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SOME WEIGHTED INTEGRAL INEQUALITIES FOR DIFFERENTIABLE h-PREINVEX FUNCTIONS

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Abstract. In this paper, by using a weighted identity for functions defined on an open invex subset of set of real numbers, by using the Hölder integral inequalituy and by using the notion of h-preinvexity, we present weighted integral inequalities of Hermite-Hadamard type for functions whose derivatives in absolute value raised to certain powers are h-preinvex functions. Some new Hermite-Hadamard type integral inequalities are obtained when h is super-additive. Inequalities of Hermite-Hadamard type for s-preinvex functions are given as well as a special case of our results.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Then the following inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds. The inequality (1.1) is known as the Hermite-Hadamard inequality for convex functions. The inequality in (1.1) holds in reversed direction if f is concave. In recent years, a number of papers have been written on Hermite-Hadamard type inequalities for convex functions by many authors see for instance [6]-[9], [12], [13], [15], [19], [23], [31], [33], [35], [36], [39], [42], [49] and the references therein.

To estimate the difference between the middle and the leftmost terms in (1.1) has been an important question in mathematical analysis. The most representative work to give the answer of the above raised question are the articles of Kırmacı [12] and Pearce and Pečarić [31]. The main results from these papers are the following.

Theorem 1. [12] *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$, and $f' \in L([a,b])$. If $\left|f'\right|$ is convex function on $[a,b]$, the following inequality holds:*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{8}\left[\left|f'(a)\right| + \left|f'(b)\right|\right]. \tag{1.2}$$

Theorem 2. [31] *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$,and $f' \in L([a,b])$ and $q \ge 1$. If $\left|f'\right|^q$ is a convex function on $[a,b]$,*

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the following inequality holds:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{4}\left[\frac{\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q}{2}\right]^{1/q}. \tag{1.3}$$

In [9], Hwang established the following results for convex functions and these results provide a weighted generalization of the results given in Theorem 1 and Theorem 2.

Theorem 3. [9] *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, where a, $b\in I^\circ$ with $a<b$ and let $g:[a,b]\to[0,\infty)$ be continuous positive mapping and symmetric to $\frac{a+b}{2}$. If $\left|f^{'}\right|$ is convex function on $[a,b]$, the following inequality holds:*

$$\left|\int_a^b f(x)g(x)dx-f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx\right| \leq\frac{b-a}{2}\left[\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right]\int_0^1 M(g;a,b,t)\,dt, \tag{1.4}$$

where $M(g;a,b,t)=\int_a^{L(a,b,t)}g(x)\,dt$ and $L(a,b,t)=\frac{1+t}{2}a+\frac{1-t}{2}b$.

Theorem 4. [9] *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, where a, $b\in I^\circ$ with $a<b$ and let $g:[a,b]\to[0,\infty)$ be continuous positive mapping and symmetric to $\frac{a+b}{2}$ and $q\geq1$. If $\left|f^{'}\right|^q$ is convex function on $[a,b]$, the following inequality holds:*

$$\left|\int_a^b f(x)g(x)dx-f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx\right| \leq(b-a)\left[\frac{\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q}{2}\right]^{\frac{1}{q}}\int_0^1 M(g;a,b,t)\,dt, \tag{1.5}$$

where $M(g;a,b,t)$ and $L(a,b,t)$ are defined as in Theorem 3.

The main purpose of the present paper is to establish new weighted Hermite-Hadamard type inequalities for functions whose derivatives in absolute value are h-preinvex. Some inequalities are deduced from the obtained results when h is super-additive and some inequalities of Hermite-Hadamard type are also obtained when the functions whose derivatives in absolute value are s-preinvex as a special case of our established results in Section 2.

2. Inequalities for h-Preinvex Functions

We want to present some basic definitions of preinvex and h-preinvex functions before we proceed to prove our main results.

Definition 1. [43] *Let K be a non-empty subset in $\mathbb{R}^n$ and $\eta: K\times K\to\mathbb{R}^n$. Let $x\in K$, then the set K is said to be invex at x with respect to $\eta(\cdot,\cdot)$, if*

$$x+t\eta(y,x)\in K,\forall x,y\in K,t\in[0,1].$$

K is said to be an invex set with respect to η if K is invex at each $x\in K$. The invex set K is also called an η-connected set.

Remark 1. *It is to be noted in the above definition that there is a path starting from a point x which is contained in K and we do not require that the point y should be one of the end point of this path. Note that if $\eta(y,x)=y-x$ then y is the end point of the path which is contained in K and consequently invexity reduces to convexity.*

Definition 2. [43] *A function $f:K\to\mathbb{R}$ on an invex set $K\subseteq\mathbb{R}^n$ is said to be preinvex with respect to η, if*

$$f(u+t\eta(v,u))\leq(1-t)\,f(u)+tf(v),\forall u,v\in K,t\in[0,1].$$

The function f is said to be preincave if and only if $-f$ is preinvex.

It is to be noted that every convex function is preinvex with respect to the map $\eta\left(x,y\right)=x-y$ but the converse is not true see for instance [1].

Definition 3. *A function $f:K\to[0,\infty)$ on an invex set $K\subseteq[0,\infty)^n$ is said to be s-preinvex with respect to η, if*

$$f(u+t\eta(v,u))\leq(1-t)\,f(u)+tf(v)$$

holds for all $u,v\in K,t\in[0,1]$ and for some fixed $s\in(0,1]$. The function f is said to be s-preincave if and only if $-f$ is preinvex.

In the paper [40], a large class of non-negative functions, the so-called h-convex functions was considered. This class contains several well-known classes of functions such as non-negative convex functions, s-convex in the second sense, Godunova Levin functions and P-functions. The definition of h-convexity is further generalized in [26] by Matloka as follows.

Definition 4. [26] *Let J be a real intervals such that $(0,1)\subseteq J$ and let $h:J\to\mathbb{R}$ be a non-negative function with $h\not\equiv 0$. A function $f:K\to\mathbb{R}$ defined on an invex subset of $\mathbb{R}^n$ is called an h-preinvex with respect, if for all $x,y\in K$ and $t\in[0,1]$*

$$f(u+t\eta(v,u))\leq h\left(1-t\right)f(u)+h\left(t\right)f(v). \tag{2.1}$$

If the inequality in (2.1) holds in reversed, then f is called h-preincave.

Remark 2. *Note, that every convex function is a h-preinvex function with respect to $\eta(v,u)=v-u$ and $h\left(t\right)=t$, $t\in[0,1]$.*

Definition 5. [2] *A function $h:J\to\mathbb{R}$ is said to be a super-additive function if*

$$h(x+y)\geq h(x)+h(y)$$

for all $x,y\in J$, when $x+y\in J$.

For several recent results on inequalities for preinvex and h-preinvex functions we refer the interested reader to [4, 11, 16, 17, 18, 20, 22, 25, 26, 27, 28, 29, 34, 44] and [45] and the references therein.

The following Lemma help us establish our main results of this section.

Lemma 1. [18] *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$. Suppose $f:K\to\mathbb{R}$ is a differentiable mapping on K such that $f^{'}\in$*

$L([a, a+\eta(b,a)])$, where $a, b \in K$ with $\eta(b,a) > 0$. If $g : [a, a+\eta(b,a)] \to [0,\infty)$ be a differentiable mapping, then the following equality holds

$$\begin{aligned}(2.2)\quad & \frac{g(a)}{2}\left[f(a)+f(b)\right]-g(a+\eta(b,a))\,f\left(\frac{a+b}{2}\right)\\ &+\frac{\eta(b,a)}{4}\int_0^1\left[f\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)+f\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right]\\ &\times\left[g^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)+g^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right]dt\\ &=\frac{\eta(b,a)}{4}\left\{\int_0^1\left[g\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)-g\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right.\right.\\ &\left.\left.+g(a+\eta(b,a))\right]\times\left[-f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)+f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right]dt\right\}.\end{aligned}$$

Remark 3. *If we take $\eta(b,a) = b-a$, then Lemma 1 reduces to Lemma 2.1 from [7].*

Now using Lemma 1, we shall propose some new upper bounds for the difference between the leftmost and the middle terms of weighted version of the Hadamard type inequality proved in [28] using h-preinvex mappings.

In what follows we use the notations $L^{'}(a,b,t) = a+\left(\frac{1-t}{2}\right)\eta(b,a)$ and $U^{'}(a,b,t) = a+\left(\frac{1+t}{2}\right)\eta(b,a)$ for our convenience.

Theorem 5. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K\times K \to \mathbb{R}$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K and $w : [a, a+\eta(b,a)] \to [0,\infty)$ be continuous and symmetric to $a+\frac{1}{2}\eta(b,a)$, where $a, b \in K$ with $\eta(b,a) > 0$. If $\left|f^{'}\right|$ is h-preinvex on K, we have the following inequality*

$$\begin{aligned}(2.3)\quad & \left|\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\int_a^{a+\eta(b,a)} w(x)\,dx\right|\\ &\le \frac{\eta(b,a)}{2}\left[\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right]\int_0^1\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]M^{'}(w;a,b,t)\,dt,\end{aligned}$$

where $M^{'}(w;a,b,t) = \int_a^{L^{'}(a,b,t)} w(x)\,dx$ for all $t \in [0,1]$.

Proof. Let $g(t)=\int_a^t w(x)\,dx$ for all $t\in[a,a+\eta(b,a)]$ in Lemma 1, we have

$$
\begin{aligned}
(2.4)\quad & -f\left(a+\frac{1}{2}\eta(b,a)\right)\int_a^{a+\eta(b,a)} w(x)\,dx \\
& +\frac{\eta(b,a)}{4}\int_0^1\left[f\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)+f\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right] \\
& \times\left[w\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)+w\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right]dt \\
& =\frac{\eta(b,a)}{4}\left\{\int_0^1\left[\int_a^{L'(a,b,t)} w(x)\,dx+\int_{U'(a,b,t)}^{a+\eta(b,a)} w(x)\,dx\right]\right. \\
& \left.\times\left[-f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)+f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right]dt\right\}.
\end{aligned}
$$

Since $w(x)$ is symmetric to $a+\frac{1}{2}\eta(b,a)$, we have

$$
(2.5)\qquad w\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)=w\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)
$$

and

$$
(2.6)\qquad \int_a^{L'(a,b,t)} w(x)\,dx=\int_{U'(a,b,t)}^{a+\eta(b,a)} w(x)\,dx
$$

for all $t\in[0,1]$. Hence by using (2.5), we have

$$
\begin{aligned}
(2.7)\quad & \frac{\eta(b,a)}{4}\int_0^1\left[f\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)+f\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right] \\
& \times\left[w\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)+w\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right]dt \\
& =\frac{\eta(b,a)}{2}\int_0^1 f\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)w\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)dt \\
& +\frac{\eta(b,a)}{2}\int_0^1 f\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)w\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)dt \\
& =\int_a^{a+\frac{1}{2}\eta(b,a)} f(x)\,w(x)\,dx+\int_{a+\frac{1}{2}\eta(b,a)}^{a+\eta(b,a)} f(x)\,w(x)\,dx=\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx.
\end{aligned}
$$

Using (2.6) and (2.7) in (2.4) and using the preinvexity of $\left|f'\right|$ on K, we have

$$
\begin{aligned}
(2.8)\quad & \left|\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx-f\left(a+\frac{1}{2}\eta(b,a)\right)\int_a^{a+\eta(b,a)} w(x)\,dx\right| \\
& \le\frac{\eta(b,a)}{2}\int_0^1 M'(w;a,b,t)\left[\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|\right. \\
& \left.+\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|\right]dt\le\frac{\eta(b,a)}{2}\left[\left|f'(a)\right|+\left|f'(b)\right|\right] \\
& \times\int_0^1\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]M'(w;a,b,t)\,dt
\end{aligned}
$$

which is the required inequality (2.3). This completes the proof of the theorem. □

Corollary 1. *In Theorem 5, if we take* $w(x) = \frac{1}{\eta(b,a)}$ *for all* $x \in [a, a+\eta(b,a)]$*, then (2.3) becomes the following inequality for h-preinvex functions*

$$\left| \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx - f\left(a + \frac{1}{2}\eta(b,a)\right) \right| \leq \frac{\eta(b,a)}{4} \left[\left|f'(a)\right| + \left|f'(b)\right| \right] \int_0^1 (1-t) \left[h\left(\frac{1-t}{2}\right) + h\left(\frac{1+t}{2}\right) \right] dt. \tag{2.9}$$

Corollary 2. *If* $\eta(b,a) = b - a$ *in Theorem 5, we have the following inequality for h-convex functions*

$$\left| \int_a^b f(x)\,w(x)\,dx - f\left(\frac{a+b}{2}\right) \int_a^b w(x)\,dx \right| \leq \frac{b-a}{2} \left[\left|f'(a)\right| + \left|f'(b)\right| \right] \int_0^1 \left[h\left(\frac{1-t}{2}\right) + h\left(\frac{1+t}{2}\right) \right] M(w;a,b,t)\,dt, \tag{2.10}$$

where $M(w;a,b,t) = \int_a^{L(a,b,t)} w(x)\,dx$ *for all* $t \in [0,1]$ *and* $L(a,b,t) = \frac{1+t}{2}a + \frac{1-t}{2}b$.

Corollary 3. *If* $\eta(b,a) = b - a$ *and* $w(x) = \frac{1}{b-a}$ *for all* $x \in [a,b]$ *in Theorem 5, we have the following inequality for h-convex functions*

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{4} \left[\left|f'(a)\right| + \left|f'(b)\right| \right] \int_0^1 (1-t) \left[h\left(\frac{1-t}{2}\right) + h\left(\frac{1+t}{2}\right) \right] dt. \tag{2.11}$$

Corollary 4. *Suppose* $h(t) = t^s$, $s \in (0,1]$ *in Corollary 1, we have the following inequality for s-preinvex functions*

$$\left| \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx - f\left(a + \frac{1}{2}\eta(b,a)\right) \right| \leq \frac{\eta(b,a)\left(2^{s+1}-1\right)}{2^{s+1}(s+1)(s+2)} \left[\left|f'(a)\right| + \left|f'(b)\right| \right]. \tag{2.12}$$

Corollary 5. *If* $\eta(b,a) = b - a$ *in Corollary 4, we have the following inequality for s-convex functions*

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)\left(2^{s+1}-1\right)}{2^{s+1}(s+1)(s+2)} \left[\left|f'(a)\right| + \left|f'(b)\right| \right]. \tag{2.13}$$

Remark 4. *If* $h(t) = t$*, the inequality (2.3) becomes the inequality proved in* [20, Theorem 8] *and the inequality (2.10) recaptures the inequality (1.4) for convex functions.*

Corollary 6. *Suppose the assumptions of Theorem 5 are satisfied. If h is super-additive, we have the inequality*

$$\left|\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\int_a^{a+\eta(b,a)} w(x)\,dx\right| \leq \frac{\eta(b,a)\,h(1)}{2}\left[\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right]\int_0^1 M^{'}(w;a,b,t)\,dt. \tag{2.14}$$

Remark 5. *If $\eta(b,a)=b-a$ in Corollary 6, we get a result when h is super-additive function.*

Theorem 6. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta: K\times K\to\mathbb{R}$. Suppose $f: K\to\mathbb{R}$ is a differentiable mapping on K and $w:[a,a+\eta(b,a)]\to[0,\infty)$ be continuous and symmetric to $a+\frac{1}{2}\eta(b,a)$, where $a, b\in K$ with $\eta(b,a)>0$. If $\left|f^{'}\right|^q$ is h-preinvex on K for $q>1$, we have the following inequality*

$$\begin{aligned}\left|\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\int_a^{a+\eta(b,a)} w(x)\,dx\right| &\leq \eta(b,a)\left[\frac{\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q}{2}\right]^{\frac{1}{q}} \\ &\times\left(\int_0^1\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]dt\right)^{\frac{1}{q}}\left(\int_0^1\left[M^{'}(w;a,b,t)\right]^p dt\right)^{\frac{1}{p}},\end{aligned} \tag{2.15}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $M^{'}(w;a,b,t)$ is defined as in Theorem 5.

Proof. Continuing from inequality (2.8) in the proof of Theorem 5 and using the Hölder's integral inequality, we have

$$\begin{aligned}&\left|\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\int_a^{a+\eta(b,a)} w(x)\,dx\right| \\ &\leq \frac{\eta(b,a)}{2}\left(\int_0^1\left[M^{'}(w;a,b,t)\right]^p dt\right)^{\frac{1}{p}}\left[\left(\int_0^1\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\ &\qquad\left.+\left(\int_0^1\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right]\end{aligned} \tag{2.16}$$

By the power-mean inequality $t^r+s^r<2^{1-r}(t+s)^r$ for $t>0$, $s>0$ and $r<1$ and by the the h-preinvexity of $\left|f^{'}\right|^q$ on K for $q>1$, we have for every $a,b\in K$

with $\eta(b,a)>0$ the following inequality

(2.17)

$$\left(\int_0^1 \left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1 \left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq 2^{1-\frac{1}{q}}\left[\int_0^1 \left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt+\int_0^1 \left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right]^{\frac{1}{q}}$$
$$\leq 2^{1-\frac{1}{q}}\left[\int_0^1 \left\{h\left(\frac{1+t}{2}\right)\left|f'(a)\right|^q+h\left(\frac{1-t}{2}\right)\left|f'(b)\right|^q\right.\right.$$
$$\left.\left.+h\left(\frac{1-t}{2}\right)\left|f'(a)\right|^q+h\left(\frac{1+t}{2}\right)\left|f'(b)\right|^q\right\}dt\right]^{\frac{1}{q}}$$
$$=2^{1-\frac{1}{q}}\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]^{\frac{1}{q}}\left(\int_0^1\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]dt\right)^{\frac{1}{q}}.$$

Using the last inequality (2.17) in (2.16), we get the desired inequality. This completes the proof of the theorem as well. □

Corollary 7. *In Theorem 6, if we take $w(x)=\frac{1}{\eta(b,a)}$ for all $x\in[a,a+\eta(b,a)]$ with $\eta(b,a)>0$, then (2.15) becomes the following inequality*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx-f\left(a+\frac{1}{2}\eta(b,a)\right)\right| \tag{2.18}$$
$$\leq\frac{\eta(b,a)}{2(1+p)^{\frac{1}{p}}}\left(\int_0^1\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]dt\right)^{\frac{1}{q}}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 8. *If we take $\eta(b,a)=b-a$ in Theorem 6, then (2.15) becomes the following inequality for h-convex functions*

$$\left|\int_a^b f(x)\,w(x)\,dx-f\left(\frac{a+b}{2}\right)\int_a^b w(x)\,dx\right| \tag{2.19}$$
$$\leq\eta(b,a)\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}\left(\int_0^1\left[M(w;a,b,t)\right]^p dt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]dt\right)^{\frac{1}{q}},$$

where $M(w;a,b,t)=\int_a^{L(a,b,t)} w(x)\,dx$, $L(a,b,t)=\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b$ for all $t\in[0,1]$ and $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 9. *Assume that all the conditions of Theorem 6 are satisfied and in addition if h is super-additive, we have the following inequality*

$$\left|\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\int_a^{a+\eta(b,a)} w(x)\,dx\right| \le \eta(b,a)\,(h(1))^{\frac{1}{q}}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}\left(\int_0^1\left[M'(w;a,b,t)\right]^p dt\right)^{\frac{1}{p}}, \tag{2.20}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Remark 6. *If $\eta(b,a)=b-a$ in Corollary 9, we have the following result when h is a super-additive function.*

Corollary 10. *Suppose that $h(t)=t^s$, $t\in(0,1]$ in Corollary 7, we have the following result from for s-preinvex functions*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\right| \le \frac{\eta(b,a)}{2^{1-\frac{1}{q}}(1+p)^{\frac{1}{p}}(s+1)^{\frac{1}{q}}}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}, \tag{2.21}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

A similar result may be stated as follows.

Theorem 7. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta: K\times K\to\mathbb{R}$. Suppose $f: K\to\mathbb{R}$ is a differentiable mapping on K and $w:[a,a+\eta(b,a)]\to[0,\infty)$ be continuous and symmetric to $a+\frac{1}{2}\eta(b,a)$, where $a,b\in K$ with $\eta(b,a)>0$. If $\left|f'\right|^q$ is preinvex on K for $q\ge 1$, we have the following inequality*

$$\left|\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\int_a^{a+\eta(b,a)} w(x)\,dx\right| \le \eta(b,a)\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}\left(\int_0^1 M'(w;a,b,t)\,dt\right)^{1-\frac{1}{q}} \times\left(\int_0^1 M'(w;a,b,t)\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]dt\right)^{\frac{1}{q}}, \tag{2.22}$$

where $M'(w;a,b,t)$ is defined as in Theorem 5.

Proof. Continuing from inequality (2.8) in the proof of Theorem 5 and using the well known Hölder's integral inequality, we have

$$\left|\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\int_a^{a+\eta(b,a)} w(x)\,dx\right| \tag{2.23}$$
$$\leq \frac{\eta(b,a)}{2}\left(\int_0^1 M'(w;a,b,t)\,dt\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left[\int_0^1 M'(w;a,b,t)\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\int_0^1 M'(w;a,b,t)\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right]^{\frac{1}{q}}\right\}.$$

By the power-mean inequality $t^r + s^r < 2^{1-r}(t+s)^r$ for $t > 0$, $s > 0$ and $r < 1$, and by the the h-preinvexity of $\left|f'\right|^q$ on K for $q > 1$, we have for every $a, b \in K$ with $\eta(b,a) > 0$ the following inequality

$$\left[\int_0^1 M'(w;a,b,t)\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right]^{\frac{1}{q}} \tag{2.24}$$
$$+\left[\int_0^1 M'(w;a,b,t)\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right]^{\frac{1}{q}}$$
$$\leq 2\left(\int_0^1 M'(w;a,b,t)\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]dt\right)^{\frac{1}{q}}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}.$$

Utilizing inequality (2.24) in (2.23), we get the inequality (2.22). This completes the proof of the theorem. □

Corollary 11. *Suppose all the assumptions of Theorem 7 are satisfied and if* $w(x) = \frac{1}{\eta(b,a)}$ *for all* $x \in [a, a+\eta(b,a)]$ *with* $\eta(b,a) > 0$*, we have the following inequality*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\right| \tag{2.25}$$
$$\leq \eta(b,a)\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}$$
$$\times\left[\int_0^1\left(\frac{1-t}{2}\right)\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]dt\right]^{\frac{1}{q}}.$$

Corollary 12. *If we take $\eta(b,a) = b - a$ in Theorem 7, we have the following weighted inequality for h-convex functions*

$$\left|\int_a^b f(x)\,w(x)\,dx - f\left(\frac{a+b}{2}\right)\int_a^b w(x)\,dx\right| \le (b-a)\left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}\left(\int_0^1 M(w;a,b,t)\,dt\right)^{1-\frac{1}{q}} \times \left(\int_0^1 M(w;a,b,t)\left[h\left(\frac{1-t}{2}\right) + h\left(\frac{1+t}{2}\right)\right]dt\right)^{\frac{1}{q}}, \tag{2.26}$$

where $M(w;a,b,t) = \int_a^{L(a,b,t)} w(x)\,dx$, $L(a,b,t) = \left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b$ for all $t \in [0,1]$.

Corollary 13. *If $w(x) = \frac{1}{b-a}$ for all $x \in [a,b]$ in Corollary 12, we have the following inequality for h-convex functions*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \le (b-a)\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right] \times \left[\int_0^1 \left(\frac{1-t}{2}\right)\left[h\left(\frac{1-t}{2}\right) + h\left(\frac{1+t}{2}\right)\right]dt\right]^{\frac{1}{q}}. \tag{2.27}$$

Remark 7. *If in Corollary 12 we take $h(t) = t$, we have the inequality 1.5 given in Theorem 4.*

Corollary 14. *Under the assumptions of Theorem 7, if h is super-additive, then we have the inequality*

$$\left|\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx - f\left(a + \frac{1}{2}\eta(b,a)\right)\int_a^{a+\eta(b,a)} w(x)\,dx\right| \le \eta(b,a)\,(h(1))^{\frac{1}{q}}\left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}\int_0^1 M'(w;a,b,t)\,dt, \tag{2.28}$$

where $M'(w;a,b,t)$ is defined as in Theorem 5.

Corollary 15. *If h is super-additive in Corollary 11, then we have the inequality*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - f\left(a + \frac{1}{2}\eta(b,a)\right)\right| \le \frac{\eta(b,a)\,(h(1))^{\frac{1}{q}}}{4}\left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}. \tag{2.29}$$

Corollary 16. *When h is super-additive in Corollary 12, we have the inequality*

$$\left|\int_a^b f(x)\,w(x)\,dx - f\left(\frac{a+b}{2}\right)\int_a^b w(x)\,dx\right| \leq (b-a)\,(h(1))^{\frac{1}{q}}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}\int_0^1 M(w;a,b,t)\,dt, \tag{2.30}$$

where $M(w;a,b,t)=\int_a^{L(a,b,t)} w(x)\,dx$, $L(a,b,t)=\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b$ *for all* $t\in[0,1]$.

Corollary 17. *In Corollary 13, if h is super-additive then we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)\,(h(1))^{\frac{1}{q}}}{4}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}. \tag{2.31}$$

Corollary 18. *If $h(t)=t^s$, $t\in(0,1]$ in Corollary 11, we have the following inequality for s-preinvex functions*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\right| \leq \eta(b,a)\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{2^{s+1}-1}{2^s(s+1)(s+2)}\right)^{\frac{1}{q}}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}. \tag{2.32}$$

Corollary 19. *If $\eta(b,a)=b-a$, we have the following inequality for s-convex functions*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq (b-a)\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{2^{s+1}-1}{2^s(s+1)(s+2)}\right)^{\frac{1}{q}}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}. \tag{2.33}$$

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