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A Class of Inequalities Involving Generalized Ostrowski and Ostrowski-Grüss inequalities and Applications

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Abstract. A main class of inequalities including generalized Ostrowski and Ostrowski-Grüss inequalities is established in $L^1[a,b]$ and $L^\infty[a,b]$ spaces. As this class can be corresponded to a weighted approximation formula, new inequalities can be obtained by using its upper and lower bounds. Some illustrative examples such as three new generalizations of Ostrowski and Ostrowski-Grüss inequalities are given in this sense. Finally, by using the aforementioned approximation formula, a three point quadrature rule involving several well known quadratures is introduced and its error bounds are obtained.

Keywords. Ostrowski and Ostrowski-Grüss type inequalities, upper and lower bounds, Weighted approximations, Kernel function, Quadrature rules.

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1. Introduction

Let $L^p[a,b]$ $(1 \le p < \infty)$ denote the space of p-power integrable functions on the interval $[a,b]$ with the standard norm

$$\| f \|_p = \left(\int_a^b | f(t) |^p \, dt \right)^{1/p},$$

and $L^\infty[a,b]$ the space of all essentially bounded functions on $[a,b]$ with the norm

$$\| f \|_\infty = ess \sup_{x \in [a,b]} | f(x) | .$$

If $h \in L^1[a,b]$ and $g \in L^\infty[a,b]$, then [17]

$$\left| \int_a^b h(x) \, g(x) \, dx \right| \le \| h \|_1 \| g \|_\infty .$$

Grüss in [6] showed that if $h, g, hg \in L^1[a,b]$ then

$$\left| \frac{1}{b-a}\int_a^b h(x)\,g(x)\,dx - \frac{1}{(b-a)^2}\left(\int_a^b h(x)\,dx\right)\left(\int_a^b g(x)\,dx\right) \right| \le \frac{1}{4}(\Gamma_1-\gamma_1)(\Gamma_2-\gamma_2)\,, \qquad (1.1)$$

where $\gamma_1, \gamma_2, \Gamma_1$ and Γ_2 are all real numbers satisfying the conditions $\gamma_1 \le h(x) \le \Gamma_1$ and $\gamma_2 \le g(x) \le \Gamma_2$ for all $x \in [a,b]$. The constant $1/4$ in (1.1) is the best possible number in the sense that it cannot be replaced by a smaller quantity.
Another well-known inequality is due to Ostrowski [19]. If $f:[a,b] \to \mathbf{R}$ is a differentiable function with bounded derivative, then for all $x \in [a,b]$ we have

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{(x-a)^2+(b-x)^2}{2(b-a)}\,\| f' \|_\infty\,. \qquad (1.2)$$

In 1997 Dragomir and Wang [3] introduced a mixture type of the inequalities (1.1) and (1.2) and named it Ostrowski-Grüss inequality. They proved that if $f:[a,b] \to \mathbf{R}$ is a differentiable function with bounded derivative and $\alpha_0 \le f'(x) \le \beta_0$ for all $x \in [a,b]$, then

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) \right| \le \frac{1}{4}(b-a)(\beta_0-\alpha_0)\,. \qquad (1.3)$$

Due to their importance in different branches of applied mathematics, many improvements and generalizations of Ostrowski and Ostrowski-Grüss inequalities have been presented in the literature up to now, e.g. [1,2,4,5,7,8,15,16]. The following theorems show some of them.

Theorem A1 (a generalization of Ostrowski inequality) **[9].** *By defining the linear kernel*

$$K_w(x;t) = \begin{cases} t - \dfrac{(b-w)f(b)-af(a)}{f(b)-f(a)} = t-\theta_1 & t \in [a,x], \\ t - \dfrac{bf(b)-(a+w)f(a)}{f(b)-f(a)} = t-\theta_2 & t \in (x,b], \end{cases} \qquad (1.4)$$

in which $w \in \mathbf{R}$, $f(b) \ne f(a)$ *and* $\theta_2 - \theta_1 = w$, *let* $f \in C^1[a,b]$ *and suppose that* $f', \alpha, \beta \in L^p[a,b]$ *are functions such that* $\alpha(t)+\beta(t)$ *is a constant function and* $\alpha(t) \le f'(t) \le \beta(t)$ *for all* $t \in [a,b]$. *Under these assumptions if* $\frac{1}{p}+\frac{1}{q}=1$ *then the following inequality holds*

$$\left| w\, f(x) - \int_a^b f(x)\,dx \right| \le \begin{cases} \left(\int_{a-\theta_1}^{x-\theta_1} |z|^q\, dz + \int_{x-\theta_2}^{b-\theta_2} |z|^q\, dz \right)^{1/q} \| f' \|_p , \\ \left(\int_{a-\theta_1}^{x-\theta_1} |z|\, dz + \int_{x-\theta_2}^{b-\theta_2} |z|\, dz \right) \| f' \|_\infty , \\ \max\left(|a-\theta_1|, |b-\theta_1|, |a-\theta_2|, |b-\theta_2| \right) \| f' \|_1 . \end{cases}$$

Theorem A2 [9]. *Let* $f \in C^1[a,b]$. *If* α^* *is a constant number such that* $\alpha^* \le f'(t)$ *for any* $t \in [a,b]$ *and* θ_1, θ_2 *are defined as the same form as in (1.4), then*

$$\left| w\, f(x) - \int_a^b f(x)\,dx - \alpha^* \left(wx - \frac{\left((b-a)^2 + 2aw\right) f(b) + \left((b-a)^2 - 2bw\right) f(a)}{2\left(f(b) - f(a) \right)} \right) \right|$$
$$\le \max\left(|a-\theta_1|, |b-\theta_1|, |a-\theta_2|, |b-\theta_2| \right) \left(f(b) - f(a) - \alpha^*(b-a) \right).$$

Theorem A3 [9]. *Let* $f \in C^1[a,b]$. *If* β^* *is a constant number such that* $f'(t) \le \beta^*$ *for any* $t \in [a,b]$ *and* θ_1, θ_2 *are defined as the same form as in (1.4), then*

$$\left| w\, f(x) - \int_a^b f(x)dx - \beta^* \left(wx - \frac{\left((b-a)^2 + 2aw\right) f(b) + \left((b-a)^2 - 2bw\right) f(a)}{2\left(f(b) - f(a) \right)} \right) \right|$$
$$\le \max\left(|a-\theta_1|, |b-\theta_1|, |a-\theta_2|, |b-\theta_2| \right) \left(\beta^*(b-a) - f(b) + f(a) \right).$$

Theorem B1 (an analogue of Ostrowski inequality) **[11].**

Let $f \in C^1[a,b]$. *If* $\alpha(x) \le f'(x) \le \beta(x)$ *for any* $\alpha, \beta \in C[a,b]$ *and* $x \in [a,b]$ *then*

$$\frac{1}{b-a}\left(\int_a^x (t-a)\,\alpha(t)\,dt + \int_x^b (t-b)\,\beta(t)\,dt \right) \le f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt$$
$$\le \frac{1}{b-a}\left(\int_a^x (t-a)\,\beta(t)\,dt + \int_x^b (t-b)\,\alpha(t)\,dt \right).$$

Theorem B2 [11]. *Let* $f \in C^1[a,b]$. *If* $\alpha(x) \le f'(x)$ *for any* $\alpha \in C[a,b]$ *and* $x \in [a,b]$ *then*

$$\frac{1}{b-a}\left(\int_a^x (t-a)\,\alpha(t)\,dt + \int_x^b (t-b)\,\alpha(t)\,dt \right) + \int_a^b \alpha(t)\,dt - \left(f(b) - f(a) \right)$$
$$\le f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \le$$
$$\frac{1}{b-a}\left(\int_a^x (t-a)\,\alpha(t)\,dt + \int_x^b (t-b)\,\alpha(t)\,dt \right) + f(b) - f(a) - \int_a^b \alpha(t)\,dt.$$

Theorem B3 [11]. *Let* $f \in C^1[a,b]$. *If* $f'(x) \le \beta(x)$ *for any* $\beta \in C[a,b]$ *and* $x \in [a,b]$ *then*

$$\frac{1}{b-a}\left(\int_a^x (t-a)\,\beta(t)\,dt + \int_x^b (t-b)\,\beta(t)\,dt\right) - \int_a^b \beta(t)\,dt + f(b) - f(a)$$

$$\le f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \le$$

$$\frac{1}{b-a}\left(\int_a^x (t-a)\,\beta(t)\,dt + \int_x^b (t-b)\,\beta(t)\,dt\right) + \int_a^b \beta(t)\,dt - \left(f(b) - f(a)\right).$$

Theorem C1 (a generalization of Ostrowski-Grüss inequality) **[10].**

Let $f \in C^1[a,b]$. *If* $\alpha(x) \le f'(x) \le \beta(x)$ *for any* $\alpha, \beta \in C[a,b]$ *and* $x \in [a,b]$ *then*

$$\frac{1}{b-a}\left(\int_{-\frac{b-a}{2}}^{\frac{a+b}{2}-x}\left(\frac{z+|z|}{2}\alpha(z+x+\frac{b-a}{2}) + \frac{z-|z|}{2}\beta(z+x+\frac{b-a}{2})\right)dz\right.$$

$$\left. + \int_{\frac{a+b}{2}-x}^{\frac{b-a}{2}}\left(\frac{z+|z|}{2}\alpha(z+x-\frac{b-a}{2}) + \frac{z-|z|}{2}\beta(z+x-\frac{b-a}{2})\right)dz\right)$$

$$\le f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}(x-\frac{a+b}{2}) \le$$

$$\frac{1}{b-a}\left(\int_{-\frac{b-a}{2}}^{\frac{a+b}{2}-x}\left(\frac{z-|z|}{2}\alpha(z+x+\frac{b-a}{2}) + \frac{z+|z|}{2}\beta(z+x+\frac{b-a}{2})\right)dz\right.$$

$$\left. + \int_{\frac{a+b}{2}-x}^{\frac{b-a}{2}}\left(\frac{z-|z|}{2}\alpha(z+x-\frac{b-a}{2}) + \frac{z+|z|}{2}\beta(z+x-\frac{b-a}{2})\right)dz\right).$$

Theorem C2 [10]. *Let* $f \in C^1[a,b]$. *If* $\alpha(x) \le f'(x)$ *for any* $\alpha \in C[a,b]$ *and* $x \in [a,b]$ *then*

$$\frac{1}{b-a}\int_a^b (t-x)\,\alpha(t)\,dt + \int_a^x \alpha(t)\,dt - \frac{f(b)-f(a)}{2}$$

$$\le f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}(x-\frac{a+b}{2}) \le$$

$$\frac{1}{b-a}\int_a^b (t-x)\,\alpha(t)\,dt - \int_x^b \alpha(t)\,dt + \frac{f(b)-f(a)}{2}.$$

Theorem C3 [10]. *Let* $f \in C^1[a,b]$. *If* $f'(x) \le \beta(x)$ *for any* $\beta \in C[a,b]$ *and* $x \in [a,b]$ *then*

$$\frac{1}{b-a}\int_a^b (t-x)\,\beta(t)\,dt-\int_x^b \beta(t)\,dt+\frac{f(b)-f(a)}{2}$$

$$\leq\ f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(b)-f(a)}{b-a}(x-\frac{a+b}{2})\ \leq$$

$$\frac{1}{b-a}\int_a^b (t-x)\,\beta(t)\,dt+\int_a^x \beta(t)\,dt-\frac{f(b)-f(a)}{2}.$$

Theorem D (a generalization of Ostrowski-Grüss inequality) **[12].**

By defining the linear kernel

$$K(x;t,\lambda)=\begin{cases} t-\lambda(x-b)-\dfrac{1}{2}(a+b)=t-r_1 & t\in[a,x], \\ t-\lambda(x-a)-\dfrac{1}{2}(a+b)=t-r_2 & t\in(x,b], \end{cases}$$

let $f\in C^1[a,b]$. *If* α_0,β_0 *are two real constants such that* $\alpha_0\leq f'(t)\leq\beta_0$ *for all* $t\in[a,b]$, *then for any* $\lambda\in[1/2,1]$ *and all* $x\in[\frac{a+(2\lambda-1)b}{2\lambda},\frac{b+(2\lambda-1)\,a}{2\lambda}]\subseteq[a,b]$ *we have*

$$\left|\ f(x)-\frac{1}{\lambda(b-a)}\int_a^b f(t)\,dt-\frac{f(b)-f(a)}{b-a}x+\frac{(2\lambda-1)\,a+b}{2\lambda(b-a)}\,f(b)-\frac{a+(2\lambda-1)\,b}{2\lambda(b-a)}\,f(a)\right|$$

$$\leq\frac{\beta_0-\alpha_0}{4(b-a)}\frac{\lambda^2+(1-\lambda)^2}{\lambda}\left((x-a)^2+(b-x)^2\right).$$

In other words, the linear approximation

$$f(x)\cong\frac{f(b)-f(a)}{b-a}x+\frac{1}{\lambda(b-a)}\left(\int_a^b f(t)\,dt+\frac{a+(2\lambda-1)\,b}{2}\,f(a)-\frac{(2\lambda-1)\,a+b}{2}\,f(b)\right),$$

has an absolute error less than

$$\frac{\beta_0-\alpha_0}{4(b-a)}\frac{\lambda^2+(1-\lambda)^2}{\lambda}\left((x-a)^2+(b-x)^2\right).$$

2. A class of inequalities for integrable functions

After reviewing the above-mentioned theorems, we can now introduce a main class of inequalities that includes the results of them together with some new improvements of Ostrowski and Ostrowski-Grüss inequalities. For this purpose, we first define the kernel function

$$\mathbf{K}(x;t,u,v;[a,b])=\begin{cases} u(t) & t\in[a,x], \\ v(t) & t\in(x,b], \end{cases} \tag{2.1}$$

on $[a,b]$ where $u(t)$ and $v(t)$ are two arbitrary integrable functions such that $u(t)\in C^1[a,x]$ and $v(t)\in C^1(x,b]$. It can be directly verified from the kernel (2.1) that

$$\begin{aligned} \mathbf{F}\left(x;f,u,v;[a,b]\right)&=\int_a^b f'(t)\,\mathbf{K}(x;t,u,v;[a,b])\,dt \\ &=\left(u(x)-v(x)\right)f(x)-\int_a^x u'(t)\,f(t)\,dt-\int_x^b v'(t)\,f(t)\,dt+v(b)\,f(b)-u(a)\,f(a). \end{aligned} \tag{2.2}$$

Also if in (2.1) $\int_a^x u(t)\,dt+\int_x^b v(t)\,dt=w^*(x)$ is assumed, then a specific subclass of integrable functions [13] is derived in the form

$$\begin{aligned} F_2\left(x;f,u,v;[a,b]\right)&=\mathbf{F}\left(x;f,u(t)-\frac{w^*(x)}{b-a},v(t)-\frac{w^*(x)}{b-a};[a,b]\right) \\ &=\mathbf{F}\left(x;f,u,v;[a,b]\right)-\frac{f(b)-f(a)}{b-a}\,w^*(x), \end{aligned} \tag{2.3}$$

which is corresponding to the kernel function

$$K_2(x;t,u,v;[a,b])=\mathbf{K}\left(x;t,u(t)-\frac{w^*(x)}{b-a},v(t)-\frac{w^*(x)}{b-a};[a,b]\right)=\begin{cases} u(t)-\dfrac{w^*(x)}{b-a} & t\in[a,x], \\ v(t)-\dfrac{w^*(x)}{b-a} & t\in(x,b]. \end{cases} \tag{2.4}$$

Note that $F_2\left(x;f,u,v;[a,b]\right)$ in (2.3) can be directly generated by replacing $h(t)=f'(t)$ and $g(t)=\mathbf{K}(x;t,u,v;[a,b])$ in the left hand side of Grüss inequality so that we have

$$\begin{aligned} F_2\left(x;f,u,v;[a,b]\right)&=\int_a^b f'(t)\,\mathbf{K}(x;t,u,v;[a,b])\,dt-\frac{1}{b-a}\int_a^b f'(t)\,dt\int_a^b \mathbf{K}(x;t,u,v;[a,b])\,dt \\ &=\mathbf{F}\left(x;f,u,v;[a,b]\right)-\frac{f(b)-f(a)}{b-a}\left(\int_a^x u(t)\,dt+\int_x^b v(t)\,dt\right). \end{aligned}$$

Recently in [13] we have applied $F_2\left(x;f,u,v;[a,b]\right)$ to study a certain class of weighted approximations. However, relation (2.3) which is indeed equivalent to

$$F_2\left(x;f,u(t)+\frac{w^*(x)}{b-a},v(t)+\frac{w^*(x)}{b-a};[a,b]\right)=\mathbf{F}\left(x;f,u,v;[a,b]\right),$$

shows that the original function is $\mathbf{F}\left(x; f, u, v; [a,b]\right)$. It is clear that this function can be considered as the error value of a weighted approximation formula in the form

$$\left(u(x)-v(x)\right) f(x) \cong \int_a^x u'(t) f(t)\,dt + \int_x^b v'(t) f(t)\,dt + u(a) f(a) - v(b) f(b). \qquad (2.5)$$

In this section, we present three main theorems (different from theorems of [13]) for error bounds of the approximation formula (2.5) in $L^1[a,b]$ and $L^\infty[a,b]$ spaces.

Theorem 1. *Let* $f \in C^1[a,b]$ *and* $u(t)$, $v(t)$ *be two arbitrary integrable functions such that* $u \in C^1(a,x)$ *and* $v \in C^1(x,b)$. *If* $\alpha(x) \le f'(x) \le \beta(x)$ *for any* $x \in [a,b]$ *and* $\alpha, \beta \in C[a,b]$, *then*

$$m_1(x) \le \mathbf{F}\left(x; f, u, v; [a,b]\right) \le M_1(x), \qquad (2.6)$$

where

$$\begin{aligned} m_1(x) = &\int_a^x \left(\frac{u(t)-|u(t)|}{2}\beta(t) + \frac{u(t)+|u(t)|}{2}\alpha(t) \right) dt \\ &+ \int_x^b \left(\frac{v(t)-|v(t)|}{2}\beta(t) + \frac{v(t)+|v(t)|}{2}\alpha(t) \right) dt, \end{aligned} \qquad (2.7)$$

and

$$\begin{aligned} M_1(x) = &\int_a^x \left(\frac{u(t)+|u(t)|}{2}\beta(t) + \frac{u(t)-|u(t)|}{2}\alpha(t) \right) dt \\ &+ \int_x^b \left(\frac{v(t)+|v(t)|}{2}\beta(t) + \frac{v(t)-|v(t)|}{2}\alpha(t) \right) dt. \end{aligned} \qquad (2.8)$$

Proof. By noting the relations (2.1) and (2.2) first we have

$$\begin{aligned} &\mathbf{F}\left(x; f, u, v; [a,b]\right) - \frac{1}{2}\left(\int_a^x u(t)(\alpha(t)+\beta(t))\,dt + \int_x^b v(t)(\alpha(t)+\beta(t))\,dt \right) \\ &= \int_a^b f'(t)\,\mathbf{K}\left(x; t, u, v; [a,b]\right) dt - \frac{1}{2}\int_a^b \mathbf{K}\left(x; t, u, v; [a,b]\right)\left(\alpha(t)+\beta(t)\right) dt \\ &= \int_a^b \mathbf{K}\left(x; t, u, v; [a,b]\right)\left(f'(t) - \frac{\alpha(t)+\beta(t)}{2} \right) dt. \end{aligned}$$

On the other hand, the given assumption $\alpha(t) \le f'(t) \le \beta(t)$ yields

$$\left| f'(t) - \frac{\alpha(t)+\beta(t)}{2} \right| \le \frac{\beta(t)-\alpha(t)}{2}.$$

Therefore

$$\left| \mathbf{F}\left(x; f,u,v;[a,b]\right) - \frac{1}{2}\left(\int_a^x u(t)\,(\alpha(t)+\beta(t))\,dt + \int_x^b v(t)\,(\alpha(t)+\beta(t))\,dt \right) \right|$$

$$= \left| \int_a^b \mathbf{K}\left(x;t,u,v;[a,b]\right)\left(f'(t) - \frac{\alpha(t)+\beta(t)}{2} \right) dt \right| \le \int_a^b \left| \mathbf{K}\left(x;t,u,v;[a,b]\right) \right| \left(\frac{\beta(t)-\alpha(t)}{2} \right) dt \quad (2.9)$$

$$= \frac{1}{2}\left(\int_a^x \left| u(t) \right| \left(\beta(t)-\alpha(t)\right) dt + \int_x^b \left| v(t) \right| \left(\beta(t)-\alpha(t)\right) dt \right).$$

After re-arranging (2.9), the main inequality (2.6) will be derived directly. ■

Remark 1. Although the condition $\alpha(x) \le f'(x) \le \beta(x)$ is straightforward in theorem 1, sometimes one might not be able to easily obtain both bounds of $\alpha(x)$ and $\beta(x)$ for f'. In this case, two analogue theorems can be considered. The first one would be helpful when f' is unbounded from above and the second one would be helpful when f' is unbounded from below.

Theorem 2. *Let* $f \in C^1[a,b]$. *If* $\alpha(x) \le f'(x)$ *for all* $x \in [a,b]$ *and* $\alpha \in C[a,b]$ *then*

$$m_2(x) \le \mathbf{F}\left(x; f,u,v;[a,b]\right) \le M_2(x)\,, \quad (2.10)$$

where

$$m_2(x) = \int_a^x u(t)\alpha(t)\,dt + \int_x^b v(t)\alpha(t)\,dt - \max_{x\in[a,b]}\left\{ \max_{t\in[a,x]}\left|u(t)\right|, \max_{t\in(x,b]}\left|v(t)\right| \right\} \left(f(b) - f(a) - \int_a^b \alpha(t)\,dt \right), \quad (2.11)$$

and

$$M_2(x) = \int_a^x u(t)\alpha(t)\,dt + \int_x^b v(t)\alpha(t)\,dt + \max_{x\in[a,b]}\left\{ \max_{t\in[a,x]}\left|u(t)\right|, \max_{t\in(x,b]}\left|v(t)\right| \right\} \left(f(b) - f(a) - \int_a^b \alpha(t)\,dt \right). \quad (2.12)$$

Proof. We have

$$\left| \mathbf{F}\left(x; f,u,v;[a,b]\right) - \int_a^x u(t)\alpha(t)\,dt - \int_x^b v(t)\alpha(t)\,dt \right| = \left| \int_a^b \mathbf{K}\left(x;t,u,v;[a,b]\right)\left(f'(t) - \alpha(t) \right) dt \right|$$

$$\le \int_a^b \left| \mathbf{K}\left(x;t,u,v;[a,b]\right) \right| \left(f'(t) - \alpha(t) \right) dt \le \max_{t\in[a,b]} \left| \mathbf{K}\left(x;t,u,v;[a,b]\right) \right| \int_a^b \left(f'(t) - \alpha(t) \right) dt$$

$$= \max_{x\in[a,b]}\left\{ \max_{t\in[a,x]}\left| u(t) \right|, \max_{t\in(x,b]}\left| v(t) \right| \right\} \left(f(b) - f(a) - \int_a^b \alpha(t)\,dt \right).$$

After re-arranging the above relation, inequality (2.10) will be derived directly. ■

Theorem 3. *Let* $f \in C^1[a,b]$. *If* $f'(x) \le \beta(x)$ *for all* $x \in [a,b]$ *and* $\beta \in C[a,b]$ *then*

$$m_3(x) \le \mathbf{F}\left(x; f,u,v;[a,b]\right) \le M_3(x)\,, \quad (2.13)$$

where

$$m_3(x)=\int_a^x u(t)\beta(t)\,dt+\int_x^b v(t)\beta(t)\,dt-\max_{x\in[a,b]}\left\{\max_{t\in[a,x]}|u(t)|,\max_{t\in(x,b]}|v(t)|\right\}\left(\int_a^b \beta(t)\,dt-f(b)+f(a)\right), \tag{2.14}$$

and

$$M_3(x)=\int_a^x u(t)\beta(t)\,dt+\int_x^b v(t)\beta(t)\,dt+\max_{x\in[a,b]}\left\{\max_{t\in[a,x]}|u(t)|,\max_{t\in(x,b]}|v(t)|\right\}\left(\int_a^b \beta(t)\,dt-f(b)+f(a)\right). \tag{2.15}$$

Proof. We have

$$\begin{aligned}&\left|\mathbf{F}\left(x;f,u,v;[a,b]\right)-\int_a^x u(t)\beta(t)\,dt-\int_x^b v(t)\beta(t)\,dt\right|=\left|\int_a^b \mathbf{K}\left(x;t,u,v;[a,b]\right)\left(f'(t)-\beta(t)\right)dt\right|\\ &\le\int_a^b\left|\mathbf{K}\left(x;t,u,v;[a,b]\right)\right|\left(\beta(t)-f'(t)\right)dt\le\max_{t\in[a,b]}\left|\mathbf{K}\left(x;t,u,v;[a,b]\right)\right|\int_a^b\left(\beta(t)-f'(t)\right)dt\\ &=\max_{x\in[a,b]}\left\{\max_{t\in[a,x]}|u(t)|,\max_{t\in(x,b]}|v(t)|\right\}\left(\int_a^b \beta(t)\,dt-f(b)+f(a)\right).\end{aligned}$$

After re-arranging the above relation, inequality (2.13) will be derived directly. ■

Remark 2. An important advantage in theorems 1, 2 and 3 is that necessary computations for obtaining the bounds $\{m_i(x)\}_{i=1}^3$ and $\{M_i(x)\}_{i=1}^3$ are only in terms of $\alpha(t),\beta(t)$ and the pre-assigned functions $u(t),v(t)$, not in terms of f and its derivatives. In other words, as far as we searched, most of presented works for obtaining such bounds contain a variety of norms (like $\|f'\|_1$, $\|f'\|_2$ and $\|f'\|_\infty$), see e.g. [1,2,3,4,7,8,9], which are rather difficult to calculate, while the computed bounds $\{m_i(x)\}_{i=1}^3$ and $\{M_i(x)\}_{i=1}^3$ are all independent of these values.

3. Illustrative Examples

With respect to pre-assigned functions $u(t)$ and $v(t)$, it is clear that there exist various cases for $\mathbf{F}\left(x;f,u,v;[a,b]\right)$ and its corresponding bounds. In this section, we consider some remarkable examples.

Example 1. (three new generalizations of Ostrowski inequality)

By choosing $u(t)=t-\dfrac{(b-w)f(b)-af(a)}{f(b)-f(a)}=t-\theta_1$ and $v(t)=t-\dfrac{bf(b)-(a+w)f(a)}{f(b)-f(a)}=t-\theta_2$, the function (2.2) changes to

$$\mathbf{F}\left(x;f,t-\theta_1,t-\theta_2;[a,b]\right)=w\,f(x)-\int_a^b f(t)\,dt. \tag{3.1}$$

Hence, substituting (3.1) in theorem 1 yields

$$\int_{a-\theta_1}^{x-\theta_1}\left(\frac{z+|z|}{2}\alpha(\theta_1+z)+\frac{z-|z|}{2}\beta(\theta_1+z)\right)dz+\int_{x-\theta_2}^{b-\theta_2}\left(\frac{z+|z|}{2}\alpha(\theta_2+z)+\frac{z-|z|}{2}\beta(\theta_2+z)\right)dz$$
$$\le w\,f(x)-\int_a^b f(t)\,dt\le$$
$$\int_{a-\theta_1}^{x-\theta_1}\left(\frac{z+|z|}{2}\beta(\theta_1+z)+\frac{z-|z|}{2}\alpha(\theta_1+z)\right)dz+\int_{x-\theta_2}^{b-\theta_2}\left(\frac{z+|z|}{2}\beta(\theta_2+z)+\frac{z-|z|}{2}\alpha(\theta_2+z)\right)dz,$$

in which $\alpha(x)\le f'(x)\le\beta(x)$ for $\alpha,\beta\in C[a,b]$ and $x\in[a,b]$. Similarly, corresponding to theorem 2 we have

$$\int_{a-\theta_1}^{x-\theta_1} z\,\alpha(\theta_1+z)\,dz+\int_{x-\theta_2}^{b-\theta_2} z\,\alpha(\theta_2+z)\,dz-\max\left\{|a-\theta_1|,|a-\theta_2|,|b-\theta_1|,|b-\theta_2|\right\}\left(f(b)-f(a)-\int_a^b\alpha(t)\,dt\right)$$
$$\le w\,f(x)-\int_a^b f(t)\,dt\le$$
$$\int_{a-\theta_1}^{x-\theta_1} z\,\alpha(\theta_1+z)\,dz+\int_{x-\theta_2}^{b-\theta_2} z\,\alpha(\theta_2+z)\,dz+\max\left\{|a-\theta_1|,|a-\theta_2|,|b-\theta_1|,|b-\theta_2|\right\}\left(f(b)-f(a)-\int_a^b\alpha(t)\,dt\right),$$

provided that $\alpha(x)\le f'(x)$ for $\alpha\in C[a,b]$ and $x\in[a,b]$, and corresponding to theorem 3 we have

$$\int_{a-\theta_1}^{x-\theta_1} z\,\beta(\theta_1+z)\,dz+\int_{x-\theta_2}^{b-\theta_2} z\,\beta(\theta_2+z)\,dz-\max\left\{|a-\theta_1|,|a-\theta_2|,|b-\theta_1|,|b-\theta_2|\right\}\left(\int_a^b\beta(t)\,dt-f(b)+f(a)\right)$$
$$\le w\,f(x)-\int_a^b f(t)\,dt\le$$
$$\int_{a-\theta_1}^{x-\theta_1} z\,\beta(\theta_1+z)\,dz+\int_{x-\theta_2}^{b-\theta_2} z\,\beta(\theta_2+z)\,dz+\max\left\{|a-\theta_1|,|a-\theta_2|,|b-\theta_1|,|b-\theta_2|\right\}\left(\int_a^b\beta(t)\,dt-f(b)+f(a)\right),$$

if $f'(x)\le\beta(x)$ for any $\alpha\in C[a,b]$ and $x\in[a,b]$. In this sense, note that if in (3.1) $w=b-a$, then

$$\mathbf{F}\left(x;f,t-a,t-b;[a,b]\right)=(b-a)\,f(x)-\int_a^b f(t)\,dt. \tag{3.2}$$

Hence, substituting (3.2) in theorems 1, 2 and 3 respectively gives theorems B1, B2 and B3. For instance, for $\alpha(x)=\alpha_0\neq0$ and $\beta(x)=\beta_0\neq0$ theorem B1 reads as

$$\frac{\alpha_0(x-a)^2-\beta_0(b-x)^2}{2(b-a)}\le f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\le\frac{\beta_0(x-a)^2-\alpha_0(b-x)^2}{2(b-a)}.$$

Example 2. (three new generalizations of Ostrowski-Grüss inequality)

By noting the assumptions of theorem D, if we choose $u(t)=t-\lambda(x-b)-\frac{1}{2}(a+b)=t-r_1$ and $v(t)=t-\lambda(x-a)-\frac{1}{2}(a+b)=t-r_2$, then the function (2.2) changes to

$$\mathbf{F}\left(x; f, t-r_1, t-r_2; [a,b]\right) =$$
$$\lambda (b-a) f(x) - \int_a^b f(t)\,dt - \lambda (f(b) - f(a))\,x + \frac{(2\lambda - 1)\,a + b}{2} f(b) - \frac{a + (2\lambda -1)\,b}{2} f(a). \tag{3.3}$$

Hence, substituting (3.3) in theorem 1 yields

$$\int_{a-r_1}^{x-r_1} \left(\frac{z+|z|}{2}\alpha(r_1+z) + \frac{z-|z|}{2}\beta(r_1+z)\right) dz + \int_{x-r_2}^{b-r_2} \left(\frac{z+|z|}{2}\alpha(r_2+z) + \frac{z-|z|}{2}\beta(r_2+z)\right) dz$$
$$\leq \lambda (b-a) f(x) - \int_a^b f(t)\,dt - \lambda (f(b) - f(a))\,x + \frac{(2\lambda - 1)\,a + b}{2} f(b) - \frac{a + (2\lambda -1)\,b}{2} f(a) \leq$$
$$\int_{a-r_1}^{x-r_1} \left(\frac{z+|z|}{2}\beta(r_1+z) + \frac{z-|z|}{2}\alpha(r_1+z)\right) dz + \int_{x-r_2}^{b-r_2} \left(\frac{z+|z|}{2}\beta(r_2+z) + \frac{z-|z|}{2}\alpha(r_2+z)\right) dz,$$

in which $\alpha(x) \leq f'(x) \leq \beta(x)$ for $\alpha, \beta \in C[a,b]$ and $x \in [a,b]$. Similarly, corresponding to theorem 2 we have

$$\int_{a-r_1}^{x-r_1} z\,\alpha(r_1+z)\,dz + \int_{x-r_2}^{b-r_2} z\,\alpha(r_2+z)\,dz - \max\left\{|a-r_1|, |a-r_2|, |b-r_1|, |b-r_2|\right\}\left(f(b) - f(a) - \int_a^b \alpha(t)\,dt\right)$$
$$\leq \lambda (b-a) f(x) - \int_a^b f(t)\,dt - \lambda (f(b) - f(a))\,x + \frac{(2\lambda - 1)\,a + b}{2} f(b) - \frac{a + (2\lambda -1)\,b}{2} f(a) \leq$$
$$\int_{a-r_1}^{x-r_1} z\,\alpha(r_1+z)\,dz + \int_{x-r_2}^{b-r_2} z\,\alpha(r_2+z)\,dz + \max\left\{|a-r_1|, |a-r_2|, |b-r_1|, |b-r_2|\right\}\left(f(b) - f(a) - \int_a^b \alpha(t)\,dt\right),$$

provided that $\alpha(x) \leq f'(x)$ for $\alpha \in C[a,b]$ and $x \in [a,b]$, and corresponding to theorem 3 we have

$$\int_{a-r_1}^{x-r_1} z\,\beta(r_1+z)\,dz + \int_{x-r_2}^{b-r_2} z\,\beta(r_2+z)\,dz - \max\left\{|a-r_1|, |a-r_2|, |b-r_1|, |b-r_2|\right\}\left(\int_a^b \beta(t)\,dt - f(b) + f(a)\right)$$
$$\leq \lambda (b-a) f(x) - \int_a^b f(t)\,dt - \lambda (f(b) - f(a))\,x + \frac{(2\lambda - 1)\,a + b}{2} f(b) - \frac{a + (2\lambda -1)\,b}{2} f(a) \leq$$
$$\int_{a-r_1}^{x-r_1} z\,\beta(r_1+z)\,dz + \int_{x-r_2}^{b-r_2} z\,\beta(r_2+z)\,dz + \max\left\{|a-r_1|, |a-r_2|, |b-r_1|, |b-r_2|\right\}\left(\int_a^b \beta(t)\,dt - f(b) + f(a)\right),$$

if $f'(x) \leq \beta(x)$ for any $\alpha \in C[a,b]$ and $x \in [a,b]$. In this sense, note that if in (3.3) $\lambda = 1$, then

$$\mathbf{F}\left(x; f, t - x + \frac{b-a}{2}, t - x - \frac{b-a}{2}; [a,b]\right) = (b-a) f(x) - \int_a^b f(t)\,dt - (f(b) - f(a))(x - \frac{a+b}{2}). \tag{3.4}$$

By substituting (3.4) in theorems 1, 2 and 3 one respectively gets theorems C1, C2 and C3.

Example 3.(An arbitrary particular case) Let $\alpha(x) = \alpha_0 \neq 0$ and $\beta(x) = \beta_0 \neq 0$. Since e.g.

$$\mathbf{F}\left(x; f, t+1, 1; [0,1]\right) = x\, f(x) - \int_0^x f(t)\, dt + f(1) - f(0),$$

applying theorem 1 yields

$$\alpha_0 (\frac{1}{2} x^2 + 1) \le x\, f(x) - \int_0^x f(t)\, dt + f(1) - f(0) \le \beta_0 (\frac{1}{2} x^2 + 1),$$

provided that $\alpha_0 \le f'(x) \le \beta_0$ for $x \in [0,1]$ and applying theorem 2 gives

$$\alpha_0 (\frac{1}{2} x^2 + 3) - 3\left(f(1) - f(0)\right) \le x\, f(x) - \int_0^x f(t)\, dt \le \alpha_0 (\frac{1}{2} x^2 - 1) + f(1) - f(0),$$

if $\alpha_0 \le f'(x) \ \ \forall x \in [0,1]$.

Remark 3. One of the advantages of theorems 1, 2 and 3 is to find the bound for some incomplete special functions. For example, by choosing $v(t) = \lambda > 0$, $u(t) = \lambda + t^r / r$ for $r > 0$, $f(t) = (1-t)^{s-1}$ for $s > 1$ and $[a,b] = [0,1]$ one gets

$$\mathbf{F}\left(x; (1-x)^{s-1}, \lambda + x^r / r, \lambda\right) = \frac{1}{r} x^r (1-x)^{s-1} - \mathbf{B}(x; r, s) - \lambda,$$

where

$$\mathbf{B}(x; r, s) = \int_0^x t^{r-1} (1-t)^{s-1} dt \qquad (r, s > 0\ ;\ x \in (0,1]),$$

is the incomplete beta function. And/or if $v(t) = \lambda > 0$, $u(t) = \lambda + t^r / r$ for $r > 0$, $f(t) = e^{-t}$ and $[a,b] = [0,b]$ then

$$\mathbf{F}\left(x; e^{-x}, \lambda + x^r / r, \lambda\right) = \frac{1}{r} x^r e^{-x} - \mathbf{\Gamma}(x; r) + \lambda (e^{-b} - 1),$$

in which

$$\mathbf{\Gamma}(x; r) = \int_0^x t^{r-1} e^{-t} dt \qquad (r, x > 0),$$

denotes the incomplete gamma function.
The mentioned remark also holds for finding upper and lower bounds of some classical integral transforms when $u(x) - v(x) = \lambda > 0$. For example, if $v(t) = e^{-st}$ for $s > 0$, $u(t) = \lambda + e^{-st}$ ($\lambda > 0$) and $[a,b] = [0,\infty)$, then

$$\mathbf{F}\left(x; f, \lambda + e^{-st}, e^{-st}; [0,\infty)\right) = \lambda\, f(x) + s\, \mathbf{L}(f(x)) - (1+\lambda)\, f(0),$$

where

$$\mathbf{L}(f(x)) = \int_0^\infty e^{-sx} f(x)\, dx \quad (s > 0),$$

shows the Laplace integral transform.

4. Applications in numerical integration

A general ($n+1$)-point weighted quadrature formula is denoted by

$$\int_a^b w(x)\, f(x)\, dx = \sum_{k=0}^{n} w_k\ f(x_k) + R_{n+1}[f], \tag{4.1}$$

where $w(x)$ is a positive function on $[a,b]$, $\{x_k\}_{k=0}^{n}$ and $\{w_k\}_{k=0}^{n}$ are respectively nodes and weight coefficients and $R_{n+1}[f]$ is the corresponding error [12].

Let $\mathbf{\Pi}_d$ be the set of algebraic polynomials of degree at most *d*. The quadrature formula (4.1) has degree of exactness *d* if for every $p \in \mathbf{\Pi}_d$ we have $R_{n+1}[p]=0$. In addition, if $R_{n+1}[p] \neq 0$ for some $\mathbf{\Pi}_{d+1}$, formula (4.1) has precise degree of exactness *d*.

The convergence order of quadrature rule (4.1) depends on the smoothness of the function f as well as on its degree of exactness. It is well known that for given $n+1$ mutually different nodes $\{x_k\}_{k=0}^{n}$ we can always achieve a degree of exactness $d=n$ by interpolating at these nodes and integrating the interpolated polynomial instead of f. Namely, taking the node polynomial

$$\Psi_{n+1}(x) = \prod_{k=0}^{n} (x - x_k)\,,$$

by integrating the Lagrange interpolation formula

$$f(x) = \sum_{k=0}^{n} f(x_k)\, L(x; x_k) + r_{n+1}(f; x)\,,$$

where

$$L(x; x_k) = \frac{\Psi_{n+1}(x)}{\Psi'_{n+1}(x_k)(x - x_k)} \qquad (k = 0,1,\ldots,n)\,,$$

we obtain (4.1), with

$$w_k = \frac{1}{\Psi'_{n+1}(x_k)} \int_a^b \frac{\Psi_{n+1}(x)\, w(x)}{x - x_k}\, dx \qquad (k = 0,1,\ldots,n)\,,$$

and

$$R_{n+1}[f] = \int_a^b r_{n+1}(f; x)\, w(x)\, dx\,.$$

It is clear that if $f \in \mathbf{\Pi}_n$ then $r_{n+1}(f;x)=0$ and therefore $R_{n+1}[f]=0$.

Quadrature formulae obtained in this way are known as interpolatory. If a quadrature is not of the interpolatory type, i.e. if it does not follow the concept of the degree of exactness, then it would be a nonstandard quadrature rule.

Usually the simplest interpolatory quadrature formula of type (4.1) with pre-determined nodes $\{x_k\}_{k=0}^n \in [a,b]$ is called a weighted Newton-Cotes formula. For $w(x)=1$ and the equidistant nodes $\{x_k\}_{k=0}^n = \{a+kh\}_{k=0}^n$ with $h=(b-a)/n$, the classical Newton-Cotes formulas are derived.

In this section, we introduce a general three point quadrature formula of the form

$$\int_a^b f(x)\,dx \cong \lambda_1 f(a) + (b-a-(\lambda_1+\lambda_2))\, f(c) + \lambda_2 f(b), \qquad (4.2)$$

where λ_1, λ_2 are two free parameters and $c \in [a,b]$. It is clear that the error value of the rule (4.2) can be written as

$$\mathbf{F}\left(c; f, t-a-\lambda_1, t-b+\lambda_2; [a,b]\right) = \lambda_1 f(a) + (b-a-(\lambda_1+\lambda_2))\, f(c) + \lambda_2 f(b) - \int_a^b f(t)\,dt. \quad (4.3)$$

According to theorems 1, 2 and 3, relation (4.3) directly gives the following corollaries.

Corollary 1. *Let* $f \in C^1[a,b]$. *If* $\alpha(x) \le f'(x) \le \beta(x)$ *for any* $x \in [a,b]$ *and* $\alpha, \beta \in C[a,b]$, *then*

$$\int_{-\lambda_1}^{c-a-\lambda_1} \left(\frac{z-|z|}{2}\beta(z+a+\lambda_1) + \frac{z+|z|}{2}\alpha(z+a+\lambda_1) \right) dz + \int_{c-b+\lambda_2}^{\lambda_2} \left(\frac{z-|z|}{2}\beta(z+b-\lambda_2) + \frac{z+|z|}{2}\alpha(z+b-\lambda_2) \right) dz$$

$$\le \lambda_1 f(a) + (b-a-(\lambda_1+\lambda_2))\, f(c) + \lambda_2 f(b) - \int_a^b f(t)\,dt \le$$

$$\int_{-\lambda_1}^{c-a-\lambda_1} \left(\frac{z+|z|}{2}\beta(z+a+\lambda_1) + \frac{z-|z|}{2}\alpha(z+a+\lambda_1) \right) dz + \int_{c-b+\lambda_2}^{\lambda_2} \left(\frac{z+|z|}{2}\beta(z+b-\lambda_2) + \frac{z-|z|}{2}\alpha(z+b-\lambda_2) \right) dz.$$

Corollary 2. *Let* $f \in C^1[a,b]$. *If* $\alpha(x) \le f'(x)$ *for all* $x \in [a,b]$ *and* $\alpha \in C[a,b]$ *then*

$$\int_a^c (t-a-\lambda_1)\alpha(t)\,dt + \int_c^b (t-b+\lambda_2)\alpha(t)\,dt - \max\left\{ |\lambda_1|, |\lambda_2|, |b-a-\lambda_1|, |a-b+\lambda_2| \right\} \left(f(b) - f(a) - \int_a^b \alpha(t)\,dt \right)$$

$$\le \lambda_1 f(a) + (b-a-(\lambda_1+\lambda_2))\, f(c) + \lambda_2 f(b) - \int_a^b f(t)\,dt \le$$

$$\int_a^c (t-a-\lambda_1)\alpha(t)\,dt + \int_c^b (t-b+\lambda_2)\alpha(t)\,dt + \max\left\{ |\lambda_1|, |\lambda_2|, |b-a-\lambda_1|, |a-b+\lambda_2| \right\} \left(f(b) - f(a) - \int_a^b \alpha(t)\,dt \right).$$

Corollary 3. *Let* $f \in C^1[a,b]$. *If* $f'(x) \le \beta(x)$ *for all* $x \in [a,b]$ *and* $\beta \in C[a,b]$ *then*

$$\int_a^c (t-a-\lambda_1)\beta(t)\,dt + \int_c^b (t-b+\lambda_2)\beta(t)\,dt - \max\left\{|\lambda_1|,|\lambda_2|,|b-a-\lambda_1|,|a-b+\lambda_2|\right\}\left(\int_a^b \beta(t)\,dt - f(b)+f(a)\right)$$

$$\leq \lambda_1\, f(a) + (b-a-(\lambda_1+\lambda_2))\, f(c) + \lambda_2\, f(b) - \int_a^b f(t)\,dt \leq$$

$$\int_a^c (t-a-\lambda_1)\beta(t)\,dt + \int_c^b (t-b+\lambda_2)\beta(t)\,dt + \max\left\{|\lambda_1|,|\lambda_2|,|b-a-\lambda_1|,|a-b+\lambda_2|\right\}\left(\int_a^b \beta(t)\,dt - f(b)+f(a)\right).$$

Many classical quadrature rules and some nonstandard rules such as

$$\int_a^b f(x)\,dx \cong (-\frac{1}{2}a+\lambda x-(\lambda-\frac{1}{2})b)\; f(a)+\lambda(b-a)\; f(c)+((\lambda-\frac{1}{2})\,a-\lambda x+\frac{1}{2}b)\; f(b),$$

which is studied in [12] in detail, are particular cases of the general quadrature (4.2) and their corresponding errors are therefore particular subsequences of corollaries 1, 2 and 3. Here we consider a few of them.

i) If $\alpha(x)=\alpha_1 x+\alpha_0 \neq 0$ and $\beta(x)=\beta_1 x+\beta_0 \neq 0$, then by choosing $\lambda_1=\lambda_2=0$ and $c=\frac{a+b}{2}\in[a,b]$ in (4.2) and applying corollary 1 we can finally obtain a new bound for the error of midpoint rule as

$$\frac{(b-a)^2}{4}\left(\frac{b-a}{6}(\alpha_1+\beta_1)+\frac{\alpha_0+a\,\alpha_1-(\beta_0+b\,\beta_1)}{2}\right) \leq (b-a)\; f(\frac{a+b}{2})-\int_a^b f(t)\,dt$$

$$\leq \frac{(b-a)^2}{4}\left(\frac{b-a}{6}(\alpha_1+\beta_1)+\frac{\beta_0+a\,\beta_1-(\alpha_0+b\,\alpha_1)}{2}\right).$$

ii) If in (4.2) $\lambda_1=\lambda_2=\frac{b-a}{2}$, then by applying corollary 1, a new bound can be derived for the trapezoidal rule error eventually as follows

$$\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}}\left(\frac{z+|z|}{2}\alpha(z+\frac{a+b}{2})+\frac{z-|z|}{2}\beta(z+\frac{a+b}{2})\right)dz$$

$$\leq \frac{b-a}{2}\left(f(a)+f(b)\right)-\int_a^b f(t)\,dt \leq$$

$$\int_{-\frac{b-a}{2}}^{\frac{b-a}{2}}\left(\frac{z-|z|}{2}\alpha(z+\frac{a+b}{2})+\frac{z+|z|}{2}\beta(z+\frac{a+b}{2})\right)dz,$$

provided that $\alpha(x)\leq f'(x)\leq \beta(x)$ for any $x\in[a,b]$ and $\alpha,\beta\in C[a,b]$. For instance, if $\alpha(x)=\alpha_0\neq 0$ and $\beta(x)=\beta_0\neq 0$, the above inequality is reduced to the well known inequality

$$\left| \frac{b-a}{2}\left(f(a)+f(b)\right) - \int_a^b f(t)\,dt \right| \le \frac{1}{8}(b-a)^2(\beta_0-\alpha_0).$$

iii) If $\alpha(x)=\alpha_1 x+\alpha_0 \ne 0$ and $\beta(x)=\beta_1 x+\beta_0 \ne 0$, then by choosing $\lambda_1=\lambda_2=\frac{b-a}{6}$ and $c=\frac{a+b}{2}\in[a,b]$ in (4.2) and applying corollary 1 we can finally obtain a new bound for the error of Simpson rule as

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right) \right| \le \frac{5(b-a)^2}{144}\left((\beta_1-\alpha_1)(a+b)+2(\beta_0-\alpha_0)\right).$$

In particular, replacing $\alpha_1=\beta_1=0$ in the above inequality leads to one of the results of [14] as

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right) \right| \le \frac{5}{72}(b-a)^2(\beta_0-\alpha_0).$$

Moreover, if only the condition $\alpha(x)=\alpha_1 x+\alpha_0 \ne 0$ is considered, then applying corollary 2 for the error of Simpson rule yields

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right) \right| \le \frac{(b-a)^2}{3}\left(\frac{f(b)-f(a)}{b-a} - \left(\alpha_0+\frac{a+b}{2}\alpha_1 \right)\right).$$

In particular, replacing $\alpha_1=0$ in the above inequality leads to theorem 1, relation (4) of [18] as

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right) \right| \le \frac{(b-a)^2}{3}\left(\frac{f(b)-f(a)}{b-a} - \alpha_0 \right).$$

Similarly if only $\beta(x)=\beta_1 x+\beta_0 \ne 0$, then applying corollary 3 for this case yields

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right) \right| \le \frac{(b-a)^2}{3}\left(\beta_0+\frac{a+b}{2}\beta_1 - \frac{f(b)-f(a)}{b-a} \right).$$

In particular, replacing $\beta_1=0$ in the above inequality leads to theorem 1, relation (5) of [18] as

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{6}\left(f(a)+4f(\frac{a+b}{2})+f(b)\right) \right| \le \frac{(b-a)^2}{3}\left(\beta_0 - \frac{f(b)-f(a)}{b-a} \right).$$

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