bf(b) - \sqrt{ab} f\left(\sqrt{ab}\right)\right]}{\ln b - \ln a} - \frac{2}{\ln b - \ln a} \int_{\sqrt{ab}}^b f(x)\,dx. \end{aligned}$$

Analogously, we have

$$I_2 = \frac{2\left[\sqrt{ab} f\left(\sqrt{ab}\right) - af(a)\right]}{\ln b - \ln a} - \frac{2}{\ln b - \ln a} \int_a^{\sqrt{ab}} f(x)\,dx. \quad (2.3)$$

Adding (2.2) and (2.3) and multiplying the result by $\frac{\ln b-\ln a}{2}$, we get the required identity. This completes the proof of the Lemma. □

We now establish new Hermite-Hadamard type inequalities for GA-convex functions, which we believe are better than those established in [19].

**Theorem 5.** *Let* $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ *be a differentiable function on* $I^\circ$ *and* $a, b \in I$ *with* $a < b$ *and* $f' \in L([a, b])$. *If* $\left|f'\right|^q$ *is GA-convex on* $[a, b]$ *for* $q \geq 1$, *we have the following inequality:*

$$\begin{aligned}&\left|bf(b)-af(a)-\int_a^b f(x)\,dx\right|\\&\leq \frac{(b-a)^{1-\frac{1}{q}}}{2^{\frac{1}{q}+1}}\left\{b\left[(L(a,b)-a)\left|f'(a)\right|^q+(2b-a-L(a,b))\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right.\\&\qquad\left.+a\left[(b-2a+L(a,b))\left|f'(a)\right|^q+(b-L(a,b))\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\}. \quad (2.4)\end{aligned}$$

*Proof.* From Lemma 1 and the Hölder's inequality, we have

$$\begin{aligned}&\left|bf(b)-af(a)-\int_a^b f(x)\,dx\right|\\&=\frac{ab(\ln b-\ln a)}{2}\left[\int_0^1\left(\frac{b}{a}\right)^t\left|f'\left(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\right)\right|dt+\int_0^1\left(\frac{a}{b}\right)^t\left|f'\left(b^{\frac{1-t}{2}}a^{\frac{1+t}{2}}\right)\right|dt\right]\\&\leq\frac{ab(\ln b-\ln a)}{2}\left\{\left[\int_0^1\left(\frac{b}{a}\right)^t dt\right]^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{b}{a}\right)^t\left|f'\left(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\&\qquad\left.+\left[\int_0^1\left(\frac{a}{b}\right)^t dt\right]^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{a}{b}\right)^t\left|f'\left(b^{\frac{1-t}{2}}a^{\frac{1+t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}}\right\}. \quad (2.5)\end{aligned}$$

By the GA-convexity of $\left|f'\right|^q$ on $[a, b]$ for $q \geq 1$ and by integration by parts, we have

$$\begin{aligned}&\int_0^1\left(\frac{b}{a}\right)^t\left|f'\left(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\right)\right|^q dt\\&\leq\left|f'(a)\right|^q\int_0^1\left(\frac{b}{a}\right)^t\left(\frac{1-t}{2}\right)dt+\left|f'(b)\right|^q\int_0^1\left(\frac{b}{a}\right)^t\left(\frac{1+t}{2}\right)dt\\&=\left|f'(a)\right|^q\frac{L(a,b)-a}{2a(\ln b-\ln a)}+\left|f'(b)\right|^q\frac{2b-a-L(a,b)}{2a(\ln b-\ln a)} \quad (2.6)\end{aligned}$$

and

$$\int_0^1 \left(\frac{a}{b}\right)^t \left|f'\left(b^{\frac{1-t}{2}}a^{\frac{1+t}{2}}\right)\right|^q dt$$
$$\leq \left|f'(a)\right|^q \int_0^1 \left(\frac{a}{b}\right)^t \left(\frac{1+t}{2}\right) dt + \left|f'(b)\right|^q \int_0^1 \left(\frac{a}{b}\right)^t \left(\frac{1-t}{2}\right) dt$$
$$= \left|f'(a)\right|^q \frac{b-2a+L(a,b)}{2b(\ln b - \ln a)} + \left|f'(b)\right|^q \frac{b-L(a,b)}{2b(\ln b - \ln a)}. \quad (2.7)$$

Using (2.6) and (2.7) in (2.5), we get the required result. This completes the proof of the theorem □

**Corollary 1.** [19, Corollary 3.2, page 232] *Under the assumptions of Theorem 5, if we take $q=1$, we have*

$$\left|bf(b) - af(a) - \int_a^b f(x)\,dx\right|$$
$$\leq \frac{1}{2}\left\{\left[L\left(a^2,b^2\right)-a^2\right]\left|f'(a)\right| + \left[b^2 - L\left(a^2,b^2\right)\right]\left|f'(b)\right|\right\}. \quad (2.8)$$

**Theorem 6.** *Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be a differentiable function on $I^\circ$ and $a,\, b \in I$ with $a<b$ and $f' \in L([a,b])$. If $\left|f'\right|^q$ is GA-convex on $[a,b]$ for $q>1$, we have the following inequality:*

$$\left|bf(b) - af(a) - \int_a^b f(x)\,dx\right| \leq \frac{(\ln b - \ln a)}{2^{1+\frac{1}{q}}}\left[L\left(a^{\frac{q}{q-1}}, b^{\frac{q}{q-1}}\right)\right]^{1-\frac{1}{q}}$$
$$\times \left\{b\left[A\left(\left|f'(a)\right|^q, 3\left|f'(b)\right|^q\right)\right]^{\frac{1}{q}} + a\left[A\left(3\left|f'(a)\right|^q, \left|f'(b)\right|^q\right)\right]^{\frac{1}{q}}\right\}. \quad (2.9)$$

*Proof.* From Lemma 1, the Hölder's inequality and the GA-convexity of $\left|f'\right|^q$ on $[a,b]$ for $q>1$, we have

$$\left|bf(b) - af(a) - \int_a^b f(x)\,dx\right|$$
$$= \frac{ab(\ln b - \ln a)}{2}\left[\int_0^1 \left(\frac{b}{a}\right)^t \left|f'\left(b^{\frac{1+t}{2}}a^{\frac{1-t}{2}}\right)\right| dt + \int_0^1 \left(\frac{a}{b}\right)^t \left|f'\left(b^{\frac{1-t}{2}}a^{\frac{1+t}{2}}\right)\right| dt\right]$$
$$\leq \frac{ab(\ln b - \ln a)}{2}\left\{\left[\int_0^1 \left(\frac{b}{a}\right)^{\frac{qt}{q-1}} dt\right]^{1-\frac{1}{q}} \left(\left|f'(a)\right|^q \int_0^1 \frac{1-t}{2}dt + \left|f'(b)\right|^q \int_0^1 \frac{1+t}{2}dt\right)^{\frac{1}{q}}\right.$$
$$\left. + \left[\int_0^1 \left(\frac{a}{b}\right)^{\frac{qt}{q-1}} dt\right]^{1-\frac{1}{q}} \left(\left|f'(a)\right|^q \int_0^1 \frac{1+t}{2}dt + \left|f'(b)\right|^q \int_0^1 \frac{1-t}{2}dt\right)^{\frac{1}{q}}\right\}$$
$$= \frac{(\ln b - \ln a)}{2^{1+\frac{1}{q}}}\left[L\left(a^{\frac{q}{q-1}}, b^{\frac{q}{q-1}}\right)\right]^{1-\frac{1}{q}} \left\{b\left[A\left(\left|f'(a)\right|^q, 3\left|f'(b)\right|^q\right)\right]^{\frac{1}{q}}\right.$$
$$\left. + a\left[A\left(3\left|f'(a)\right|^q, \left|f'(b)\right|^q\right)\right]^{\frac{1}{q}}\right\}. \quad (2.10)$$

Hence the proof of the theorem is completed. □

**Theorem 7.** *Let $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ be a differentiable function on $I^\circ$ and $a,\ b \in I^\circ$ with $a < b$ and $f' \in L([a, b])$. If $\left|f'\right|^q$ is GA-convex on $[a, b]$ for $q \geq 1$, we have the following inequality:*

$$\left|bf(b) - af(a) - \int_a^b f(x)\,dx\right| \leq \frac{(\ln b - \ln a)^{1-\frac{1}{q}}}{2(2q)^{\frac{1}{q}}}$$
$$\times \left\{ b\left[ \left(L(a^q, b^q) - a^q\right)\left|f'(a)\right|^q + \left(2b^q - a^q - L(a^q, b^q)\right)\left|f'(b)\right|^q \right]^{\frac{1}{q}} \right.$$
$$\left. + a\left[ \left(b^q - 2a^q + L(a^q, b^q)\right)\left|f'(a)\right|^q + \left(b^q - L(a^q, b^q)\right)\left|f'(b)\right|^q \right]^{\frac{1}{q}} \right\}. \quad (2.11)$$

*Proof.* From Lemma 1 and the Hölder's inequality, we have

$$\left|bf(b) - af(a) - \int_a^b f(x)\,dx\right|$$
$$\leq \frac{ab(\ln b - \ln a)}{2} \left\{ \left[\int_0^1 1dt\right]^{1-\frac{1}{q}} \left(\int_0^1 \left(\frac{b}{a}\right)^{qt} \left|f'\left(b^{\frac{1+t}{2}} a^{\frac{1-t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}} \right.$$
$$\left. + \left[\int_0^1 1dt\right]^{1-\frac{1}{q}} \left(\int_0^1 \left(\frac{a}{b}\right)^{qt} \left|f'\left(b^{\frac{1-t}{2}} a^{\frac{1+t}{2}}\right)\right|^q dt\right)^{\frac{1}{q}} \right\}. \quad (2.12)$$

By the GA-convexity of $\left|f'\right|^q$ on $[a, b]$ for $q \geq 1$ and by integration by parts, we have

$$\int_0^1 \left(\frac{b}{a}\right)^{qt} \left|f'\left(b^{\frac{1+t}{2}} a^{\frac{1-t}{2}}\right)\right|^q dt$$
$$\leq \left|f'(a)\right|^q \int_0^1 \left(\frac{b}{a}\right)^{qt} \left(\frac{1-t}{2}\right) dt + \left|f'(b)\right|^q \int_0^1 \left(\frac{b}{a}\right)^{qt} \left(\frac{1+t}{2}\right) dt$$
$$= \left|f'(a)\right|^q \frac{L(a^q, b^q) - a^q}{2qa^q(\ln b - \ln a)} + \left|f'(b)\right|^q \frac{2b^q - a^q - L(a^q, b^q)}{2a^q(\ln b - \ln a)}. \quad (2.13)$$

Similarly, we also have

$$\int_0^1 \left(\frac{a}{b}\right)^{qt} \left|f'\left(b^{\frac{1-t}{2}} a^{\frac{1+t}{2}}\right)\right|^q dt$$
$$\leq \left|f'(a)\right|^q \int_0^1 \left(\frac{a}{b}\right)^{qt} \left(\frac{1+t}{2}\right) dt + \left|f'(b)\right|^q \int_0^1 \left(\frac{a}{b}\right)^{qt} \left(\frac{1-t}{2}\right) dt$$
$$= \left|f'(a)\right|^q \frac{b^q - 2a^q + L(a^q, b^q)}{2qb^q(\ln b - \ln a)} + \left|f'(b)\right|^q \frac{b^q - L(a^q, b^q)}{2qb^q(\ln b - \ln a)}. \quad (2.14)$$

Using (2.13) and (2.14) in (2.12), we get the required inequality. This completes the proof of the theorem. □

**Remark 1.** *Under the assumptions of Theorem 7, if $q = 1$ we get the result established in Corollary 1.*

**Theorem 8.** *Let $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ be a differentiable function on $I^\circ$ and $a, b \in I$ with $a < b$ and $f' \in L([a, b])$. If $\left|f'\right|^q$ is GA-convex on $[a, b]$ for $q > 1$ and $q > p > 0$, we have the following inequality:*

$$\left|bf(b) - af(a) - \int_a^b f(x)\,dx\right| \leq \frac{\left[(\ln b - \ln a) L\left(a^{\frac{q-p}{q-1}}, b^{\frac{q-p}{q-1}}\right)\right]^{1-\frac{1}{q}}}{2(2p)^{\frac{1}{q}}}$$
$$\times \left\{a^{1-\frac{p}{q}} b\left[(L(a^p, b^p) - a^p)\left|f'(a)\right|^q + (2b^p - a^p - L(a^p, b^p))\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right.$$
$$\left. + ab^{1-\frac{p}{q}}\left[(b^p - 2a^p + L(a^p, b^p))\left|f'(a)\right|^q + (b^p - L(a^p, b^p))\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\}. \quad (2.15)$$

*Proof.* From Lemma 1, the Hölder's inequality and the GA-convexity of $\left|f'\right|^q$ on $[a, b]$ for $q > 1$, we have

$$\left|bf(b) - af(a) - \int_a^b f(x)\,dx\right| \leq \frac{ab(\ln b - \ln a)}{2}\left\{\left[\int_0^1 \left(\frac{b}{a}\right)^{\frac{(q-p)t}{q-1}} dt\right]^{1-\frac{1}{q}}\right.$$
$$\times \left(\left|f'(a)\right|^q \int_0^1 \left(\frac{b}{a}\right)^{pt}\left(\frac{1-t}{2}\right) dt + \left|f'(b)\right|^q \int_0^1 \left(\frac{b}{a}\right)^{pt}\left(\frac{1+t}{2}\right) dt\right)^{\frac{1}{q}}$$
$$+ \left[\int_0^1 \left(\frac{a}{b}\right)^{\frac{(q-p)t}{q-1}} dt\right]^{1-\frac{1}{q}} \left(\left|f'(a)\right|^q \int_0^1 \left(\frac{a}{b}\right)^{pt}\left(\frac{1+t}{2}\right) dt\right.$$
$$\left.\left. + \left|f'(b)\right|^q \int_0^1 \left(\frac{a}{b}\right)^{pt}\left(\frac{1-t}{2}\right) dt\right)^{\frac{1}{q}}\right\} = \frac{ab(\ln b - \ln a)}{2}\left[\frac{(q-1)\left(b^{\frac{q-p}{q-1}} - a^{\frac{q-p}{q-1}}\right)}{(q-p)(\ln b - \ln a)}\right]^{1-\frac{1}{q}}$$
$$\times \left\{\left(\frac{(L(a^p, b^p) - a^p)\left|f'(a)\right|^q + (2b^p - a^p - L(a^p, b^p))\left|f'(b)\right|^q}{2pa^p(\ln b - \ln a)}\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\frac{(b^p - 2a^p + L(a^p, b^p))\left|f'(a)\right|^q + (b^p - L(a^p, b^p))\left|f'(b)\right|^q}{2pb^p(\ln b - \ln a)}\right)^{\frac{1}{q}}\right\}$$
$$= \frac{\left[(\ln b - \ln a) L\left(b^{\frac{q-p}{q-1}}, a^{\frac{q-p}{q-1}}\right)\right]^{1-\frac{1}{q}}}{2(2p)^{\frac{1}{q}}}$$
$$\times \left\{a^{1-\frac{p}{q}} b\left[(L(a^p, b^p) - a^p)\left|f'(a)\right|^q + (2b^p - a^p - L(a^p, b^p))\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right.$$
$$\left. + ab^{1-\frac{p}{q}}\left[(b^p - 2a^p + L(a^p, b^p))\left|f'(a)\right|^q + (b^p - L(a^p, b^p))\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\}. \quad (2.16)$$

Hence the proof of the theorem is completed. □

**Remark 2.** *Under the assumptions of Theorem 8, if $p \to q$, we have by using L'Hospital rule that $L\left(a^{\frac{q-p}{q-1}}, b^{\frac{q-p}{q-1}}\right) \to 1$ and hence we get the inequality proved in Theorem 7.*

## 3. Applications to Special Means

In this section we apply our results to establish several new inequalities for special means.

**Theorem 9.** *For $a > b > 0$, $s > 0$, $q \geq 1$ and $sq \neq 1$, we have*

$$\begin{aligned} &\left[L_{s+1}(a,b)\right]^{s+1} \\ &\quad \leq \frac{1}{2^{\frac{1}{q}+1}}\left\{b\left[b^{sq}+(sq+1)\left[L_{sq}(a,b)\right]^{sq}-sq\left[L_{sq-1}(a,b)\right]^{sq-1}L(a,b)\right]^{\frac{1}{q}}\right. \\ &\qquad \left. +a\left[a^{sq}+(sq+1)\left[L_{sq}(a,b)\right]^{sq}-sq\left[L_{sq-1}(a,b)\right]^{sq-1}L(a,b)\right]^{\frac{1}{q}}\right\}. \quad (3.1) \end{aligned}$$

*Proof.* Consider the functions

$$f(x) = \frac{x^{s+1}}{s+1}, \ x \in \mathbb{R}_+, \ s > 0.$$

Then $\left|f'(x)\right|^q = x^{sq}$ is a GA-convex function on $\mathbb{R}_+$. Now the left side of the inequality (2.4) becomes

$$\left|bf(b) - af(a) - \int_a^b f(x)\,dx\right| = \frac{b^{s+2}-a^{s+2}}{s+2} = (b-a)\left[L_{s+1}(a,b)\right]^{s+1} \quad (3.2)$$

and the right side of (2.4) will be

$$\begin{aligned} &\frac{(b-a)^{1-\frac{1}{q}}}{2^{\frac{1}{q}+1}}\left\{b\left[(L(a,b)-a)\left|f'(a)\right|^q+(2b-a-L(a,b))\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right. \\ &\qquad \left. +a\left[(b-2a+L(a,b))\left|f'(a)\right|^q+(b-L(a,b))\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\} \\ &\quad = \frac{(b-a)^{1-\frac{1}{q}}}{2^{\frac{1}{q}+1}}\left\{b\left[(L(a,b)-a)a^{sq}+(2b-a-L(a,b))b^{sq}\right]^{\frac{1}{q}}\right. \\ &\qquad \left. +a\left[(b-2a+L(a,b))a^{sq}+(b-L(a,b))b^{sq}\right]^{\frac{1}{q}}\right\} \\ &= \frac{(b-a)}{2^{\frac{1}{q}+1}}\left\{b\left[b^{sq}+(sq+1)\left[L_{sq}(a,b)\right]^{sq}-sq\left[L_{sq-1}(a,b)\right]^{sq-1}L(a,b)\right]^{\frac{1}{q}}\right. \\ &\qquad \left. +a\left[a^{sq}+(sq+1)\left[L_{sq}(a,b)\right]^{sq}-sq\left[L_{sq-1}(a,b)\right]^{sq-1}L(a,b)\right]\right\}^{\frac{1}{q}}. \quad (3.3) \end{aligned}$$

A combination of (3.2) and (3.3) gives (3.1). □

**Theorem 10.** *For $a > b > 0$, $s > 0$, $q > 1$, we have*

$$\begin{aligned} &L(a,b)\left[L_{s+1}(a,b)\right]^{s+1} \\ &\quad \leq \frac{1}{2^{1+\frac{1}{q}}}\left[L\left(a^{\frac{q}{q-1}}, b^{\frac{q}{q-1}}\right)\right]^{1-\frac{1}{q}}\left\{b\left[A\left(a^{sq}, 3b^{sq}\right)\right]^{\frac{1}{q}}+a\left[A\left(3a^{sq}, b^{sq}\right)\right]^{\frac{1}{q}}\right\}. \quad (3.4) \end{aligned}$$

*Proof.* Applying the function

$$f(x)=\frac{x^{s+1}}{s+1},\ x\in\mathbb{R}_{+},\ s>0.$$

to the upper bound of inequality (2.9) in Theorem 6 results in

$$\begin{aligned}&\frac{(\ln b-\ln a)}{2^{1+\frac{1}{q}}}\left[L\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}}\right)\right]^{1-\frac{1}{q}}\\&\quad\times\left\{b\left[A\left(\left|f^{'}(a)\right|^{q},3\left|f^{'}(b)\right|^{q}\right)\right]^{\frac{1}{q}}+a\left[A\left(3\left|f^{'}(a)\right|^{q},\left|f^{'}(b)\right|^{q}\right)\right]^{\frac{1}{q}}\right\}\\&=\frac{(\ln b-\ln a)}{2^{1+\frac{1}{q}}}\left[L\left(a^{\frac{q}{q-1}},b^{\frac{q}{q-1}}\right)\right]^{1-\frac{1}{q}}\left\{b\left[A\left(a^{sq},3b^{sq}\right)\right]^{\frac{1}{q}}+a\left[A\left(3a^{sq},b^{sq}\right)\right]^{\frac{1}{q}}\right\}.\end{aligned}\tag{3.5}$$

Combining (3.5) with (3.2) yields the required result. □

**Theorem 11.** *For $a>b>0$, $s>0$, $q\geq 1$ and $sq\neq 1$, we have*

$$\begin{aligned}&[L(a,b)]^{1-\frac{1}{q}}\left[L_{s+1}(a,b)\right]^{s+1}\leq\frac{1}{2^{\frac{1}{q}+1}}\left\{b\left[(s+1)\left[L_{sq+q-1}(a,b)\right]^{sq+q-1}\right.\right.\\&\left.+b^{sq}\left[L_{q-1}(a,b)\right]^{q-1}-s\left[L_{sq-1}(a,b)\right]^{sq-1}L\left(a^{q},b^{q}\right)\right]^{\frac{1}{q}}+a\left[a^{sq}\left[L_{q-1}(a,b)\right]^{q-1}\right.\\&\left.\left.(s+1)\left[L_{sq+q-1}(a,b)\right]^{sq+q-1}+s\left[L_{sq-1}(a,b)\right]^{sq-1}L\left(a^{q},b^{q}\right)\right]^{\frac{1}{q}}\right\}.\end{aligned}\tag{3.6}$$

*Proof.* Applying the function

$$f(x)=\frac{x^{s+1}}{s+1},\ x\in\mathbb{R}_{+},\ s>0.$$

to the upper bound of inequality (2.11) in Theorem 7 gives

$$\begin{aligned}&\frac{(\ln b-\ln a)^{1-\frac{1}{q}}}{2(2q)^{\frac{1}{q}}}\left\{b\left[\left(L\left(a^{q},b^{q}\right)-a^{q}\right)\left|f^{'}(a)\right|^{q}+\left(2b^{q}-a^{q}-L\left(a^{q},b^{q}\right)\right)\left|f^{'}(b)\right|^{q}\right]^{\frac{1}{q}}\right.\\&\quad\left.+\ a\left[\left(b^{q}-2a^{q}+L\left(a^{q},b^{q}\right)\right)\left|f^{'}(a)\right|^{q}+\left(b^{q}-L\left(a^{q},b^{q}\right)\right)\left|f^{'}(b)\right|^{q}\right]^{\frac{1}{q}}\right\}\\&=\frac{(\ln b-\ln a)^{1-\frac{1}{q}}}{2(2q)^{\frac{1}{q}}}\left\{b\left[\left(L\left(a^{q},b^{q}\right)-a^{q}\right)a^{sq}+\left(2b^{q}-a^{q}-L\left(a^{q},b^{q}\right)\right)b^{sq}\right]^{\frac{1}{q}}\right.\\&\quad\left.+a\left[\left(b^{q}-2a^{q}+L\left(a^{q},b^{q}\right)\right)a^{sq}+\left(b^{q}-L\left(a^{q},b^{q}\right)\right)b^{sq}\right]^{\frac{1}{q}}\right\}\\&=\frac{(b-a)[L(a,b)]^{\frac{1}{q}-1}}{2^{\frac{1}{q}+1}}\left\{b\left[(s+1)\left[L_{sq+q-1}(a,b)\right]^{sq+q-1}+b^{sq}\left[L_{q-1}(a,b)\right]^{q-1}\right.\right.\\&\quad\left.-s\left[L_{sq-1}(a,b)\right]^{sq-1}L\left(a^{q},b^{q}\right)\right]^{\frac{1}{q}}+a\left[(s+1)\left[L_{sq+q-1}(a,b)\right]^{sq+q-1}\right.\\&\quad\left.\left.+a^{sq}\left[L_{q-1}(a,b)\right]^{q-1}+s\left[L_{sq-1}(a,b)\right]^{sq-1}L\left(a^{q},b^{q}\right)\right]^{\frac{1}{q}}\right\}\end{aligned}\tag{3.7}$$

A combination of (3.7) and (3.2) gives the desired result. □

**Theorem 12.** *For $a>b>0$, $s>0$, $q>p>0$ and $sq\neq 1$, we have*

$$\begin{aligned}
\left[L\left(a,b\right)\right]^{1-\frac{1}{q}}\left[L_{s+1}\left(a,b\right)\right]^{s+1} &\leq \frac{\left[L\left(a^{\frac{q-p}{q-1}},b^{\frac{q-p}{q-1}}\right)\right]^{1-\frac{1}{q}}}{2\left(2p\right)^{\frac{1}{q}}} \\
&\times\left\{a^{1-\frac{p}{q}}b\left[\left(p+sq\right)\left[L_{sq+p-1}\left(a,b\right)\right]^{sq+p-1}+pb^{sq}\left[L_{p-1}\left(a,b\right)\right]^{p-1}\right.\right. \\
&-sqL\left(a^{p},b^{p}\right)\left[L_{sq-1}\left(a,b\right)\right]^{sq-1}\Big]^{\frac{1}{q}}+ab^{1-\frac{p}{q}}\Big[\left(p+sq\right)\left[L_{sq+p-1}\left(a,b\right)\right]^{sq+p-1} \\
&\left.\left.+pa^{sq}\left[L_{p-1}\left(a,b\right)\right]^{p-1}-sqL\left(a^{p},b^{p}\right)\left[L_{sq-1}\left(a,b\right)\right]^{sq-1}\right]^{\frac{1}{q}}\right\}. \qquad (3.8)
\end{aligned}$$

*Proof.* The proof is similar to that of Theorem 11 and we omit the details for the interested reader. □

**Remark 3.** *If $p\to q$, $L\left(a^{\frac{q-p}{q-1}},b^{\frac{q-p}{q-1}}\right)\to 1$ and the inequality (3.8) becomes the inequality (3.6) proved in Theorem 11.*

## References


[1] P. S. Bullen, *Handbook of Means and Their Inequalities*, Mathematics and its Applications, Volume 560, Kluwer Academic Publishers,Dordrecht/Boston/London, 2003.

[2] S. S. Dragomir, Hermite-Hadamard's type inequalities for convex functions of selfadjoint operators in Hilbert spaces, Linear Algebra Appl. 436 (2012), no. 5, 1503–1515.

[3] S. S. Dragomir, Superadditivity and monotonicity of some functionals associated with the Hermite-Hadamard inequality for convex functions in linear spaces, Rocky Mountain J. Math. 42 (2012), no. 5, 1447–1459.

[4] H. Hudzik and L. Maligranda, Some remarks on $s$-convex functions, Aequationes Math. 48 (1994), no. 1, 100–111. V. N. Huy and N. T. Chung, Some generalizations of the Fejér and Hermite-Hadamard inequalities in Hölder spaces, J. Appl. Math. Inform. 29 (2011), no. 3-4, 859–868.

[5] D. A. Ion, Some estimates on the Hermite-Hadamard inequality through quasi-convex functions, An. Univ. Craiova Ser. Mat. Inform. 34 (2007), 83–88.

[6] A. P. Ji, T. Y. Zhang, F. Qi, Integral Inequalities of Hermite-Hadamard Type for $(\alpha,m)$-GA-Convex Functions, Journal of Function Spaces and Applications, 2013 (2013), Article ID 823856, 8 pages.

[7] M. Klaričić Bakula and J. Pečarić, Note on some Hadamard-type inequalities, JIPAM. J. Inequal. Pure Appl. Math. 5(2004), no. 3, Article 74, 9 pp.

[8] M. Klaričić Bakula, M. E. Özdemir and J. Pečarić, Hadamard type inequalities for $m$-convex and $(\alpha,m)$-convex functions, JIPAM. J. Inequal. Pure Appl. Math. 9 (2008), no. 4, Article 96, 12 pp.

[9] V. G. Mihesan, A generalization of the convexity, Seminar on Functional Equations, Approx. and Convex., Cluj-Napoca, Romania, 1993.

[10] C. P. Niculescu, Convexity according to the geometric mean, Math. Inequal. Appl. 3 (2000), no. 2, 155–167.

[11] C. P. Niculescu, Convexity according to means, Math. Inequal. Appl. 6 (2003), no. 4, 571–579.

[12] J. Park, Hermite-Hadamard-type inequalities for real $\alpha$-star $s$-convex mappings, J. Appl. Math. Inform. 28 (2010), no. 5-6, 1507–1518.

[13] J. Park, Generalizations of Hermite-Hadamard-like type inequalities for co-ordinated convex and concave mappings, Far East J. Math. Sci. (FJMS) 63 (2012), no. 2, 235–251.

[14] F. Qi, Generalized abstracted mean values, J. Inequal. Pure Appl. Math. 1 (1) (2000), Art. 4.

[15] F. Qi, Generalized weighted mean values with two parameters, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 454 (1978) (1998), 2723–2732.

[16] M. Z. Sarikaya, On new Hermite Hadamard Fejér type integral inequalities, Stud. Univ. Babeş-Bolyai Math. 57 (2012), no. 3, 377–386.

[17] M. Z. Sarikaya, A. Saglam and H. Yildirim, On some Hadamard-type inequalities for $h$-convex functions, J. Math. Inequal. 2 (2008), no. 3, 335–341.
[18] Y. Shuang, H. P. Yin, F. Qi, Hermite-Hadamard type integral inequalities for geometric-arithmetically $s$-convex functions, Analysis 33, (2013), 1001–1010.
[19] T. Y. Zhang, A. P. Ji, F. Qi, Some inequalities of Hermite-Hadamard type for GA-Convex functions with applications to means. Le Matematiche, 48 (2013), no. 1, 229–239.

School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa
*E-mail address*: m_amer_latif@hotmail.com