Received 01/10/14

INEQUALITIES OF HERMITE-HADAMARD TYPE FOR n-TIMES DIFFERENTIABLE (α, m)-LOGARITHMICALLY CONVEX FUNCTIONS

M. A. LATIF, S. S. DRAGOMIR[1,2], AND E. MOMONIAT

Abstract. In this paper, some new integral inequalities of Hermite-Hadamard type are presented for functions whose nth derivatives in absolute value are (α, m)-logarithmically convex. From our results, several inequalities of Hermite-Hadamard type can be derived in terms of functions whose first and second derivatives in absolute value are (α, m)-logarithmically convex functions as special cases. Our results may provide refinements of some results for (α, m)-logarithmically convex functions already exist in the most recent concerned literature of inequalities.

1. Introduction

Let us first refresh our knowledge how the following definition of classical convex functions is generalized.

Definition 1. *A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $t \in [0, 1]$. The inequalities in (1.1) are swapped if f is a concave function.

The definition of convex functions plays an important role in the theory of convex analysis and in many other branches of pure and applied mathematics. A number of remarkable and significant results in the theory of inequality hinge on the this definition.

One of the momentous results which uses the notion of convexity is the stated as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1.2}$$

where $f : \emptyset \neq I \subseteq \mathbb{R} \to \mathbb{R}$, is a convex function of single variable, a, $b \in I$ with $a < b$. The inequalities in (1.2) are celebrated as Hermite-Hadamard inequality and are overturned if f is a concave function.

The inequalities (1.2) have been target of extensive research because of its usefulness and usages in the theory of inequalities and in various other branches of

---

Date: Today.

2000 *Mathematics Subject Classification.* Primary 26D15, 26D20, 26E60; Secondary 41A55.

Key words and phrases. Hermite-Hadamard's inequality, s-logarithmically convex function, Hölder integral inequality.

This paper is in final form and no version of it will be submitted for publication elsewhere.

mathematics. A vast literature is reported on the Hermite-Hadamard type inequalities during the past few years which generalize, improve and extend the inequalities (1.2), see for example [6, 12, 13, 14, 15, 17, 19, 23, 28] and closely related references therein.

The classical convexity has been generalized in diverse ways such as s-convexity, m-convexity, (α, m)-convexity, h-convexity, logarithmic-convexity, s- logarithmic convexity, (α, m)- logarithmic convexity and h-logarithmic-convexity but we will focus on the following generalizations of the classical convexity to prove our results.

Definition 2. [2, 33, 34] *If a function $f : I \subseteq \mathbb{R} \to (0, \infty)$ satisfies*

$$f(\lambda x + (1-\lambda) y) \leq [f(x)]^{\lambda} [f(y)]^{1-\lambda} \tag{1.3}$$

for all x, $y \in I$, $\lambda \in [0, 1]$, the function f is called logarithmically convex on I. If the inequality (1.3) reverses, the function f is called logarithmically concave on I.

The above stated concept logarithmically convex functions is further generalized as in the definitions below.

Definition 3. [9] *A function $f : [0, b] \to (0, \infty)$ is said to be m-logarithmically convex if*

$$f(tx + m(1-t) y) \leq [f(x)]^{t} [f(y)]^{m(1-t)}$$

holds for all x, $y \in [0, b]$, $t \in [0, 1]$ and $m \in (0, 1]$.

Definition 4. [9] *A function $f : [0, b] \to (0, \infty)$ is said to be (α, m)-logarithmically convex if*

$$f(tx + m(1-t) y) \leq [f(x)]^{t^{\alpha}} [f(y)]^{m(1-t^{\alpha})}$$

holds for all x, $y \in [0, b]$, $t \in [0, 1]$ and $(\alpha, m) \in (0, 1] \times (0, 1]$.

It is also obvious that if $m = 1$ in Definition 3 and if $(\alpha, m) = (1, 1)$ in Definition 4, the notion of m-logarthmic convexity and (α, m)-logarithmic convexity recapture the notion of usual logarithmic convexity.

Many papers have been written by a number of mathematicians concerning Hermite-Hadamard type inequalities for different classes of convex functions see for instance the recent papers [2, 3, 4, 7, 8, 9, 16, 18, 24, 25, 27, 29, 31, 32, 33, 35] and the references within these papers.

The main purpose of the present paper is to establish new Hermite-Hadamard type integral inequalities by using the notion of m- and (α, m)-logarthmically convex functions and a new identity for n-times differentiable functions from [19]in Section 2.

2. Main Results

We will use the following Lemmas to establish our main results in this section.

Lemma 1. [19] *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a function such that $f^{(n)}$ exists on I° and $f^{(n)} \in L([a,b])$ for $n \in \mathbb{N}$, where $a, b \in I^{\circ}$ with $a < b$, we have the identity*

$$\begin{aligned}
(2.1)\quad & \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=1}^{n-1} \frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) \\
&= \frac{(b-a)^n}{2^{n+1}n!}\int_0^1 (1-t)^{n-1}(n-1+t) f^{(n)}\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) dt \\
&\quad + \frac{(-1)^n (b-a)^n}{2^{n+1}n!}\int_0^1 (1-t)^{n-1}(n-1+t) f^{(n)}\left(\frac{1-t}{2}b + \frac{1+t}{2}a\right) dt,
\end{aligned}$$

where an empty sum is understood to be nil.

Lemma 2. [20] *If $\mu > 0$ and $n \in \mathbb{N}$, then*

$$(2.2)\qquad \int_0^1 t^n \mu^t dt = \begin{cases} \frac{(-1)^{n+1} n!}{(\ln\mu)^{n+1}} + n!\mu \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!(\ln\mu)^{k+1}}, & \mu \neq 1 \\ \frac{1}{n+1}, & \mu = 1. \end{cases}$$

Lemma 3. *If $\mu > 0$ and $n \in \mathbb{N}$, then*

(2.3)

$$E(n;\mu) := \int_0^1 (1-t)^n \mu^t dt = \begin{cases} \frac{n!\mu}{(\ln\mu)^{n+1}} - n!\sum_{k=0}^{n} \frac{1}{(n-k)!(\ln\mu)^{k+1}}, & \mu \neq 1 \\ \frac{1}{n+1}, & \mu = 1. \end{cases}$$

Proof. By making the substitution $t = 1-u$ in Lemma 2, we get (2.3). □

Lemma 4. [7] *For $\alpha > 0$ and $\mu > 0$, we have*

$$(2.4)\qquad G(\alpha;\mu) := \int_0^1 (1-t)^{\alpha-1}\mu^t dt = \sum_{k=1}^{\infty} \frac{(\ln\mu)^{k-1}}{(\alpha)_k} < \infty,$$

where

$$(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+k-1).$$

From Lemma 3 and Lemma 4, by simple computations we get the following results.

Lemma 5. *If $\mu > 0$ and $n \in \mathbb{N}$, then*

$$\begin{aligned}
(2.5)\quad F(n;\mu) &:= nE(n-1;\mu) - E(n;\mu) \\
&= \begin{cases} \frac{n!\mu(\ln\mu - 1)}{(\ln\mu)^{n+1}} + \frac{1}{\ln\mu} - n!\sum_{k=1}^{n} \frac{\ln\mu - 1}{(n-k)!(\ln\mu)^{k+1}}, & \mu \neq 1 \\ \frac{n}{n+1}, & \mu = 1. \end{cases}
\end{aligned}$$

Lemma 6. *For $\alpha > 0$ and $\mu > 0$, we have*

$$(2.6)\quad H(\alpha;\mu) := nG(\alpha;\mu) - G(\alpha+1;\mu) = \sum_{k=1}^{\infty} \frac{(n\alpha + nk - \alpha)(\ln\mu)^{k-1}}{(\alpha)_{k+1}} < \infty,$$

where

$$(\alpha)_{k+1} = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+k).$$

Lemma 7. [35] *Let* $0 < \xi \leq 1 \leq \eta$, $0 \leq \lambda \leq 1$ *and* $0 < s \leq 1$. *Then*

$$\xi^{\lambda^s} \leq \xi^{s\lambda} \text{ and } \eta^{\lambda^s} \leq \eta^{s\lambda+1-s}. \tag{2.7}$$

Theorem 1. *Let* $f : I \subset [0, \infty) \to (0, \infty)$ *be a function such that* $f^{(n)}$ *exists on* I° *and* $f^{(n)} \in L([a, b])$ *for* $n \in \mathbb{N}$, *where* $a, b \in I^\circ$ *with* $0 \leq a < b < \infty$. *If* $\left|f^{(n)}\right|^q$ *is* (α, m)*-logarithmically convex on* $\left[a, \frac{b}{m}\right]$ *for* $(\alpha, m) \in (0, 1] \times (0, 1]$ *and* $q \in [1, \infty)$. *Then*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right. \\ &\left. \qquad - \sum_{k=1}^{n-1} \frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right)\right| \\ &\leq \frac{(b-a)^n}{2^{n+1}n!}\left(\frac{n}{n+1}\right)^{1-\frac{1}{q}} \left[f^{(n)}\left(\frac{b}{m}\right)\right]^m \mu^\theta \left\{\left[F\left(n; \mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q} + \left[F\left(n; \mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}, \end{aligned} \tag{2.8}$$

where $F(n; \xi)$ *is defined in Lemma 5,* $\mu = \frac{f^{(n)}(a)}{\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m}$ *and*

$$\theta = \begin{cases} \frac{\alpha}{2}, & 0 < \mu \leq 1 \\ 1 - \frac{\alpha}{2}, & \mu > 1. \end{cases}$$

Proof. From Lemma 1, the Hölder inequality and using the fact that $\left|f^{(n)}\right|^q$ is (α, m)-logarithmically convex on $\left[a, \frac{b}{m}\right]$, we have

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \sum_{k=1}^{n-1} \frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) \right. \\ &\left. - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^n}{2^{n+1}n!}\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m \left(\int_0^1 (1-t)^{n-1}(n-1+t)\,dt\right)^{1-\frac{1}{q}} \\ &\qquad \times \left\{\left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{q\left(\frac{1-t}{2}\right)^\alpha}dt\right)^{1/q}\right. \\ &\qquad\qquad \left. + \left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{q\left(\frac{1+t}{2}\right)^\alpha}dt\right)^{1/q}\right\}, \end{aligned} \tag{2.9}$$

where $\mu = \frac{f^{(n)}(a)}{\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m}$.
It is obvious that

$$\int_0^1 (1-t)^{n-1}(n-1+t)\,dt = \frac{n}{n+1}. \tag{2.10}$$

Also when $0 < \mu \leq 1$. By using Lemma 5 and Lemma 7, we obtain

$$\begin{aligned}
(2.11)\quad & \left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{q\left(\frac{1-t}{2}\right)^{\alpha}}dt\right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{q\left(\frac{1+t}{2}\right)^{\alpha}}dt\right)^{\frac{1}{q}} \\
\leq & \left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{\alpha q\left(\frac{1-t}{2}\right)}dt\right)^{\frac{1}{q}} + \left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{\alpha q\left(\frac{1+t}{2}\right)}dt\right)^{\frac{1}{q}} \\
= & \mu^{\frac{\alpha}{2}}\left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{-\frac{\alpha q t}{2}}dt\right)^{\frac{1}{q}} + \mu^{\frac{\alpha}{2}}\left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{\frac{\alpha q t}{2}}dt\right)^{\frac{1}{q}} \\
&= \mu^{\frac{\alpha}{2}}\left\{\left[F\left(n;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q} + \left[F\left(n;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}.
\end{aligned}$$

When $\mu > 1$. By using Lemma 5 and Lemma 7, we have

$$\begin{aligned}
(2.12)\quad & \left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{q\left(\frac{1-t}{2}\right)^{\alpha}}dt\right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{q\left(\frac{1+t}{2}\right)^{\alpha}}dt\right)^{\frac{1}{q}} \\
&\leq \left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{\alpha q\left(\frac{1-t}{2}\right)+q-\alpha q}dt\right)^{\frac{1}{q}} \\
&+ \left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{\alpha q\left(\frac{1+t}{2}\right)+q-\alpha q}dt\right)^{\frac{1}{q}} \\
&= \mu^{1-\frac{\alpha}{2}}\left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{-\frac{\alpha q t}{2}}dt\right)^{\frac{1}{q}} \\
&+ \mu^{1-\frac{\alpha}{2}}\left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{\frac{\alpha q t}{2}}dt\right)^{\frac{1}{q}} \\
&= \mu^{1-\frac{\alpha}{2}}\left\{\left[F\left(n;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q} + \left[F\left(n;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}.
\end{aligned}$$

A combination of (2.9)-(2.12) gives the desired result. □

Corollary 1. *Suppose the assumptions of Theorem 1 are satisfied and if $q = 1$, we have*

$$\begin{aligned}
(2.13)\quad & \left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right. \\
&\quad \left. - \sum_{k=1}^{n-1} \frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right)\right| \\
&\quad \leq \frac{(b-a)^n}{2^{n+1}n!}\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m \mu^{\theta}\left\{F\left(n;\mu^{-\frac{\alpha}{2}}\right) + F\left(n;\mu^{\frac{\alpha}{2}}\right)\right\},
\end{aligned}$$

where $F(n;\xi)$ *is defined in Lemma 5,* $\mu = \frac{f^{(n)}(a)}{\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m}$ *and* θ *is defined in Theorem 1.*

Corollary 2. *Under the assumptions of Theorem 1, if* $n = 1$*, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.14}$$
$$\leq \frac{(b-a)}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[f'\left(\frac{b}{m}\right)\right]^m \mu^\theta \left\{\left[F\left(1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q} + \left[F\left(1;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\},$$

where $\mu = \frac{f'(a)}{\left[f'\left(\frac{b}{m}\right)\right]^m}$*,* θ *is defined in Theorem 1 and*

$$F(1;\xi) = \begin{cases} \frac{1}{\ln\xi}\left[\xi + \frac{1-\xi}{\ln\xi}\right], & \xi \neq 1 \\ \frac{1}{2}, & \xi = 1. \end{cases}$$

Corollary 3. *Corollary 2 with* $q = 1$ *gives the following result*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.15}$$
$$\leq \frac{(b-a)}{4}\left[f'\left(\frac{b}{m}\right)\right]^m \mu^\theta \left\{\left[F\left(1;\mu^{-\frac{\alpha}{2}}\right)\right] + \left[F\left(1;\mu^{\frac{\alpha}{2}}\right)\right]\right\},$$

where $F(1;\xi)$ *and* μ *are defined as in Corollary 2 and* θ *is as defined in Theorem 1.*

Corollary 4. *Suppose the assumptions of Theorem 1 are fulfilled and if* $n = 2$*, we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.16}$$
$$\leq \frac{(b-a)^2}{16}\left(\frac{2}{3}\right)^{1-\frac{1}{q}}\left[f''\left(\frac{b}{m}\right)\right]^m \mu^\theta \left\{\left[F\left(2;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q} + \left[F\left(2;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\},$$

where $\mu = \frac{f''(a)}{\left[f''\left(\frac{b}{m}\right)\right]^m}$*,* θ *is as defined in Theorem 1 and*

$$F_1(\xi,2) = \begin{cases} \frac{2\xi\ln\xi - (\ln\xi)^2 - 2\xi + 2}{(\ln\xi)^3}, & \xi \neq 1, \\ \frac{2}{3}, & \xi = 1. \end{cases}$$

Corollary 5. *If* $q = 1$ *in Corollary 4, we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.17}$$
$$\leq \frac{(b-a)^2}{16}\left[f''\left(\frac{b}{m}\right)\right]^m \mu^\theta \left\{F\left(2;\mu^{-\frac{\alpha}{2}}\right) + F\left(2;\mu^{\frac{\alpha}{2}}\right)\right\},$$

where θ *is defined in Theorem 1 and* μ*,* $F_1(\xi,2)$ *are defined in Corollary 4.*

Theorem 2. *Let $f : I \subset [0,\infty) \to (0,\infty)$ be a function such that $f^{(n)}$ exists on I° and $f^{(n)} \in L([a,b])$ for $n \in \mathbb{N}$, where $a,\ b \in I^{\circ}$ with $a < b$. If $\left|f^{(n)}\right|^q$ is (α,m)-logarithmically convex on $\left[a, \frac{b}{m}\right]$ for $(\alpha,m) \in (0,1] \times (0,1]$ and $q \in (1,\infty)$, we have*

$$\begin{aligned}(2.18)\quad &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right.\\ &\left. - \sum_{k=1}^{n-1} \frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right)\right|\\ &\le \frac{(b-a)^n\left[n^{(2q-1)/(q-1)} - (n-1)^{(2q-1)/(q-1)}\right]^{1-\frac{1}{q}}}{2^{n+1}n!}\left(\frac{q-1}{2q-1}\right)^{1/q}\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m \mu^{\theta}\\ &\times\left\{\left[G\left(nq-q+1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}} + \left[G\left(nq-q+1;\mu^{\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}\right\},\end{aligned}$$

where $\mu = \frac{f^{(n)}(a)}{\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m}$, $G(\alpha;\xi)$ is defined in Lemma 4 and θ is defined in Theorem 1.

Proof. Using Lemma 1, the Hölder inequality and the (α,m)-logarithmic convexity of $\left|f^{(n)}\right|^q$ on $\left[a,\frac{b}{m}\right]$, we have

$$\begin{aligned}(2.19)\quad &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right.\\ &\left. - \sum_{k=1}^{n-1} \frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right)\right|\\ &\le \frac{(b-a)^n}{2^{n+1}n!}\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m\left(\int_0^1 (n-1+t)^{\frac{q}{q-1}}\,dt\right)^{1-\frac{1}{q}}\\ &\times\left\{\left(\int_0^1 (1-t)^{q(n-1)}\,\mu^{q\left(\frac{1-t}{2}\right)^{\alpha}}dt\right)^{1/q} + \left(\int_0^1 (1-t)^{q(n-1)}\,\mu^{q\left(\frac{1+t}{2}\right)^{\alpha}}dt\right)\right\}^{1/q}.\end{aligned}$$

The proof follows by using similar arguments as in proving Theorem 1, using Lemma 4 and Lemma 7. □

Corollary 6. *Under the assumptions of Theorem 2, if $n = 1$, we have the inequality*

$$\begin{aligned}(2.20)\quad &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ &\le \frac{(b-a)}{4}\left(\frac{q-1}{2q-1}\right)^{1/q}\left[f'\left(\frac{b}{m}\right)\right]^m \mu^{\theta}\left\{\left[G\left(1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}} + \left[G\left(1;\mu^{\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}\right\},\end{aligned}$$

where $\mu = \frac{f'(a)}{\left[f''\left(\frac{b}{m}\right)\right]^m}$,

$$G(1;\xi) = \sum_{k=1}^{\infty} \frac{(\ln\xi)^{k-1}}{k!} < \infty$$

and θ is defined in Theorem 1.

Corollary 7. *Under the assumptions of Theorem 2, if $n=2$, we have the inequality*

$$\begin{aligned}(2.21)\quad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\le \frac{(b-a)^2\left[2^{(2q-1)/(q-1)}-1\right]^{1-\frac{1}{q}}}{16}\left(\frac{q-1}{2q-1}\right)^{1/q}\left[f''\left(\frac{b}{m}\right)\right]^m\mu^\theta \\ &\qquad\times\left\{\left[G\left(q+1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}+\left[G\left(q+1;\mu^{\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}\right\},\end{aligned}$$

where $\mu=\frac{f''(a)}{\left[f''\left(\frac{b}{m}\right)\right]^m}$,

$$G(q+1;\xi)=\sum_{k=1}^{\infty}\frac{(\ln\xi)^{k-1}}{(q+1)_k}<\infty$$

and θ is defined in Theorem 1.

Theorem 3. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a function such that $f^{(n)}$ exists on I° and $f^{(n)}\in L([a,b])$ for $n\in\mathbb{N}$, where $a,\ b\in I^\circ$ with $a<b$. If $\left|f^{(n)}\right|^q$ is (α,m)-logarithmically convex on $\left[a,\frac{b}{m}\right]$ for $(\alpha,m)\in(0,1]\times(0,1]$ and $q\in(1,\infty)$, we have*

$$\begin{aligned}(2.22)\quad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right. \\ &\left.\qquad-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right| \\ &\le\frac{n^{n+1-\frac{1}{q}}(b-a)^n}{2^{n+1}n!}\left[B\left(\frac{1}{n};\frac{nq-1}{q-1},\frac{2q-1}{q-1}\right)\right]^{1-\frac{1}{q}} \\ &\qquad\times\mu^\theta\left\{\left[F_3\left(\mu^{-\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}+\left[F_3\left(\mu^{\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}\right\},\end{aligned}$$

where $\mu=\frac{f^{(n)}(a)}{\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m}$,

$$F_3(\xi)=\begin{cases}\frac{\xi-1}{\ln\xi}, & \xi\neq 1\\ 1, & \xi=1\end{cases},$$

$$B(z;\alpha,\beta)=\int_0^z z^{\alpha-1}(1-z)^{\beta-1}\,dt, 0\le z\le 1,\alpha>0,\beta>0$$

is the incomplete Beta function and θ is defined in Theorem 1.

Proof. Using Lemma 1, the Hölder inequality and the (α, m)-logarithmic convexity of $\left|f^{(n)}\right|^q$ on $\left[a, \frac{b}{m}\right]$, we have

$$\begin{aligned}(2.23)\quad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right.\\&\qquad\left.-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|\\&\le\frac{(b-a)^n}{2^{n+1}n!}\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m\left(\int_0^1(1-t)^{q(n-1)/(q-1)}(n-1+t)^{q/(q-1)}\,dt\right)^{1-1/q}\\&\qquad\times\left\{\left(\int_0^1\mu^{q\left(\frac{1-t}{2}\right)^\alpha}dt\right)^{1/q}+\left(\int_0^1\mu^{q\left(\frac{1+t}{2}\right)^\alpha}dt\right)^{1/q}\right\}.\end{aligned}$$

By using Lemma 7 and the fact that

$$\begin{aligned}&\int_0^1(1-t)^{q(n-1)/(q-1)}(n-1+t)^{q/(q-1)}\,dt\\&=n^{\frac{nq+q-1}{q-1}}\int_0^{\frac{1}{n}}t^{\frac{(n-1)q}{q-1}}(1-t)^{\frac{q}{q-1}}\,dt=n^{\frac{nq+q-1}{q-1}}B\left(\frac{1}{n};\frac{nq-1}{q-1},\frac{2q-1}{q-1}\right),\end{aligned}$$

we get the required inequality (2.22) from (2.23). □

Corollary 8. *Suppose the assumptions of Theorem 3 are satisfied and if $n = 1$, we have the inequality*

$$\begin{aligned}(2.24)\quad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\&\le\frac{(b-a)}{4}\left(\frac{2q-1}{q-1}\right)^{1-\frac{1}{q}}\mu^\theta\left\{\left[F_3\left(\mu^{-\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}+\left[F_3\left(\mu^{\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}\right\},\end{aligned}$$

where $\mu=\frac{f'(a)}{\left[f'\left(\frac{b}{m}\right)\right]^m}$,

$$F_3(\xi)=\begin{cases}\frac{\xi-1}{\ln\xi}, & \xi\ne1\\ 1, & \xi=1\end{cases}$$

and θ are defined as in Theorem 1.

Corollary 9. *Suppose the assumptions of Theorem 3 are satisfied and if $n = 2$, we have the inequality*

$$\begin{aligned}(2.25)\quad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\&\le\frac{(b-a)^2}{2^{1+\frac{1}{q}}}\left[B\left(\frac{1}{2};\frac{2q-1}{q-1},\frac{2q-1}{q-1}\right)\right]^{1-\frac{1}{q}}\mu^\theta\left\{\left[F_3\left(\mu^{-\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}+\left[F_3\left(\mu^{\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}\right\},\end{aligned}$$

where $\mu=\frac{f''(a)}{\left[f''\left(\frac{b}{m}\right)\right]^m}$,

$$F_3(\xi)=\begin{cases}\frac{\xi-1}{\ln\xi}, & \xi\ne1\\ 1, & \xi=1\end{cases},$$

$B(z;a,b)$ is the incomplete Beta function as defined in Theorem 3 and θ is defined as in Theorem 1.

Theorem 4. *Let $f : I \subset [0,\infty) \to (0,\infty)$ be a function such that $f^{(n)}$ exists on I° and $f^{(n)} \in L([a,b])$ for $n \in \mathbb{N}$, where $a, b \in I^{\circ}$ with $a < b$. If $\left|f^{(n)}\right|^{q}$ is (α,m)-logarithmically convex on $\left[a, \frac{b}{m}\right]$ for $(\alpha,m) \in (0,1] \times (0,1]$, $q \in (1,\infty)$ and $0 \le r \le (n-1)q$. Then*

$$\begin{aligned}(2.26)\quad &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right.\\ &\left. - \sum_{k=1}^{n-1} \frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right)\right|\\ &\le \frac{(b-a)^n}{2^{n+1}n!}\left[\frac{(q-1)\left(n^2q-nr-2n+r+1\right)}{(nq-r-1)(nq+q-r-2)}\right]^{1-\frac{1}{q}}\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m \mu^{\theta}\\ &\times\left\{\left[H\left(r+1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q} + \left[H\left(r+1;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}.\end{aligned}$$

$\mu = \frac{f^{(n)}(a)}{\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m}$, θ is defined in Theorem 1 and $H(\alpha;\xi)$ is defined in Lemma 6.

Proof. From Lemma 1, the Hölder inequality and using the fact that $\left|f^{(n)}\right|^{q}$ is (α,m)-logarithmically convex on $\left[a, \frac{b}{m}\right]$, we have

$$\begin{aligned}(2.27)\quad &\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right.\\ &\left. - \sum_{k=1}^{n-1} \frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right)\right|\\ &\le \frac{(b-a)^n}{2^{n+1}n!}\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m \left(\int_0^1 (1-t)^{(nq-q-r)/(q-1)}(n-1+t)\,dt\right)^{1-\frac{1}{q}}\\ &\times\left\{\left(\int_0^1 (1-t)^r (n-1+t)\,\mu^{q\left(\frac{1-t}{2}\right)^{\alpha}}dt\right)^{1/q}\right.\\ &\left. + \left(\int_0^1 (1-t)^r (n-1+t)\,\mu^{q\left(\frac{1+t}{2}\right)^{\alpha}}dt\right)^{1/q}\right\},\end{aligned}$$

The rest of the proof is similar to that of the proof of Theorem 2 by using Lemma 6 and Lemma 7. □

Corollary 10. *Suppose the assumptions of Theorem 4 are fulfilled and if $r = 0$, we have*

$$\begin{aligned}(2.28)\quad & \left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right.\\ &\left.-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|\\ &\le\frac{(b-a)^n}{2^{n+1}n!}\left[\frac{(q-1)\left(n^2q-2n+1\right)}{(nq-1)(nq+q-2)}\right]^{1-\frac{1}{q}}\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m\mu^\theta\\ &\times\left\{\left[H\left(1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q}+\left[H\left(1;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}.\end{aligned}$$

Corollary 11. *Suppose the assumptions of Theorem 4 are fulfilled and if $r = (n-1)q$, we have*

$$\begin{aligned}(2.29)\quad & \left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right.\\ &\left.-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|\\ &\le\frac{(b-a)^n}{2^{n+1}n!}\left[\frac{q-2n+2r+1}{2(q-1)}\right]^{1-\frac{1}{q}}\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m\mu^\theta\\ &\times\left\{\left[H\left((n-1)q+1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q}+\left[H\left((n-1)q+1;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}.\end{aligned}$$

Remark 1. *Several interesting inequalities for m-logarithmically convex functions can be obtained by setting $\alpha = 1$ in the results presented in this section. However, we leave the details to the interested reader.*

Remark 2. *We can get several interesting inequalities for logarithmically convex functions by setting $\alpha = 1$ and $m = 1$ in the results proved above. However, the details are left to the interested reader.*

References

[1] M. W. Alomari, M. Darus and U. S. Kirmaci, Some inequalities of Hadamard type inequalities for s-convex , Acts Math. Sci. Ser. B Engl. Ed. 31 (2011), no. 4, 1643-1652.

[2] A. O. Akdemir and M. Tunç, On some inegral inequalities for s-logarithmically convex and their applications, arXiv:1212.1584 [math.FA].

[3] S. S. Dragomir, On some new inequalities of Hermite-Hadamard type for m-convex functions, Tamkang J. Math. 33 (2002) 45–55.

[4] S. S. Dragomir and S. Fitzpatric, The Hadamard inequalities for s-convex functions in the second sense, Demonstratio Math. 32 (1999), no. 4 687-696.

[5] S. S. Dragomir and C. E. M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000.

[6] S. S. Dragomir, R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. Lett. 11 (1998) 91–95.

[7] J. Deng and J. R. Wang, Fractional Hermite-Hadamard inequalities for (α, m)-logarithmically convex functions, J Inequal Appl 2013, 2013:364.
[8] W.-D. Jiang, D.-W. Niu, Y. Hua, and F. Qi, Generalizations of Hermite-Hadamard inequality to n-time differentiable functions which are s-convex in the second sense, Analysis (Munich) 32 (2012), 1001–1012.
[9] R.- F. Bai, F. Qi and B. -Y. Xi, Hermite-Hadamard type inequalities for the m- and (α, m)-logarithmically convex functions, Filomat 27:1 (2013), 1-7.
[10] J. Hadamard, Étude sur les propriétés des fonctions entières et en particulier d'une fonction considerée par Riemann, J. Math Pures Appl., 58 (1893), 171–215.
[11] Ch. Hermite, Sur deux limites d'une intégrale définie, Mathesis 3 (1883), 82.
[12] D.-Y. Hwang, Some inequalities for n-time differentiable mappings and applications, Kyugpook Math. J. 43(2003), 335-343.
[13] U.S. Kırmacı, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp., 147 (2004), 137-146.
[14] U.S. Kırmacı and M.E. Özdemir, On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp., 153 (2004), 361-368.
[15] U.S. Kırmacı, Improvement and further generalization of inequalities for differentiable mappings and applications, Comp and Math. with Appl., 55 (2008), 485-493.
[16] H. Kavurmaci and M. Tunç, On some inequalities for s-logarithmically convex in the second sense via fractional inegrals, arXiv:1212.1604 [math.FA].
[17] M. A. Latif and S. S. Dragomir, New inequalities of Hermite-Hadamard type for functions whose derivatives in absolute value are convex with applications, Acta Univ. M. Belii, ser. Math. (2013), 24–39.
[18] M. A. Latif and S. S. Dragomir, On Hermite-Hadamard type integral inequalities for n-times differentiable (α, m)-logarithmically convex functions, RGMIA Research Report Collection, 17(2014), Article 14, 16 pp.
[19] M. A. Latif and S. S. Dragomir, New inequalities of Hermite-Hadamard type for n-times differentiable convex and concave functions with applications. (Accepted)
[20] M. A. Latif and S. S. Dragomir, On Hermite-Hadamard type integral inequalities for n-times differentiable log-preinvex functions. (Accepted)
[21] M. A. Latif and S. S. Dragomir,G eneralization of Hermite-Hadamard type inequalities for n-times differentiable functions which are s-preinvex in the second sense with applications. (Accepted)
[22] D. S. Mitrinović and I. B. Lacković, Hermite and convexity, Aequationes Math. 28 (1985), 229–232.
[23] C.E.M. Pearce and J. Pečarić, Inequalities for differentiable mappings with application to special means and quadrature formulae, Appl. Math. Lett., 13(2) (2000), 51–55.
[24] M. A. Noor, F. Qi and M. U. Awan, Some Hermite-Hadamard type inequalities for log-h-convex functions, Analysis 12/2013, 33(4):367–375. DOI:10.1524/anly.2013.1223
[25] M. Z. Sarikaya, A. Saglam and H. Yıldırım, On some Hadamard-type inequalities for h-convex functions, J Math Inequal, Vol. 2, No. 3 (2008), 335-341.
[26] M. Z. Sarikaya and N. Aktan, On the generalization some integral inequalities and their applications Mathematical and Computer Modelling, Volume 54, Issues 9-10, November 2011, Pages 2175-2182.
[27] M. Z. Sarikaya, E. Set and M. E. Ozdemir, On some new inequalities of Hadamard type involving h-convex functions, Acta Mathematica Universitatis Comenianae, Vol. LXXIX, 2(2010), pp. 265-272.
[28] A. Saglam, M. Z. Sarikaya and H. Yildirim, Some new inequalities of Hermite-Hadamard's type, Kyungpook Mathematical Journal, 50(2010), 399-410.
[29] M. Tunç and A. O. Akdemir, Ostrowski type inequalities for s-logarithmically convex functions in the second sense with applications, arXiv:1301.3041 [math.CA].
[30] J. Wang, J. Deng, M. Fečkan, Exploring s-e-condition and applications to some Ostrowski type inequalities via Hadamard fractional integrals. Math. Slovaca (2013, in press).
[31] S.-H. Wang, B.-Y. Xi, F. Qi, Some new inequalities of Hermite-Hadamard type for n-time differentiable functions which are m-convex, Analysis (Munich) 32 (2012) 247–262.
[32] B.-Y. Xi, R.-F. Bai and F. Qi, Hermite-Hadamard type inequalities for the m- and (α, m)-geometrically convex functions, Aequationes Math. 39 (2012), in press.

[33] B.-Y. Xi and F. Qi, Some integral inequalities of Hermite-Hadamard type for s-logarithmically convex functions. (Accepted)
[34] T.-Y. Zhang, A.-P. Ji, F. Qi, On integral inequalities of Hermite-Hadamard type for s-geometrically convex functions, Abstr. Appl. Anal., 2012 (2012).
[35] T.-Y. Zhang, M. Tunç, A.-P. Ji and B.-Y. Xi, Erratum to "On integral inequalities of Hermite-Hadamard type for s-geometrically convex functions", Abstr. Appl. Anal., Volume 2014, Article ID 294739, 5 pageshttp://dx.doi.org/10.1155/2014/294739.

School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa
E-mail address: `m_amer_latif@hotmail.com`

[1] School of Engineering and Science, Victoria University, P. O. Box 14428, Melbourne City, MC8001, Australia

[2]School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa
E-mail address: `sever.dragomir@vu.edu.au`

Center for Differential Equations, Continuum Mechanics and Applications, School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa
E-mail address: `ebrahim.momoniat@wits.ac.za`