Received 14/10/14

FEJER TYPE INEQUALITIES FOR LOGARITHMICALLY CONVEX FUNCTIONS

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Abstract. In this paper, we proved some new integral inequalities of Fejer type by using an integral identity and Riemann-Liouville fractional integrals. We also gave some reduced results by selecting special values of the parameters and functions.

1. INTRODUCTION

If $[a,b] \subseteq I$, then the following double inequality holds for any convex function f:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This result is known as the Hermite-Hadamard inequality for convex function. Note that some of classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f. In [10], Fejer proved the following inequality that is a weighted version of Hadamard inequality:

Theorem 1. *Let $f : I \to \mathbb{R}$ be a convex function on I and let $a, b \in I$ with $a < b$. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \leq \int_a^b f(x)\,g(x)\,dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx$$

holds, where $g : [a;b] \to \mathbb{R}$ is nonnegative, integrable, and symmetric to $x = \frac{a+b}{2}$.

In [2], Pečarić *et al.* mentioned $\log -$ convex functions as following:

A function $f : I \to [0,\infty)$ is said to be $\log -$convex or multiplicatively convex if $\log f$ is convex, or, equivalently, for all $x, y \in I$ and $t \in [0,1]$ one has the inequality

$$f(tx+(1-t)y) \leq f(x)^t f(y)^{1-t}.$$

Example 1. *The function $f(x) = \frac{1}{x}$, $x \in (0,1)$ is $\log -$convex on $(0,1)$. The function $f(x) = x^x$, $x > 0$ or $f(x) = e^x + 1$, $x \in \mathbb{R}$, etc.*

Many different extensions, generalizations and improvements related to $\log -$convex functions can be found in [1]-[9].

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

1991 *Mathematics Subject Classification.* 26D15, 26D10.

Key words and phrases. $\log -$ Convex Functions, Fejer Inequality.

Definition 1. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \qquad x > a$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \qquad x < b$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} u^{\alpha-1} du$, here is $J_{a^+}^{0} f(x) = J_{b^-}^{0} f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. Properties of this operator can be found in the references [15]-[17].

The main purpose of this paper is to give some new integral inequalities of Fejer type for logarithmically convex functions by using Lemma 1.

2. MAIN RESULTS

In order to prove our main result we will use the following Lemma that have been obtained by Set et.al. (See [11]). Let I be an interval on $\mathbb{R}$ and let $\|g\|_{[a,b],\infty} = \sup_{t \in [a,b]} |g(x)|$ for the continuous function $g : [a;b] \to \mathbb{R}$.

Lemma 1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$ and let $g : [a,b] \to \mathbb{R}$. If $f', g \in L[a,b]$, then the following equality for fractional integrals holds:*

$$f\left(\frac{a+b}{2}\right)\left[J_{\left(\frac{a+b}{2}\right)-}^{\alpha} g(a) + J_{\left(\frac{a+b}{2}\right)+}^{\alpha} g(b)\right] - \left[J_{\left(\frac{a+b}{2}\right)-}^{\alpha} (fg)(a) + J_{\left(\frac{a+b}{2}\right)+}^{\alpha} (fg)(b)\right]$$
$$= \frac{1}{\Gamma(\alpha)} \int_a^b k(t) f'(t) dt$$

where

$$k(t) = \begin{cases} \displaystyle\int_a^t (s-a)^{\alpha-1} g(s) ds \quad , & t \in \left[a, \frac{a+b}{2}\right] \\ \displaystyle\int_t^b (b-s)^{\alpha-1} g(s) ds & t \in \left[a, \frac{a+b}{2}\right] \end{cases}.$$

In terms of simplicity of reading, we will denote

$$\kappa = \left(\frac{|f'(b)|^{\frac{1}{b-a}}}{|f'(a)|^{\frac{1}{b-a}}}\right)$$

Also in the sequel of the paper, we set $\kappa \neq 1$.

Theorem 2. *Let $f : I \to \mathbb{R}$ is a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$, $f' \in L[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous. If $|f'(x)|$ is logarithmically convex on*

$[a,b]$, then the following inequality holds for fractional integrals:

(2.1)
$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}g(b)\right]-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}(fg)(b)\right]\right| \\
\leq\ & \frac{1}{\Gamma(\alpha+1)}\left(\frac{|f'(a)|^{b}}{|f'(b)|^{a}}\right)^{\frac{1}{b-a}}\left\{\|g\|_{\left[a,\frac{a+b}{2}\right],\infty}\mu_1(\kappa,\alpha)\right. \\
&\left.+\|g\|_{\left[\frac{a+b}{2},b\right],\infty}\mu_2(\kappa,\alpha)\right\}
\end{aligned}$$

where

$$\begin{aligned}
\mu_1 &= -\frac{\kappa^{a}\Gamma(1+\alpha)(-\log\kappa)^{\alpha}}{\log\kappa}+\frac{(b-a)^{\alpha}\kappa^{a}\Gamma\left(1+\alpha,\frac{a-b}{2}\log\kappa\right)(a-b)(-\log\kappa)^{\alpha}}{\log\kappa} \\
\mu_2 &= \frac{\kappa^{b}}{2}\left[-2^{-\alpha}(b-a)^{1+\alpha}2^{1+\alpha}\right. \\
&\quad\left.\left((b-a)(\log\kappa)^{-\alpha-1}\Gamma\left(1+\alpha,\frac{b-a}{2}\log\kappa\right)\right)+2\Gamma\left(1+\alpha,(\log\kappa)^{-\alpha-1}\right)\right]
\end{aligned}$$

with $\alpha>0$.

Proof. From Lemma 1, we can write

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}g(b)\right]-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}(fg)(b)\right]\right| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\left\{\int_a^{\frac{a+b}{2}}\left|\int_a^t(s-a)^{\alpha-1}g(s)\,ds\right|\left|f'\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|dt\right. \\
&\left.+\int_{\frac{a+b}{2}}^{b}\left|\int_t^b(b-s)^{\alpha-1}g(s)\,ds\right|\left|f'\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|dt\right\}.
\end{aligned}$$

Since $|f'(t)|$ is logarithmically convex on $[a,b]$, we have

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}g(b)\right]-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}(fg)(b)\right]\right| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\left(\frac{|f'(a)|^{b}}{|f'(b)|^{a}}\right)^{\frac{1}{b-a}}\left\{\int_a^{\frac{a+b}{2}}\left|\int_a^t(s-a)^{\alpha-1}g(s)\,ds\right|\left(\frac{|f'(b)|^{\frac{1}{b-a}}}{|f'(a)|^{\frac{1}{b-a}}}\right)^{t}dt\right. \\
&\left.+\int_{\frac{a+b}{2}}^{b}\left|\int_t^b(b-s)^{\alpha-1}g(s)\,ds\right|\left(\frac{|f'(b)|^{\frac{1}{b-a}}}{|f'(a)|^{\frac{1}{b-a}}}\right)^{t}dt\right\}.
\end{aligned}$$

It is easy to see that

$$\left| f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}g(b)\right]-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}(fg)(b)\right]\right|$$

$$\leq \frac{1}{\Gamma(\alpha+1)}\left(\frac{|f'(a)|^{b}}{|f'(b)|^{a}}\right)^{\frac{1}{b-a}}\left\{\|g\|_{\left[a,\frac{a+b}{2}\right],\infty}\int_{a}^{\frac{a+b}{2}}(t-a)^{\alpha}\left(\frac{|f'(b)|^{\frac{1}{b-a}}}{|f'(a)|^{\frac{1}{b-a}}}\right)^{t}dt\right.$$

$$\left.+\|g\|_{\left[\frac{a+b}{2},b\right],\infty}\int_{\frac{a+b}{2}}^{b}(b-t)^{\alpha}\left(\frac{|f'(b)|^{\frac{1}{b-a}}}{|f'(a)|^{\frac{1}{b-a}}}\right)^{t}dt\right\}.$$

By making use of necessary process. The proof is completed. □

Corollary 1. *Let $f : I \to \mathbb{R}$ is a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$, $f' \in L[a, b]$ and $g : [a, b] \to \mathbb{R}$ is continuous. If $|f'(x)|$ is logarithmically convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_{a}^{b}f(x)g(x)\,dx-\frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_{a}^{b}g(x)\,dx\right|$$

$$\leq \left(\frac{|f'(a)|^{b}}{|f'(b)|^{a}}\right)^{\frac{1}{b-a}}\left\{\|g\|_{\left[a,\frac{a+b}{2}\right],\infty}\left[\kappa^{a}\left(1+(b-a)^{2}\Gamma\left(2,\frac{a-b}{2}\log\kappa\right)\right)\right]\right.$$

$$\left.+\|g\|_{\left[\frac{a+b}{2},b\right],\infty}\left[(a-b)^{3}\kappa^{b}\left((\log\kappa)^{-2}\Gamma\left(2,\frac{b-a}{2}\log\kappa\right)\right)+\kappa^{b}\Gamma\left(2,(\log\kappa)^{-2}\right)\right]\right\}.$$

Proof. If we take $\alpha = 1$ in Theorem 2, we obtain the result. □

Corollary 2. *Under the assumptions of Corollary 1, if we choose $g(x) = 1$, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_{a}^{b}f(x)\,dx-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \left(\frac{|f'(a)|^{b}}{|f'(b)|^{a}}\right)^{\frac{1}{b-a}}\left\{\left[\kappa^{a}\left(1+(b-a)^{2}\Gamma\left(2,\frac{a-b}{2}\log\kappa\right)\right)\right]\right.$$

$$\left.+\left[(a-b)^{3}\kappa^{b}\left((\log\kappa)^{-2}\Gamma\left(2,\frac{b-a}{2}\log\kappa\right)\right)+\kappa^{b}\Gamma\left(2,(\log\kappa)^{-2}\right)\right]\right\}.$$

Theorem 3. *Let $f : I \to \mathbb{R}$ is a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$, $f' \in L[a, b]$ and $g : [a, b] \to \mathbb{R}$ is continuous. If $|f'(x)|^{q}$ is logarithmically convex on*

$[a,b]$, then the following inequality holds for fractional integrals with $\alpha>0$:

(2.2)
$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}g(b)\right]-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}(fg)(b)\right]\right|\\
\leq\ &\frac{(b-a)^{\alpha+\frac{1}{p}}}{2^{\alpha+\frac{1}{p}}(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\left(\frac{|f'(a)|^{b}}{|f'(b)|^{a}}\right)^{\frac{1}{b-a}}\left\{\|g\|_{\left[a,\frac{a+b}{2}\right],\infty}\left(\frac{\kappa^{q\frac{a+b}{2}}-\kappa^{qa}}{q\ln\kappa}\right)^{\frac{1}{q}}\right.\\
&\left.+\|g\|_{\left[a,\frac{a+b}{2}\right],\infty}\left(\frac{\kappa^{qb}-\kappa^{q\frac{a+b}{2}}}{q\ln\kappa}\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

for $q>1$ and $p^{-1}+q^{-1}=1$.

Proof. By using the Hölder integral inequality and the identity that is given in Lemma 1, we have

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}g(b)\right]-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}(fg)(b)\right]\right|\\
\leq\ &\frac{1}{\Gamma(\alpha)}\left\{\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t(s-a)^{\alpha-1}g(s)\,ds\right|^p dt\right)^{\frac{1}{p}}\left(\int_a^{\frac{a+b}{2}}|f'(t)|^q\,dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{a+b}{2}}^b\left|\int_t^b(b-s)^{\alpha-1}g(s)\,ds\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{a+b}{2}}^b|f'(t)|^q\,dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

From the definition of logarithmically convex functions, we obtain

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}g(b)\right]-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}(fg)(b)\right]\right|\\
\leq\ &\frac{1}{\Gamma(\alpha)}\left\{\|g\|_{\left[a,\frac{a+b}{2}\right],\infty}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t(s-a)^{\alpha-1}ds\right|^p dt\right)^{\frac{1}{p}}\left(\int_a^{\frac{a+b}{2}}|f'(a)|^{q\frac{b-t}{b-a}}|f'(b)|^{q\frac{t-a}{b-a}}\,dt\right)^{\frac{1}{q}}\right.\\
&\left.+\|g\|_{\left[a,\frac{a+b}{2}\right],\infty}\left(\int_{\frac{a+b}{2}}^b\left|\int_t^b(b-s)^{\alpha-1}ds\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{a+b}{2}}^b|f'(a)|^{q\frac{b-t}{b-a}}|f'(b)|^{q\frac{t-a}{b-a}}\,dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

By computing the above integrals, we get the desired result. □

Corollary 3. *Let $f:I\to\mathbb{R}$ is a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$, $f'\in L[a,b]$ and $g:[a,b]\to\mathbb{R}$ is continuous. If $|f'(x)|^q$ is logarithmically convex on*

$[a,b]$, *then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx\right| \\
&\leq \frac{(b-a)^{1+\frac{1}{p}}}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^b}{|f'(b)|^a}\right)^{\frac{1}{b-a}}\left\{\|g\|_{\left[a,\frac{a+b}{2}\right],\infty}\left(\frac{\kappa^{q\frac{a+b}{2}}-\kappa^q}{q\ln\kappa}\right)^{\frac{1}{q}}\right. \\
&\left.+\|g\|_{\left[a,\frac{a+b}{2}\right],\infty}\left(\frac{\kappa^{qb}-\kappa^{q\frac{a+b}{2}}}{q\ln\kappa}\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

for $q>1$ *and* $p^{-1}+q^{-1}=1$.

Proof. By taking $\alpha=1$ in Theorem 3, we get the desired result. □

Corollary 4. *Under the assumptions of Corollary 3, if we choose* $g(x)=1$, *then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{(b-a)^{1+\frac{1}{p}}}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^b}{|f'(b)|^a}\right)^{\frac{1}{b-a}}\left[\left(\frac{\kappa^{q\frac{a+b}{2}}-\kappa^q}{q\ln\kappa}\right)^{\frac{1}{q}}+\left(\frac{\kappa^{qb}-\kappa^{q\frac{a+b}{2}}}{q\ln\kappa}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Theorem 4. *Let* $f:I\to\mathbb{R}$ *is a differentiable mapping on* I°, $a,b\in I^\circ$ *with* $a<b$, $f'\in L[a,b]$ *and* $g:[a,b]\to\mathbb{R}$ *is continuous. If* $|f'(x)|^q$ *is logarithmically convex on* $[a,b]$, *then the following inequality holds for fractional integrals with* $\alpha>0$*:*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)\left[J^\alpha_{\left(\frac{a+b}{2}\right)-}g(a)+J^\alpha_{\left(\frac{a+b}{2}\right)+}g(b)\right]-\left[J^\alpha_{\left(\frac{a+b}{2}\right)-}(fg)(a)+J^\alpha_{\left(\frac{a+b}{2}\right)+}(fg)(b)\right]\right| \\
&\leq \frac{1}{\Gamma(\alpha+1)}\left(\frac{|f'(a)|^b}{|f'(b)|^a}\right)^{\frac{1}{b-a}}\left(\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}\right)^{1-\frac{1}{q}}\left\{\|g\|_{\left[a,\frac{a+b}{2}\right],\infty}(\mu_3(\kappa,\alpha))^{\frac{1}{q}}\right. \\
&\left.+\|g\|_{\left[\frac{a+b}{2},b\right],\infty}(\mu_4(\kappa,\alpha))^{\frac{1}{q}}\right\}
\end{aligned}$$

where

$$\begin{aligned}
\mu_3(\kappa,\alpha) &= -\frac{\kappa^{qa}\Gamma(1+\alpha)(-q\log\kappa)^\alpha}{\log\kappa} \\
&\quad+\frac{(b-a)^\alpha\kappa^{qa}\Gamma\left(1+\alpha,q\frac{a-b}{2}\log\kappa\right)(a-b)(-q\log\kappa)^\alpha}{q\log\kappa} \\
\mu_4(\kappa,\alpha) &= -\frac{\kappa^{qb}}{2}-2^{-\alpha}(b-a)^{1+\alpha}\left[2^{1+\alpha}\left((b-a)(q\log\kappa)^{-\alpha-1}\Gamma\left(1+\alpha,q\frac{b-a}{2}\log\kappa\right)\right)\right. \\
&\quad\left.+2\Gamma\left(1+\alpha,(q\log\kappa)^{-\alpha-1}\right)\right]
\end{aligned}$$

for $q\geq1$.

Proof. By using the Power-mean inequality and the identity that is given in Lemma 1, we can write

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}g(b)\right]-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}(fg)(b)\right]\right| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\left\{\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\alpha-1}g(s)\,ds\right|dt\right)^{1-\frac{1}{q}}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\alpha-1}g(s)\,ds\right|\left|f'(t)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{a+b}{2}}^{b}\left|\int_t^b (b-s)^{\alpha-1}g(s)\,ds\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{a+b}{2}}^{b}\left|\int_t^b (b-s)^{\alpha-1}g(s)\,ds\right|\left|f'(t)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Since $|f'(x)|^q$ is logarithmically convex functions, we get

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}g(b)\right]-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)+}(fg)(b)\right]\right| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\left\{\|g\|_{\left[a,\frac{a+b}{2}\right],\infty}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\alpha-1}ds\right|dt\right)^{1-\frac{1}{q}}\right. \\
&\times\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\alpha-1}ds\right|\left|f'(a)\right|^{q\frac{b-t}{b-a}}\left|f'(b)\right|^{q\frac{t-a}{b-a}}dt\right)^{\frac{1}{q}} \\
&+\|g\|_{\left[\frac{a+b}{2},b\right],\infty}\left(\int_{\frac{a+b}{2}}^{b}\left|\int_t^b (b-s)^{\alpha-1}ds\right|dt\right)^{1-\frac{1}{q}} \\
&\left.\left(\int_{\frac{a+b}{2}}^{b}\left|\int_t^b (b-s)^{\alpha-1}ds\right|\left|f'(a)\right|^{q\frac{b-t}{b-a}}\left|f'(b)\right|^{q\frac{t-a}{b-a}}dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

By computing the above integrals, we get the desired result. □

Corollary 5. *Let $f: I\to\mathbb{R}$ is a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$, $f'\in L[a,b]$ and $g:[a,b]\to\mathbb{R}$ is continuous. If $|f'(x)|^q$ is logarithmically convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx-\frac{1}{b-a}f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx\right| \\
\leq\ & \left(\frac{|f'(a)|^b}{|f'(b)|^a}\right)^{\frac{1}{b-a}}\left(\frac{(b-a)^2}{8}\right)^{1-\frac{1}{q}}\left\{\|g\|_{\left[a,\frac{a+b}{2}\right],\infty}\left(\mu_3(\kappa,1)\right)^{\frac{1}{q}}\right. \\
&\left.+\|g\|_{\left[\frac{a+b}{2},b\right],\infty}\left(\mu_4(\kappa,1)\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

where

$$\mu_3(\kappa,1) = -\frac{\kappa^{qa}(-q\log\kappa)}{\log\kappa} + \frac{(b-a)\kappa^{qa}\Gamma\left(2, q\frac{a-b}{2}\log\kappa\right)(a-b)(-q\log\kappa)}{q\log\kappa}$$

$$\mu_4(\kappa,1) = -\frac{\kappa^{qb}}{2} - 2^{-1}(b-a)^2 \times\left[4\left((b-a)(q\log\kappa)^{-2}\Gamma\left(2, q\frac{b-a}{2}\log\kappa\right)\right) + 2\Gamma\left(2,(q\log\kappa)^{-2}\right)\right]$$

for $q \geq 1$.

Proof. If we take $\alpha = 1$ in Theorem 4, we obtain the result. □

Corollary 6. *Under the assumptions of Corollary 5, if we take* $g(x) = 1$, *we obtain yhe following inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq \left(\frac{|f'(a)|^b}{|f'(b)|^a}\right)^{\frac{1}{b-a}}\left(\frac{(b-a)^2}{8}\right)^{1-\frac{1}{q}}\left[(\mu_3(\kappa,1))^{\frac{1}{q}} + (\mu_4(\kappa,1))^{\frac{1}{q}}\right]$$

where $\mu_3(\kappa,1)$ *and* $\mu_4(\kappa,1)$ *as in Corollary 5.*

Remark 1. *Several applications can be given based on our results to the special means of real numbers and to numerical integration via mid-point formula, we omit the details.*

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