

# Annular Regions Containing all the Zeros of a Polynomial

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## Abstract

In this paper, ring shaped regions containing all the zeros of a polynomial with complex coefficients involving binomial coefficients and Fibonacci–Pell numbers are given. Furthermore, bounds for positive polynomials involving their derivatives are also presented. Finally, using Maple, some examples illustrating the bounds proposed are computed and compared with other existing explicit bounds for the zeros.

**Keywords:** Inequalities; polynomials location of the zeros; Fibonacci and Pell numbers; positive polynomials; Maple.

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## 1 Introduction

Problems involving polynomials in general, and location of their zeros in particular, have a long history. But, recently come under reexamination because of their importance in many areas of applied mathematics such as control theory, signal processing and electrical networks, coding theory, cryptography, combinatorics, number theory, and engineering among others [16, 17, 12, 13]. Specially, the zeros of polynomials play an important role to solve control engineering problems [2], digital audio signal processing problems [18], eigenvalue problems in mathematical physics [10], and in mathematical biology, where polynomials with positive coefficients have efficiently used [3]. A lot of methods to approximate the actual value of the zeros of polynomials with real or complex coefficients, such as Sturm sequence method start with an estimate of an upper bound for the moduli of the zeros. If we have an accurate estimation for the bound, then the amount of work needed to search the range of possible values to begin

with, can be considerably reduced in comparison with the classical starting searchers such as the bisection methods. Therefore, it should be important to have a set of available bounds to choose what is most accurate to begin the computation. It is well-known that one of the first explicit bounds for the zeros is a classical result due to Cauchy (1829) [5] and since then a lot of papers devoted to the subject can be found in the literature. Some of them that will be used in the computations performed to obtain the numerical results presented at the end of this paper, and for ease of reference, are summarized in the following three theorems.

**Theorem A (Upper bounds for the zeros)** *Let  $A(z) = \sum_{k=0}^n a_k z^k$ , ( $a_k \neq 0$ ) be a nonconstant polynomial with complex coefficients. Then, all its zeros lie in the disc  $\mathcal{C} = \{z \in \mathbb{C} : |z| \leq r\}$ , where*

$$(i) \quad r < 1 + \max_{0 \leq k \leq n-1} \{|a_k|\} \quad (\text{Cauchy [5]})$$

$$(ii) \quad r = \max \left\{ 1, \sum_{k=0}^{n-1} |a_k| \right\} \quad (\text{Tikoo [14]})$$

(iii)  $r < 1 + \delta_3$ , where  $\delta_3$  is the unique positive root of the equation

$$Q_3(x) = x^3 + (2 - |a_{n-1}|)x^2 + (1 - |a_{n-1}| - |a_{n-2}|)x - A = 0$$

$$\text{Here } A = \max_{0 \leq k \leq n-1} |a_k| \quad (\text{Sun et al. [11]})$$

(iv)  $r < \sqrt{1 + A}$ , where  $A = \max_{0 \leq k \leq n-1} \{|a_k^2 + 2(-1)^k(B - C)|\}$  and

$$B = \sum_{\substack{0 \leq i < j \leq [n/2] \\ i+j=k}} a_{2i} a_{2j} \quad \text{and} \quad C = \sum_{\substack{0 \leq i < j \leq [(n-1)/2] \\ i+j=k-1}} a_{2i+1} a_{2j+1} \quad (\text{Zilovic [17]})$$

$$(v) \quad r = \sum_{k=0}^{n-1} |a_k|^{1/n-k} \quad (\text{Walsh [15]})$$

$$(vi) \quad r = \sqrt{1 + \sum_{k=0}^{n-1} |a_k|^2} \quad (\text{Carmichael et. al. [4]})$$

Another refinement of the explicit bound of Cauchy was given by Díaz-Barrero [6], by proving the following

**Theorem B** *Let  $A(z) = \sum_{k=0}^n a_k z^k$ , ( $a_k \neq 0$ ) be a nonconstant polynomial with complex coefficients. Then, all its zeros lie in the annulus  $\mathcal{C} = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}$ , where*

$$r_1 = \frac{3}{2} \min_{1 \leq k \leq n} \left\{ \frac{2^n F_k C(n, k)}{F_{4n}} \left| \frac{a_0}{a_k} \right| \right\}^{1/k},$$

and

$$r_2 = \frac{2}{3} \max_{1 \leq k \leq n} \left\{ \frac{F_{4n}}{2^n F_k C(n, k)} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{1/k}$$

Here  $F_k$  are Fibonacci's numbers, namely,  $F_0 = 0$ ,  $F_1 = 1$ , and for  $k \geq 2$ ,  $F_k = F_{k-1} + F_{k-2}$ . Furthermore,  $C(n, k) = \frac{n!}{k!(n-k)!}$  are the binomial coefficients also note as  $\binom{n}{k}$ .

Recently Affane-Aji et al. [1] have generalized the results obtained in [11], by given the following

**Theorem C** Let  $A(z) = \sum_{k=0}^n a_k z^k$ , ( $a_k \neq 0$ ) be a nonconstant polynomial with complex coefficients. Then, all its zeros lie in the annulus  $\mathcal{C} = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}$ , where

$$r_1 = \min_{1 \leq k \leq n} \left\{ \frac{k C(n, k)}{n 2^{n-1}} \left| \frac{a_0}{a_k} \right| \right\}^{1/k},$$

and

$$r_2 = 1 + \delta_k.$$

Here,  $\delta_k$  is the unique positive root of the equation

$$Q_k(z) = z^k + \sum_{\nu=2}^k \left[ C(k-1, k-\nu) - \sum_{j=1}^{\nu-1} C(k-j-1, k-\nu) |a_{n-j}| \right] z^{k+1-\nu} - A = 0,$$

where  $A = \max_{0 \leq k \leq n-1} |a_k|$ ,  $a_k = 0$  if  $k < 0$ .

Our goal in this paper is to present some ring-shaped regions in the complex plane containing all the zeros of a polynomial involving the elements of second order recurrences such as the Fibonacci and Pell numbers and binomial coefficients. Moreover, bounds polynomials with positive coefficients are also obtained. To illustrate the results given, numerical computations using a Maple code are performed and the results obtained are exhibited in the last section where they are compared with other bounds appeared previously.

## 2 Main results

In the following some theorems on the location of the zeros are given. We begin with

**Theorem 1** Let  $A(z) = \sum_{k=0}^n a_k z^k$ , ( $a_k \neq 0$ ) be a nonconstant polynomial with complex coefficients. Then, all its zeros lie in the ring shaped region  $\mathcal{C} = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}$ , where

$$r_1 = \min_{1 \leq k \leq n} \left\{ 2^{3(1-n)} \binom{2n}{n+k} \mathcal{P}_k^2 \left| \frac{a_0}{a_k} \right| \right\}^{1/k} \quad (1)$$

and

$$r_2 = \max_{1 \leq k \leq n} \left\{ 2^{3(n-1)} \binom{2n}{n+k}^{-1} \mathcal{P}_k^{-2} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{1/k} \quad (2)$$

Here  $\mathcal{P}_k$  stands for the  $k^{\text{th}}$  Pell number defined by  $\mathcal{P}_0 = 0$ ,  $\mathcal{P}_1 = 1$ , and for  $n \geq 2$ ,  $\mathcal{P}_n = 2\mathcal{P}_{n-1} + \mathcal{P}_{n-2}$ .

*Proof.* To prove the preceding result, we will use the identity

$$\sum_{k=0}^n \binom{2n}{n+k} (z^k + z^{-k}) = \binom{2n}{n} + z^{-n} (z+1)^{2n}, \quad (3)$$

valid for all nonnegative integer  $n$  and for all nonzero complex number  $z$ . In fact, developing the LHS of the preceding expression, we have

$$\begin{aligned} z^n \sum_{k=0}^n \binom{2n}{n+k} (z^k + z^{-k}) &= 2 \binom{2n}{n} z^n + \binom{2n}{n+1} (z^{n+1} + z^{n-1}) + \dots + \binom{2n}{2n} z^n (z^n + z^{-n}) \\ &= 2 \binom{2n}{n} z^n + \binom{2n}{n+1} z^{n+1} + \dots + \binom{2n}{2n} z^{2n} + \binom{2n}{n-1} z^{n-1} + \dots + \binom{2n}{0} z^0 \\ &= \binom{2n}{n} z^n + \sum_{k=0}^{2n} \binom{2n}{k} z^k = \binom{2n}{n} z^n + (z+1)^{2n} \end{aligned}$$

and (3) is proven after dividing by  $z^n$ . Now, using the above result we can obtain

$$\sum_{k=0}^n \binom{2n}{n+k} \mathcal{P}_k^2 = 2^{3(n-1)} \quad (4)$$

Indeed, putting  $z = -1$  in (3) the identity becomes

$$\sum_{k=0}^n (-1)^k \binom{2n}{n+k} = \frac{1}{2} \binom{2n}{n} \quad (5)$$

Putting  $z = \alpha^2$ , where  $\alpha = 1 + \sqrt{2}$ , a characteristic root of the second order recurrence  $x^2 - 2x - 1 = 0$  that defines Pell's numbers in (3), we get

$$\sum_{k=0}^n \binom{2n}{n+k} (\alpha^{2k} + \alpha^{-2k}) = \frac{1}{\alpha^{2n}} (\alpha^2 + 1)^{2n} + \binom{2n}{n} \quad (6)$$

On account of Binet's formulae, we have  $\mathcal{P}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{2\sqrt{2}}$ , where  $\alpha = 1 + \sqrt{2}$ , and  $\beta = 1 - \sqrt{2}$ . Squaring the last expression, and taking into account that  $\alpha\beta = -1$ , yields

$$\begin{aligned} \mathcal{P}_n^2 &= \frac{1}{2^3} (\alpha^n - \beta^n)^2 = \frac{1}{2^3} (\alpha^{2n} + \beta^{2n} - 2(\alpha\beta)^n) \\ &= \frac{1}{2^3} \left( \alpha^{2n} + \left(\frac{-1}{\alpha}\right)^{2n} - 2(-1)^n \right) = \frac{1}{2^3} (\alpha^{2n} + \alpha^{-2n} - 2(-1)^n) \end{aligned}$$

from which follows

$$\begin{aligned} \sum_{k=0}^n \binom{2n}{n+k} \mathcal{P}_k^2 &= \sum_{k=0}^n \frac{1}{2^3} \binom{2n}{n+k} (\alpha^{2k} + \alpha^{-2k} - 2(-1)^k) \\ &= \frac{1}{2^3} \sum_{k=0}^n \binom{2n}{n+k} (\alpha^{2k} + \alpha^{-2k}) - \frac{1}{2^2} \sum_{k=0}^n (-1)^k \binom{2n}{n+k} \end{aligned}$$

Finally,

$$\begin{aligned} \sum_{k=0}^n \binom{2n}{n+k} \mathcal{P}_k^2 &= \frac{1}{2^3} \left[ \binom{2n}{n} + \frac{1}{\alpha^{2n}} (\alpha^2 + 1)^{2n} \right] - \frac{1}{2^3} \binom{2n}{n} \\ &= \frac{1}{2^3} \left( \frac{1}{\alpha} + \alpha \right)^{2n} = 2^{3(n-1)} \end{aligned}$$

on account of (5) and (6) and the fact that  $\alpha + \alpha^{-1} = 2\sqrt{2}$ , as can be easily checked.

From (1), it follows for  $1 \leq k \leq n$ ,

$$r_1^k \leq 2^{3(1-n)} \binom{2n}{n+k} \mathcal{P}_k^2 \left| \frac{a_0}{a_k} \right| \quad (7)$$

Suppose that  $|z| < r_1$ , then

$$\begin{aligned} |A(z)| &= \left| \sum_{k=0}^n a_k z^k \right| \geq |a_0| - \sum_{k=1}^n |a_k| |z|^k > |a_0| - \sum_{k=1}^n |a_k| r_1^k \\ &= |a_0| \left( 1 - \sum_{k=1}^n \left| \frac{a_k}{a_0} \right| r_1^k \right) \geq |a_0| \left( 1 - \sum_{k=1}^n 2^{3(1-n)} \binom{2n}{n+k} \mathcal{P}_k^2 \right) = 0 \end{aligned}$$

on account of (4) and (7). Consequently,  $A(z)$  does not have zeros in  $\{z \in \mathbb{C} : |z| < r_1\}$ .

It is known ([5], [7]) that all the zeros of  $A(z)$  have modulus less or equal than the unique positive root of the equation

$$G(z) = |a_n|z^n - |a_{n-1}|z^{n-1} - \dots - |a_1|z - |a_0|$$

Hence, the second part of our statement will be proved if we show that  $G(r_2) \geq 0$ . From (2), it follows for  $1 \leq k \leq n$ ,

$$\left| \frac{a_{n-k}}{a_n} \right| \leq 2^{3(1-n)} \binom{2n}{n+k} \mathcal{P}_k^2 r_2^k \quad (8)$$

Then,

$$\begin{aligned} G(r_2) &= |a_n| \left[ r_2^n - \sum_{k=1}^n \left| \frac{a_{n-k}}{a_n} \right| r_2^{n-k} \right] \\ &\geq |a_n| \left[ r_2^n - \sum_{k=1}^n \left( 2^{3(1-n)} \binom{2n}{n+k} \mathcal{P}_k^2 r_2^k \right) r_2^{n-k} \right] \\ &= |a_n| r_2^n \left( 1 - \sum_{k=1}^n 2^{3(1-n)} \binom{2n}{n+k} \mathcal{P}_k^2 \right) = 0, \end{aligned}$$

and the proof is complete.  $\square$

Using Fibonacci numbers instead of Pell's numbers, we state and prove the following

**Theorem 2** *Let  $A(z) = \sum_{k=0}^n a_k z^k$ , ( $a_k \neq 0$ ) be a nonconstant polynomial with complex coefficients. Then, all its zeros lie in the ring shaped region  $\mathcal{C} = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}$ , where*

$$r_1 = \min_{1 \leq k \leq n} \left\{ 5^{1-n} \binom{2n}{n+k} F_k^2 \left| \frac{a_0}{a_k} \right| \right\}^{1/k}$$

and

$$r_2 = \max_{1 \leq k \leq n} \left\{ 5^{n-1} \binom{2n}{n+k}^{-1} F_k^{-2} \left| \frac{a_{n-k}}{a_n} \right| \right\}^{1/k}$$

Here  $F_k$  stands for the  $k^{\text{th}}$  Fibonacci number defined by  $F_0 = 0, F_1 = 1$ , and for all  $n \geq 2$ ,  $F_n = F_{n-1} + F_{n-2}$ .

*Proof.* To prove the preceding result we will use the same technique as in the proof of Theorem 1. So, we begin claiming that

$$\sum_{k=0}^n \binom{2n}{n+k} F_k^2 = 5^{n-1} \quad (9)$$

Indeed, since the roots of the characteristic equation of the second order recurrence for Fibonacci numbers,  $x^2 - x - 1 = 0$  are  $\alpha = \frac{1 + \sqrt{5}}{2}$  and  $\beta = \frac{1 - \sqrt{5}}{2}$ , then we obtain  $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\sqrt{5}}$ , as it is well-known. From the preceding immediately follows

$$F_n^2 = \frac{1}{5} (\alpha^n - \beta^n)^2 = \frac{1}{5} (\alpha^{2n} + \alpha^{-2n} - 2(-1)^n)$$

and

$$\begin{aligned} \sum_{k=0}^n \binom{2n}{n+k} F_k^2 &= \sum_{k=0}^n \binom{2n}{n+k} \frac{1}{5} (\alpha^{2k} + \alpha^{-2k} - 2(-1)^k) \\ &= \frac{1}{5} \sum_{k=0}^n \binom{2n}{n+k} (\alpha^{2k} + \alpha^{-2k}) - \frac{2}{5} \sum_{k=0}^n (-1)^k \binom{2n}{n+k} \end{aligned}$$

Putting  $z = \alpha^2$ , where  $\alpha^2 = \frac{1 + \sqrt{5}}{2}$  and  $z = -1$  in (3) and substituting respectively into the right hand side of the preceding expression, follows

$$\sum_{k=0}^n \binom{2n}{n+k} F_k^2 = \frac{1}{5} \left[ \frac{1}{\alpha^{2n}} (\alpha^2 + 1)^{2n} + \binom{2n}{n} \right] - \frac{1}{5} \binom{2n}{n}$$

$$= \frac{1}{5} \left( \frac{1}{\alpha} + \alpha \right)^{2n} = 5^{n-1}$$

because  $\alpha + \alpha^{-1} = \sqrt{5}$ , as can be easily checked.

Now using (9) instead of (4), and carrying out step by step the proof of Theorem 1 the proof of Theorem 2 can be easily completed. We omit the details.  $\square$

Hereafter, we state and prove two results involving explicit bounds for the zeros of polynomials with positive coefficients. We begin with a bound involving the derivative of the polynomial. It is stated in the following

**Theorem 3** *Let  $A(z) = \sum_{k=0}^n a_k z^k$  be a nonconstant polynomial with positive coefficients. Then, all its zeros lie in the ring shaped region  $\mathcal{C} = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}$ , where*

$$r_1 = \min_{1 \leq k \leq n} \left\{ k p^{k-1} \left( \frac{a_0}{A'(p)} \right) \right\}^{1/k} \quad (10)$$

and

$$r_2 = \max_{1 \leq k \leq n} \left\{ \frac{q^{k-n}}{n-k+1} \left( \frac{B'(q)}{a_n} \right) \right\}^{1/k} \quad (11)$$

Here  $p, q$  are positive real numbers and  $B(z) = z(A(z) - a_n z^n)$ .

*Proof.* We will argue as in the proof of Theorem 1. So, we begin assuming that  $|z| < r_1$ , and we have

$$\begin{aligned} |A(z)| &= \left| \sum_{k=0}^n a_k z^k \right| \geq |a_0| - \sum_{k=1}^n |a_k| |z|^k > |a_0| - \sum_{k=1}^n |a_k| r_1^k \\ &= |a_0| \left( 1 - \sum_{k=1}^n \left| \frac{a_k}{a_0} \right| r_1^k \right) = a_0 \left( 1 - \sum_{k=1}^n \left( \frac{a_k}{a_0} \right) r_1^k \right) \end{aligned}$$

From (10), we have for  $1 \leq k \leq n$ ,

$$\left( \frac{a_k}{a_0} \right) r_1^k \leq k p^{k-1} \left( \frac{a_k}{A'(p)} \right)$$

Substituting in the preceding, yields

$$|A(z)| > a_0 \left( 1 - \sum_{k=1}^n \left( \frac{a_k}{a_0} \right) r_1^k \right) \geq a_0 \left( 1 - \sum_{k=1}^n k p^{k-1} \left( \frac{a_k}{A'(p)} \right) \right) = 0$$

Therefore,  $A(z)$  does not have zeros in  $\{z \in \mathbb{C} : |z| < r_1\}$  and we are done with the lower bound. For the upper bound we have to see that all the zeros of  $A(z)$  have modulus less than or equal to the unique positive root of the equation  $G(z) = |a_n|z^n - |a_{n-1}|z^{n-1} - \dots - |a_1|z - |a_0| = 0$ . So, it

would be sufficient to prove that  $G(r_2) \geq 0$ . Indeed, from (11) we get for  $1 \leq k \leq n$ ,

$$\frac{a_{n-k}}{a_n} \leq (n-k+1)q^{n-k} \left( \frac{a_{n-k}}{B'(q)} \right) r_2^k$$

Since  $G(r_2) = a_n \left( r_2^n - \sum_{k=1}^n \frac{a_{n-k}}{a_n} r_2^{n-k} \right)$ , then immediately follows

$$\begin{aligned} G(r_2) &\geq a_n \left( r_2^n - \sum_{k=1}^n \left( \frac{a_{n-k}(n-k+1)q^{n-k}r_2^k}{B'(q)} \right) r_2^{n-k} \right) \\ &= a_n r_2^n \left( 1 - \sum_{k=1}^n \frac{a_{n-k}(n-k+1)q^{n-k}}{B'(q)} \right) = 0 \end{aligned}$$

on account that  $B'(z) = \sum_{k=1}^n (n-k+1)a_{n-k}z^{n-k}$ . This completes the proof.  $\square$

Finally, we close the section stating and proving a result involving higher order derivatives of the polynomial.

**Theorem 4** *Let  $A(z) = \sum_{k=0}^n a_k z^k$  be a nonconstant polynomial with positive coefficients. Then, all its zeros lie in the ring shaped region  $\mathcal{C} = \{z \in \mathbb{C} : r_1 \leq |z| \leq r_2\}$ , where*

$$r_1 = \min_{1 \leq k \leq n} \left\{ \frac{(p-q)^k A^{(k)}(q)}{k! [A(p) - A(q)]} \left( \frac{a_0}{a_k} \right) \right\}^{1/k} \quad (12)$$

and

$$r_2 = \max_{1 \leq k \leq n} \left\{ \frac{k! [A(r) - A(s)]}{A^{(k)}(s) (r-s)^k} \left( \frac{a_{n-k}}{a_n} \right) \right\}^{1/k} \quad (13)$$

Here  $p > q > 0$  and  $r > s > 0$ .

*Proof.* Arguing out as in the proof of Theorem 1, if we suppose that  $|z| < r_1$ , then we have

$$\begin{aligned} |A(z)| &= \left| \sum_{k=0}^n a_k z^k \right| \geq a_0 - \sum_{k=1}^n a_k |z|^k > a_0 - \sum_{k=1}^n a_k r_1^k \\ &= a_0 \left( 1 - \sum_{k=1}^n \frac{a_k}{a_0} r_1^k \right) \end{aligned}$$

From (12), we get for  $1 \leq k \leq n$ ,  $\left( \frac{a_k}{a_0} \right) r_1^k \leq \frac{A^{(k)}(q) (p-q)^k}{k! [A(p) - A(q)]}$ . Substituting in the preceding expression, we obtain

$$\begin{aligned} |A(z)| &> a_0 \left( 1 - \sum_{k=1}^n \frac{a_k}{a_0} r_1^k \right) \\ &\geq a_0 \left( 1 - \sum_{k=1}^n \frac{A^{(k)}(q) (p-q)^k}{k! [A(p) - A(q)]} \right) = 0, \end{aligned}$$



because

$$A(p) - A(q) = A'(q)(p - q) + \frac{A''(q)}{2!}(p - q)^2 + \dots + \frac{A^{(n)}(q)}{n!}(p - q)^n$$

on account of Taylor's formulae applied to  $A(p)$  at point  $q$ . Consequently,  $A(z)$  does not have zeros in  $\{z \in \mathbb{C} : |z| < r_1\}$ . For the upper bound we have to find the unique positive root of the equation

$$G(z) = |a_n|z^n - |a_{n-1}|z^{n-1} - \dots - |a_1|z - |a_0| = 0$$

as it is well-known ([7], [8], [9]). Hence, we have to prove that  $G(r_2) \geq 0$  and we are done. From (13), we get for  $1 \leq k \leq n$ ,

$$\frac{a_{n-k}}{a_n} \leq \frac{A^{(k)}(s)(r-s)^k}{k! [A(r) - A(s)]} r_2^k$$

Since  $G(r_2) = a_n \left( r_2^n - \sum_{k=1}^n \frac{a_{n-k}}{a_n} r_2^{n-k} \right)$ , then

$$\begin{aligned} G(r_2) &\geq a_n \left[ r_2^n - \sum_{k=1}^n \left( \frac{A^{(k)}(s)(r-s)^k}{k! [A(r) - A(s)]} r_2^k \right) r_2^{n-k} \right] \\ &= a_n r_2^n \left( 1 - \sum_{k=1}^n \frac{A^{(k)}(s)(r-s)^k}{k! [A(r) - A(s)]} \right) \\ &= 0, \end{aligned}$$

because

$$A(r) - A(s) = A'(s)(r-s) + \frac{A''(s)}{2!}(r-s)^2 + \dots + \frac{A^{(n)}(s)}{n!}(r-s)^n$$

on account of Taylor's formulae applied to  $A(r)$  at point  $s$ . This completes the proof.  $\square$

### 3 Applications using Maple

In this section, we use a Maple code to compute the inner and outer radius of a ring shaped region containing all the zeros for the polynomials  $A_1(z) = z^3 + 0.1z^2 + 0.3z + 0.7$  and  $A_2(z) = z^3 + 0.1z^2 + 0.1z + 0.04$ , obtainable by the results presented in this paper. Also, the annuli obtained from the known results given in Theorem A, Theorem B and Theorem C are computed and compared with our results. These numerical results are shown in Table 1. Furthermore, by using Maple code, we also computed the actual zeros of both polynomials. We found that for  $A_1(z)$  they all lie in the annulus  $\mathcal{C} = \{z \in \mathbb{C} : 0.80579 \leq |z| \leq 0.93205\}$  which has area equal to 0.68933. For  $A_2(z)$  the annulus containing all the zeros is  $\mathcal{C} = \{z \in \mathbb{C} : 0.27228 \leq |z| \leq 0.38328\}$  with area equal to 0.22862. These

values has been used to express in terms of percentage error the area of the annuli obtained. They can be found in columns fourth and seventh of Table 1.

Bounds	$A_1(z)$			$A_2(z)$		
	$r_1$	$r_2$	Error	$r_1$	$r_2$	Error
Cauchy [5]	0.41176	1.7	1139.83	0.03846	1.1	1560.69
Tikoo [14]	0.49999	1.1	337.51	0.03333	1.0	1272.63
Sun et al. [11]	0.58693	1.43399	680.16	0.21858	1.1	1497.07
Zilovic [17]	0.57346	1.26095	474.76	0.03997	1.09087	1533.05
Walsh [15]	0.51739	1.53563	852.71	0.14275	0.75822	662.00
Carmichael et al. [4]	0.55514	1.26095	484.18	0.03957	1.01074	1301.68
Díaz-Barrero [6]	0.58333	1.23127	435.85	0.1	0.51640	252.70
Affane-Aji et al. [1]	0.55934	1.17510	386.73	0.1	1.09564	1535.82
Theorem 1	0.54687	1.21463	436.08	0.09375	0.51640	254.36
Theorem 2	0.48203	1.63553	1013.21	0.18566	0.64550	425.19
Theorem 3	0.74413	1.05601	155.87	0.22557	0.49763	170.37
Theorem 4	0.74665	1.03928	138.18	0.22046	0.49037	163.64

**Table 1.** Inner radius  $r_1$ , outer radius  $r_2$  and Error for polynomials  $A_1(z)$  and  $A_2(z)$ .

In the computation of the values presented in Table 1 for  $A_1(z)$  using Theorem 3 we have taken  $p = 0.43$ ,  $q = 0.61$ , and using Theorem 4 we have taken  $p = 0.75$ ,  $q = 0.0001$ ,  $r = 2.07$  and  $s = 0.24$ . For the polynomial  $A_2(z)$  the values used for Theorem 3 were  $p = 0.13$ ,  $q = 0.29$ , and for Theorem 4 we have taken  $p = 0.24$ ,  $q = 0.016$ ,  $r = 3.00$  and  $s = 0.90$ .

Table 2 shows the results obtained for the polynomials  $A_3(z) = z^5 + 0.009z^4 + 0.75z^3 + 0.5z^2 + 0.32z + 0.8$  and  $A_4(z) = z^3 + (1+i)z^2 + iz + 3.4$ .

Bounds	$A_3(z)$			$A_4(z)$		
	$r_1$	$r_2$	Error	$r_1$	$r_2$	Error
Cauchy [5]	0.44444	1.8	902.27	0.70624	4.4	656.37
Tikoo [14]	0.31020	2.379	1732.74	0.99584	5.81421	1215.89
Sun et al. [11]	0.57578	1.51862	550.51	0.72346	2.60820	151.81
Zilovic [17]	0.54688	1.58111	625.02	0.72640	3.54400	382.52
Walsh [15]	0.28244	3.37720	3630.99	0.62341	3.91791	499.98
Carmichael et al. [4]	0.50049	1.59843	659.16	0.86193	3.94462	494.20
Díaz-Barrero [6]	0.08869	2.65459	2218.83	0.85	5.36685	1154.29
Affane-Aji et al. [1]	0.15625	1.48590	619.30	0.85	2.60820	143.83
Theorem 1	0.12817	2.52982	2002.92	0.79687	6.03398	1334.61
Theorem 2	0.50238	1.97642	1103.68	0.75960	2.76983	184.52
Theorem 3	0.61818	1.40333	422.87	-	-	-
Theorem 4	0.66258	1.67427	678.82	-	-	-

**Table 2.** Inner radius  $r_1$ , outer radius  $r_2$  and Error for polynomials  $A_3(z)$  and  $A_4(z)$ .

In the computation of the values presented in Table 2 for  $A_3(z)$  using Theorem 3 we have taken  $p = 0.41$ ,  $q = 0.92$ , and using Theorem 4 we have taken  $p = 0.68$ ,  $q = 0.01$ ,  $r = 2.99$  and  $s = 1.25$ . For the polynomial  $A_4(z)$  the values used for Theorem 3 were  $p = 0.61$ ,  $q = 1.30$ , and for Theorem 4 we have taken  $p = 2.16$ ,  $q = 0.52$ ,  $r = 1.08$  and  $s = 0.32$ .

*Remark:* We point out that the lower bounds in Theorem A was obtained computing the upper bounds of the corresponding self-inversive polynomials  $A_i^*(z)$ .

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