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# WEIGHTED GENERALIZATION OF SOME INEQUALITIES FOR DIFFERENTIABLE CO-ORDINATED CONVEX FUNCTIONS WITH APPLICATIONS

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ABSTRACT. In this paper, a new weighted identity for functions defined on a rectangle from the plane is established. By using the obtained identity and analysis, some new weighted integral inequalities for the classes of co-ordinated convex, co-ordinated wright-convex and co-ordinated quasi-convex functions on the rectangle from the plane are established which provide weighted generalization of some recent results proved for co-ordinated convex functions. Some applications of our results to random variables and $2D$ weighted quadrature formula are given as well.

## 1. INTRODUCTION

The following definition is well known in mathematical analysis:

A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on $I$ if the inequality

$$f(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) f(y),$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$.

A number of results have been established for the class of convex functions but the most famous is the Hermite-Hadamard's inequality (see for instance [7]). This double inequality is stated as:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$ a convex function, $a, b \in I$ with $a < b$. The inequalities in (1.1) are reversed if $f$ is a concave function.

The inequalities (1.1) have become an important cornerstone in mathematical analysis and optimization and many uses of these inequalities have been discovered in a variety of settings. Moreover, many inequalities of special means can be obtained for a particular choice of the function $f$. Due to the rich geometrical significance of Hermite-Hadamard's inequality (1.1), there is growing literature providing its new proofs, extensions, refinements and generalizations, see for example [2, 4, 5, 6, 9, 21, 22] and the references therein.

Let us consider now a bidimensional interval $[a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$.

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A mapping $f:[a,b]\times[c,d]\rightarrow\mathbb{R}$ is said to be convex on $[a,b]\times[c,d]$ if the inequality

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\leq\lambda f(x,y)+(1-\lambda)f(z,w)$$

holds for all $(x,y),(z,w)\in[a,b]\times[c,d]$ and $\lambda\in[0,1]$ .

A modification for convex functions on $[a,b]\times[c,d]$, which are also known as co-ordinated convex functions, was initiated by Dragomir [4, 6] as follows:

A function $f:[a,b]\times[c,d]\rightarrow\mathbb{R}$ is said to be convex on the co-ordinates on $[a,b]\times[c,d]$ if the partial mappings $f_y:[a,b]\rightarrow\mathbb{R}, f_y(u)=f(u,y)$ and $f_x:[c,d]\rightarrow\mathbb{R}, f_x(v)=f(x,v)$ are convex where defined for all $x\in[a,b], y\in[c,d]$.

A formal definition for co-ordinated convex functions may be stated as follows:

**Definition 1.** [13] *A function $f:[a,b]\times[c,d]\rightarrow\mathbb{R}$ is said to be convex on the co-ordinates on $[a,b]\times[c,d]$ if the inequality*

$$\begin{aligned}&f(tx+(1-t)y,sz+(1-s)w)\\&\leq tsf(x,z)+t(1-s)f(x,w)+s(1-t)f(y,z)+(1-t)(1-s)f(y,w)\end{aligned}$$

*holds for all $(t,s)\in[0,1]\times[0,1]$ and $(x,z),(y,w)\in[a,b]\times[c,d]$.*

It has been proved in [4] that every convex mapping $f:[a,b]\times[c,d]\rightarrow\mathbb{R}$ is convex on the co-ordinates. Furthermore, there exists co-ordinated convex function which is not convex, (see for example [4, 6]).

The following Hermite-Hadamard type inequality for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$ was also proved in [4]:

**Theorem 1.** [4] *Suppose that $f:[a,b]\times[c,d]\rightarrow\mathbb{R}$ is co-ordinated convex on $[a,b]\times[c,d]$. Then one has the inequalities:*

$$\begin{aligned}(1.2)\quad &f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\&\leq\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\&\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\\&\leq\frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.\\&\left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\\&\leq\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.\end{aligned}$$

*The above inequalities are sharp.*

Sarikaya et al. [23], proved the following Hermite-Hadamard type inequalities.

**Theorem 2.** [23] *Let $f:[a,b]\times[c,d]\subset\mathbb{R}^2\rightarrow\mathbb{R}$ be a partial differentiable mapping on $[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$, $c<d$. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ is convex on the co-ordinates on*

$[a,b]\times[c,d]$, *then one has the inequalities:*

$$\begin{aligned}(1.3)\quad &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.\\ &-\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.\\ &\left.\left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\right|\\ &\leq\frac{(b-a)(d-c)}{16}\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right].\end{aligned}$$

The next two results from [23] involve powers of the absolute value of $\frac{\partial^2 f}{\partial s\partial t}$.

**Theorem 3.** [23] *Let* $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ *be a partial differentiable mapping on* $[a,b]\times[c,d]$ *in* $\mathbb{R}^2$ *with* $a<b$, $c<d$. *If* $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$, $q\geq 1$, *is convex on the co-ordinates on* $[a,b]\times[c,d]$, *then one has the inequalities:*

$$\begin{aligned}(1.4)\quad &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.\\ &-\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.\\ &\left.\left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\right|\\ &\leq\frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\left[\frac{\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q}{4}\right]^{\frac{1}{q}},\end{aligned}$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

**Theorem 4.** [23] *Let* $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ *be a partial differentiable mapping on* $[a,b]\times[c,d]$ *in* $\mathbb{R}^2$ *with* $a<b$, $c<d$. *If* $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$, $q>1$, *is convex on the co-ordinates on* $[a,b]\times[c,d]$, *then one has the inequalities:*

$$\begin{aligned}(1.5)\quad &\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx+\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.\\ &-\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.\\ &\left.\left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\right|\\ &\leq\frac{(b-a)(d-c)}{16}\left[\frac{\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q}{4}\right]^{\frac{1}{q}}.\end{aligned}$$

In a recent paper [22], M. E. Özdemir et al. give the notion of co-ordinated quasi-convex functions which generalize the notion of co-ordinated convex functions.

**Definition 2.** [20] *A function $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ is said to be quasi-convex on $[a,b]\times[c,d]$ if the inequality*

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\leq\max\{f(x,y),f(z,w)\}$$

*holds for all $(x,y),(z,w)\in[a,b]\times[c,d]$ and $\lambda\in[0,1]$ .*

A function $f:[a,b]\times[c,d]\to\mathbb{R}$ is said to be quasi-convex on the co-ordinates on $[a,b]\times[c,d]$ if the partial mappings $f_y:[a,b]\to\mathbb{R}, f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}, f_x(v)=f(x,v)$ are quasi-convex where defined for all $x\in[a,b], y\in[c,d]$.

Another way of dedscribing the definition of co-ordinated quasi-convex functions is given below.

**Definition 3.** [16] *A function $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ is said to be quasi-convex on the co-ordinates on $[a,b]\times[c,d]$ if*

$$f(tx+(1-t)z,sy+(1-s)w)\leq\max\{f(x,y),f(x,w),f(z,y),f(z,w)\}$$

*for all $(x,y),(z,w)\in[a,b]\times[c,d]$ and $(s,t)\in[0,1]\times[0,1]$ .*

The class of co-ordinated quasi-convex functions on $[a,b]\times[c,d]$ is denoted by $QC([a,b]\times[c,d])$. It has also been proved in [20] that every quasi-convex functions on $[a,b]\times[c,d]$ is quasi-convex on the co-ordinates on $[a,b]\times[c,d]$. The following example reveals that there exists quasi-convex function on the co-ordinates which is not quasi-convex.

**Example 1.** [16] *The function $f:[-2,2]^2\to\mathbb{R}$, defined by $f(x,y)=\lfloor x\rfloor\lfloor y\rfloor$, where $\lfloor.\rfloor$ is the floor function. This function is quasi-convex on the co-ordinates on $[-2,2]^2$ but is not quasi-convex on $[0,1]^2$.*
*For example, take $(x,y)=(-2,1)$, $(z,w)=(1,-1)$ and $\lambda=\frac{1}{2}$, then*

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)=f\left(-\frac{1}{2},0\right)=0,$$

*on the other hand*

$$\max\{f(x,y),f(z,w)\}=\max\{f(-2,1),f(1,-1)\}=-1,$$

*which shows that $f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)>\max\{f(x,y),f(z,w)\}$.*

Another generalization of the notion of the co-ordinated convex functions is the concept of wright-convex functions which is given in the definition below.

**Definition 4.** [20] *A function $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ is said to be wright-convex on $[a,b]\times[c,d]$ if the inequality*

$$\begin{aligned}&f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)+f((1-\lambda)x+\lambda z,(1-\lambda)y+\lambda w)\\&\leq\max\{f(x,z),f(y,w)\},\end{aligned}$$

*holds for all $(x,z),(y,w)\in[a,b]\times[c,d]$ and $\lambda\in[0,1]$ .*

A function $f:[a,b]\times[c,d]\to\mathbb{R}$ is said to be wright-convex on the co-ordinates on $[a,b]\times[c,d]$ if the partial mappings $f_y:[a,b]\to\mathbb{R}, f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}, f_x(v)=f(x,v)$ are wright-convex where defined for all $x\in[a,b], y\in[c,d]$.

The above definition of wright-convex functions on the co-ordinates can be reformulated as follows.

**Definition 5.** [20] *A function $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ is said to be wright-convex on the co-ordinates on $[a,b]\times[c,d]$ if*

$$\begin{aligned}&f(tx+(1-t)z,sy+(1-s)w)+f((1-t)x+tz,(1-s)y+sw)\\&\leq f(x,y)+f(z,y)+f(x,w)+f(z,w)\end{aligned}$$

*for all $(x,z),(y,w)\in[a,b]\times[c,d]$ and $(s,t)\in[0,1]\times[0,1]$ .*

The class of co-ordinated wright-convex functions on $[a,b]\times[c,d]$ is represented by $W([a,b]\times[c,d])$. It has also been proved in [20] that every wright-convex functions on $[a,b]\times[c,d]$ is wright-convex on the co-ordinates on $[a,b]\times[c,d]$.

For more recent results on co-ordinated convex, co-ordinated quasi-convex, co-ordinated $m$-convex, co-ordinated $(\alpha\ ,m)$-convex and co-ordinated $s$-convex functions on a rectangle $[a,b]\times[c,d]$ from the plane $\mathbb{R}^2$, we refer the readers to [1, 5, 8], [10]-[20].

In the present paper, we establish a new weighted identity for differentiable mappings defined on a rectangle $[a,b]\times[c,d]$ from the plane $\mathbb{R}^2$ and by using the obtained identity and analysis, some new weighted integral inequalities for differentiable co-ordinated convex, co-ordinated wright-convex and co-ordinated quasi convex functions are proved. The results proved in the paper provide a weighted generalization of the results given in Theorem 2, Theorem 3 and Theorem 4. Applications of our results to random variables and $2D$ weighted quadrature formula are provided as well.

## 2. Main Results

We need the following lemma to prove our results.

**Lemma 1.** *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a twice differentiable mapping on $\Delta^\circ$ and $p:[a,b]\times[c,d]\to[0,\infty)$ be continuous and symmetric to $\frac{a+b}{2}$ and $\frac{c+d}{2}$ for $[a,b]\times[c,d]\subset\Delta^\circ$ with $a<b$, $c<d$. If $\frac{\partial^2 f}{\partial s\partial t}\in L([a,b]\times[c,d])$, then*

$$\begin{aligned}&\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_c^d\int_a^b p(x,y)\,dxdy \qquad (2.1)\\&-\frac{1}{2}\int_c^d\int_a^b[f(x,c)+f(x,d)]\,p(x,y)\,dxdy\\&-\frac{1}{2}\int_c^d\int_a^b[f(a,y)+f(b,y)]\,p(x,y)\,dxdy+\int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy\\&=\frac{(b-a)(d-c)}{16}\int_0^1\int_0^1\left[\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)}p(x,y)\,dxdy\right]\left[\frac{\partial^2}{\partial s\partial t}f(U_1(t),U_2(s))\right.\\&\left.-\frac{\partial^2}{\partial s\partial t}f(U_1(t),L_2(s))-\frac{\partial^2}{\partial s\partial t}f(L_1(t),U_2(s))+\frac{\partial^2}{\partial s\partial t}f(L_1(t),L_2(s))\right]dsdt,\end{aligned}$$

*where*

$$\begin{aligned}U_1(t)&=\frac{1-t}{2}a+\frac{1+t}{2}b,L_1(t)=\frac{1+t}{2}a+\frac{1-t}{2}b,\\U_2(s)&=\frac{1-s}{2}c+\frac{1+s}{2}d,L_2(s)=\frac{1+s}{2}c+\frac{1-s}{2}d.\end{aligned}$$

*Proof.* Let

$$I = \frac{(b-a)(d-c)}{16}\int_0^1\int_0^1\left[\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdy\right]\left[\frac{\partial^2}{\partial s\partial t}f(U_1(t),U_2(s))\right.$$
$$\left. -\frac{\partial^2}{\partial s\partial t}f(U_1(t),L_2(s)) - \frac{\partial^2}{\partial s\partial t}f(L_1(t),U_2(s)) + \frac{\partial^2}{\partial s\partial t}f(L_1(t),L_2(s))\right]dsdt$$

and

$$\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdy = q(t,s).$$

then

$$I = \frac{(b-a)(d-c)}{16}\int_0^1\int_0^1 q(t,s)\left[\frac{\partial^2}{\partial s\partial t}f(U_1(t),U_2(s)) - \frac{\partial^2}{\partial s\partial t}f(U_1(t),L_2(s))\right.$$
$$\left. -\frac{\partial^2}{\partial s\partial t}f(L_1(t),U_2(s)) + \frac{\partial^2}{\partial s\partial t}f(L_1(t),L_2(s))\right]dsdt.$$

Now by integration by parts and by using the symmetry of $p(x,y)$ about $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$, we have

$$\frac{(b-a)(d-c)}{16}\int_0^1\int_0^1 q(t,s)\frac{\partial^2}{\partial s\partial t}f(U_1(t),U_2(s))\,dsdt \tag{2.2}$$
$$= \frac{(b-a)(d-c)}{16}\int_0^1\left[\int_0^1 q(t,s)\frac{\partial^2}{\partial s\partial t}f(U_1(t),U_2(s))\,ds\right]dt$$
$$= \frac{(b-a)(d-c)}{16}\int_0^1\left[\frac{2}{d-c}q(t,s)\frac{\partial}{\partial t}f(U_1(t),U_2(s))\Big|_0^1\right.$$
$$\left. -\frac{2}{d-c}\int_0^1\frac{\partial}{\partial s}q(t,s)\frac{\partial}{\partial t}f(U_1(t),U_2(s))\,ds\right]dt$$
$$= \frac{(b-a)}{8}\int_0^1\left[\frac{\partial}{\partial t}f(U_1(t),d)\left(\int_c^d\int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdy\right)\right.$$
$$\left. -(d-c)\int_0^1\left(\int_{L_1(t)}^{U_1(t)} p(x,U_2(s))\,dx\right)\frac{\partial}{\partial t}f(U_1(t),U_2(s))\,ds\right]dt$$
$$= \frac{(b-a)}{8}\int_0^1\frac{\partial}{\partial t}f(U_1(t),d)\left(\int_c^d\int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdy\right)dt$$
$$-\frac{(b-a)}{4}\int_{\frac{c+d}{2}}^d\int_0^1\left(\int_{L_1(t)}^{U_1(t)} p(x,y)\,dx\right)\frac{\partial}{\partial t}f(U_1(t),y)\,dtdy$$
$$= \frac{1}{4}f(b,d)\int_c^d\int_a^b p(x,y)\,dxdy - \frac{1}{2}\int_c^d\int_{\frac{a+b}{2}}^b p(x,y)f(x,d)\,dxdy$$
$$-\frac{1}{2}\int_{\frac{c+d}{2}}^d\int_a^b p(x,y)f(b,y)\,dxdy + \int_{\frac{c+d}{2}}^d\int_{\frac{a+b}{2}}^b p(x,y)f(x,y)\,dxdy.$$

Similarly, we have

$$\begin{aligned}(2.3)\quad & -\frac{(b-a)(d-c)}{16}\int_0^1\int_0^1 q(t,s)\frac{\partial^2}{\partial s\partial t}f(U_1(t),L_2(s))\,dsdt \\ &= \frac{1}{4}f(b,c)\int_c^d\int_a^b p(x,y)\,dxdy-\frac{1}{2}\int_c^d\int_{\frac{a+b}{2}}^b p(x,y)f(x,c)\,dxdy \\ &\quad -\frac{1}{2}\int_c^{\frac{c+d}{2}}\int_a^b p(x,y)f(b,y)\,dxdy+\int_c^{\frac{c+d}{2}}\int_{\frac{a+b}{2}}^b p(x,y)f(x,y)\,dxdy,\end{aligned}$$

$$\begin{aligned}(2.4)\quad & -\frac{(b-a)(d-c)}{16}\int_0^1\int_0^1 q(t,s)\frac{\partial^2}{\partial s\partial t}f(L_1(t),U_2(s))\,dsdt \\ &= \frac{1}{4}f(a,d)\int_c^d\int_a^b p(x,y)\,dxdy-\frac{1}{2}\int_c^d\int_a^{\frac{a+b}{2}} p(x,y)f(x,d)\,dxdy \\ &\quad -\frac{1}{2}\int_{\frac{c+d}{2}}^d\int_a^b p(x,y)f(a,y)\,dxdy+\int_{\frac{c+d}{2}}^d\int_a^{\frac{a+b}{2}} p(x,y)f(x,y)\,dxdy\end{aligned}$$

and

$$\begin{aligned}(2.5)\quad & \frac{(b-a)(d-c)}{16}\int_0^1\int_0^1 q(t,s)\frac{\partial^2}{\partial s\partial t}f(L_1(t),L_2(s))\,dsdt \\ &= \frac{1}{4}f(a,c)\int_c^d\int_a^b p(x,y)\,dxdy-\frac{1}{2}\int_c^d\int_a^{\frac{a+b}{2}} p(x,y)f(x,c)\,dxdy \\ &\quad -\frac{1}{2}\int_c^{\frac{c+d}{2}}\int_a^b p(x,y)f(a,y)\,dxdy+\int_c^{\frac{c+d}{2}}\int_a^{\frac{a+b}{2}} p(x,y)f(x,y)\,dxdy.\end{aligned}$$

Adding (2.2)-(2.5), we get the desired result. □

**Remark 1.** *If we take $p(x,y)=\frac{1}{(b-a)(d-c)}$ for all $(x,y)\in[a,b]\times[c,d]$ in Lemma 1, we get Lemma 1 from* [23, page 139].

Now by using lemma 1, we present the main results of this section.

**Theorem 5.** *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a twice differentiable mapping on $\Delta^\circ$ and $p:[a,b]\times[c,d]\to[0,\infty)$ be continuous and symmetric to $\frac{a+b}{2}$ and $\frac{c+d}{2}$ for $[a,b]\times[c,d]\subset\Delta^\circ$ with $a<b$, $c<d$. If $\frac{\partial^2 f}{\partial s\partial t}\in L([a,b]\times[c,d])$ and $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ is convex on the*

*co-ordinates on* $[a,b]\times[c,d]$, *then*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_c^d\int_a^b p(x,y)\,dxdy\right.$$
$$-\frac{1}{2}\int_c^d\int_a^b[f(x,c)+f(x,d)]\,p(x,y)\,dxdy$$
$$\left.-\frac{1}{2}\int_c^d\int_a^b[f(a,y)+f(b,y)]\,p(x,y)\,dxdy+\int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy\right|$$
$$\leq\frac{(b-a)(d-c)}{16}\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right]$$
$$\times\int_0^1\int_0^1\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)}p(x,y)\,dxdydtds. \tag{2.6}$$

*Proof.* Taking absolute value on both sides of (2.1) and using the properties of absolute value, we have

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_c^d\int_a^b p(x,y)\,dxdy\right.$$
$$-\frac{1}{2}\int_c^d\int_a^b[f(x,c)+f(x,d)]\,p(x,y)\,dxdy$$
$$\left.-\frac{1}{2}\int_c^d\int_a^b[f(a,y)+f(b,y)]\,p(x,y)\,dxdy+\int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy\right|$$
$$\leq\frac{(b-a)(d-c)}{16}\int_0^1\int_0^1\left[\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)}p(x,y)\,dxdy\right]\left[\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),U_2(s))\right|\right.$$
$$\left.+\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),L_2(s))\right|+\left|\frac{\partial^2}{\partial s\partial t}f(L_1(t),U_2(s))\right|+\left|\frac{\partial^2}{\partial s\partial t}f(L_1(t),L_2(s))\right|\right]dsdt. \tag{2.7}$$

By the convexity of $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ on the co-ordinates on $[a,b]\times[c,d]$, we have

$$\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),U_2(s))\right|$$
$$\leq\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|$$
$$+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|, \tag{2.8}$$

$$\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),L_2(s))\right|$$
$$\leq\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|$$
$$+\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|, \tag{2.9}$$

$$
\begin{aligned}
(2.10)\quad & \left|\frac{\partial^2}{\partial s\partial t} f\left(L_1(t), U_2(s)\right)\right| \\
& \leq \left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right| + \left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right| \\
& \quad + \left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right| + \left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|,
\end{aligned}
$$

and

$$
\begin{aligned}
(2.11)\quad & \left|\frac{\partial^2}{\partial s\partial t} f\left(L_1(t), L_2(s)\right)\right| \\
& \leq \left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right| + \left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right| \\
& \quad + \left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right| + \left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|.
\end{aligned}
$$

Using (2.8)-(2.11) in (2.7), we get (2.6). □

**Remark 2.** *If we take* $p(x,y) = \frac{1}{(b-a)(d-c)}$ *for all* $(x,y) \in [a,b]\times[c,d]$ *in Theorem 5, we get Theorem 2 from* [23].

A more general result is given in the following theorem.

**Theorem 6.** *Let* $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ *be a twice differentiable mapping on* $\Delta^\circ$ *and* $p : [a,b]\times[c,d] \to [0,\infty)$ *be continuous and symmetric to* $\frac{a+b}{2}$ *and* $\frac{c+d}{2}$ *for* $[a,b]\times[c,d] \subset \Delta^\circ$ *with* $a<b$, $c<d$. *If* $\frac{\partial^2 f}{\partial s\partial t} \in L([a,b]\times[c,d])$ *and* $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ *is convex on the co-ordinates on* $[a,b]\times[c,d]$ *for* $q\geq 1$, *then*

$$
\begin{aligned}
(2.12)\quad & \left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_c^d\int_a^b p(x,y)\,dxdy\right. \\
& -\frac{1}{2}\int_c^d\int_a^b [f(x,c)+f(x,d)]\,p(x,y)\,dxdy \\
& \left.-\frac{1}{2}\int_c^d\int_a^b [f(a,y)+f(b,y)]\,p(x,y)\,dxdy + \int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy\right| \\
& \leq \frac{(b-a)(d-c)}{4}\left[\frac{\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q}{4}\right]^{\frac{1}{q}} \\
& \quad \times \int_0^1\int_0^1\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdydsdt.
\end{aligned}
$$

*Proof.* Taking absolute value on both sides of (2.1), by using the properties of absolute value and the Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_c^d\int_a^b p(x,y)\,dxdy\right.\\
&\qquad -\frac{1}{2}\int_c^d\int_a^b [f(x,c)+f(x,d)]\,p(x,y)\,dxdy\\
&\left.-\frac{1}{2}\int_c^d\int_a^b [f(a,y)+f(b,y)]\,p(x,y)\,dxdy+\int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy\right|\\
&\leq \frac{(b-a)(d-c)}{16}\left(\int_0^1\int_0^1\left[\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdy\right]dsdt\right)^{1-\frac{1}{q}}\\
&\times\left[\left(\int_0^1\int_0^1\left[\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdy\right]\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),U_2(s))\right|^q dsdt\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^1\int_0^1\left[\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdy\right]\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),L_2(s))\right|^q dsdt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\int_0^1\left[\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdy\right]\left|\frac{\partial^2}{\partial s\partial t}f(L_1(t),U_2(s))\right|^q dsdt\right)^{\frac{1}{q}}\\
&\left.+\left(\int_0^1\int_0^1\left[\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdy\right]\left|\frac{\partial^2}{\partial s\partial t}f(L_1(t),L_2(s))\right|^q dsdt\right)^{\frac{1}{q}}\right].
\end{aligned}\tag{2.13}$$

By the power-mean inequality $(a_1^r+a_2^r+a_3^r+a_4^r\leq 4^{1-r}(a_1+a_2+a_3+a_4)^r$ for $a_1$, $a_2$, $a_3$, $a_4>0$ and $r<1)$ and using the convexity of $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ on the co-ordinates on $[a,b]\times[c,d]$ for $q\geq 1$, we have

$$\begin{aligned}
&\left(\int_0^1\int_0^1\left[\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdy\right]\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),U_2(s))\right|^q dsdt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\int_0^1\left[\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdy\right]\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),L_2(s))\right|^q dsdt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\int_0^1\left[\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdy\right]\left|\frac{\partial^2}{\partial s\partial t}f(L_1(t),U_2(s))\right|^q dsdt\right)^{\frac{1}{q}}\\
&\left.+\left(\int_0^1\int_0^1\left[\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdy\right]\left|\frac{\partial^2}{\partial s\partial t}f(L_1(t),L_2(s))\right|^q dsdt\right)^{\frac{1}{q}}\right]\\
&\qquad\leq 4^{1-\frac{1}{q}}\left\{\int_0^1\int_0^1\left(\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdy\right)\right.
\end{aligned}\tag{2.14}$$

$$\times\left[\left|\frac{\partial^2}{\partial s\partial t}f\left(U_1\left(t\right),U_2\left(s\right)\right)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f\left(U_1\left(t\right),L_2\left(s\right)\right)\right|^q\right.$$
$$\left.\left.+\left|\frac{\partial^2}{\partial s\partial t}f\left(L_1\left(t\right),U_2\left(s\right)\right)\right|^q+\left|\frac{\partial^2}{\partial s\partial t}f\left(L_1\left(t\right),L_2\left(s\right)\right)\right|^q\right]dsdt\right\}^{\frac{1}{q}}$$
$$\leq 4^{1-\frac{1}{q}}\left[\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,c\right)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,d\right)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,c\right)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,d\right)\right|\right]^{\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)}p\left(x,y\right)dxdydsdt\right)^{\frac{1}{q}}.$$

A usage of (2.14) in (2.13) yields the desired result. □

**Remark 3.** *If we take* $p\left(x,y\right)=\frac{1}{(b-a)(d-c)}$ *for all* $(x,y)\in[a,b]\times[c,d]$ *in Theorem 6, we get Theorem 4.*

A different approach leads to the following result.

**Theorem 7.** *Let* $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ *be a twice differentiable mapping on* $\Delta^\circ$ *and* $p:[a,b]\times[c,d]\to[0,\infty)$ *be continuous and symmetric to* $\frac{a+b}{2}$ *and* $\frac{c+d}{2}$ *for* $[a,b]\times[c,d]\subset\Delta^\circ$ *with* $a<b$, $c<d$. *If* $\frac{\partial^2 f}{\partial s\partial t}\in L\left([a,b]\times[c,d]\right)$ *and* $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ *is convex on the co-ordinates on* $[a,b]\times[c,d]$ *for* $q>1$, *then*

$$(2.15)\qquad \left|\frac{f\left(a,c\right)+f\left(a,d\right)+f\left(b,c\right)+f\left(b,d\right)}{4}\int_c^d\int_a^b p\left(x,y\right)dxdy\right.$$
$$-\frac{1}{2}\int_c^d\int_a^b\left[f\left(x,c\right)+f\left(x,d\right)\right]p\left(x,y\right)dxdy$$
$$\left.-\frac{1}{2}\int_c^d\int_a^b\left[f\left(a,y\right)+f\left(b,y\right)\right]p\left(x,y\right)dxdy+\int_c^d\int_a^b f\left(x,y\right)p\left(x,y\right)dxdy\right|$$
$$\leq\frac{\left(b-a\right)\left(d-c\right)}{4}\left[\frac{\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,c\right)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,d\right)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,c\right)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,d\right)\right|^q}{4}\right]^{\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1\left[\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)}p\left(x,y\right)dxdy\right]^p dsdt\right)^{\frac{1}{p}},$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* From Lemma 1 and the Hölder inequality, we have

$$\begin{aligned}(2.16)\quad &\Bigg|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_c^d\int_a^b p(x,y)\,dxdy \\ &-\frac{1}{2}\int_c^d\int_a^b [f(x,c)+f(x,d)]\,p(x,y)\,dxdy \\ &-\frac{1}{2}\int_c^d\int_a^b [f(a,y)+f(b,y)]\,p(x,y)\,dxdy+\int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy\Bigg| \\ &\le \frac{(b-a)(d-c)}{16}\left(\int_0^1\int_0^1\left[\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdy\right]^p dsdt\right)^{\frac{1}{p}} \\ &\times\Bigg[\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),U_2(s))\right|^q dsdt\right)^{\frac{1}{q}} \\ &+\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),L_2(s))\right|^q dsdt\right)^{\frac{1}{q}} \\ &+\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(L_1(t),U_2(s))\right|^q dsdt\right)^{\frac{1}{q}} \\ &+\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(L_1(t),L_2(s))\right|^q dsdt\right)^{\frac{1}{q}}\Bigg].\end{aligned}$$

By the power-mean inequality $(a_1^r+a_2^r+a_3^r+a_4^r\le 4^{1-r}(a_1+a_2+a_3+a_4)^r$ for $a_1$, $a_2$, $a_3$, $a_4>0$ and $r<1)$ and using the convexity of $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ on the co-ordinates on $[a,b]\times[c,d]$ for $q>1$, we have

(2.17)

$$\begin{aligned}&\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),U_2(s))\right|^q dsdt\right)^{\frac{1}{q}}+\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),L_2(s))\right|^q dsdt\right)^{\frac{1}{q}} \\ &+\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(L_1(t),U_2(s))\right|^q dsdt\right)^{\frac{1}{q}}+\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(L_1(t),L_2(s))\right|^q dsdt\right)^{\frac{1}{q}} \\ &\le 4^{1-\frac{1}{q}}\Bigg[\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),U_2(s))\right|^q dsdt \\ &+\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),L_2(s))\right|^q dsdt+\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(L_1(t),U_2(s))\right|^q dsdt \\ &+\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(L_1(t),L_2(s))\right|^q dsdt\Bigg]^{\frac{1}{q}} \\ &\le 4\left[\frac{\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q}{4}\right]^{\frac{1}{q}}.\end{aligned}$$

From (2.16) and (2.17), we get (2.15). □

**Remark 4.** *If we take $p(x,y)=\frac{1}{(b-a)(d-c)}$ for all $(x,y)\in[a,b]\times[c,d]$ in Theorem 7, we get Theorem 3.*

**Remark 5.** *Theorem 5-Theorem 7 continue to hold true if in their statements we replace the condition "convex on the co-ordinates" with the condition "wright-convex on the co-ordinates". However, the details are left to the interested reader.*

In what follows we give our results for the quasi-convex mappings on the co-ordinates on $[a,b]\times[c,d]$.

**Theorem 8.** *Suppose the assumptions of Theorem 5 are satisfied. If the mapping $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ is quasi-convex on the co-ordinates on $[a,b]\times[c,d]$, then the following inequality holds*

$$\begin{aligned}(2.18)\quad &\Bigg|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_c^d\int_a^b p(x,y)\,dxdy\\ &-\frac{1}{2}\int_c^d\int_a^b[f(x,c)+f(x,d)]\,p(x,y)\,dxdy\\ &-\frac{1}{2}\int_c^d\int_a^b[f(a,y)+f(b,y)]\,p(x,y)\,dxdy+\int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy\Bigg|\\ &\le\frac{(b-a)(d-c)}{16}\Bigg[\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|,\right.\\ &\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}\\ &+\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|,\right.\\ &\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}\\ &+\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|,\right.\\ &\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}\\ &+\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|,\right.\\ &\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}\Bigg]\int_0^1\int_0^1\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)}p(x,y)\,dxdydtds.\end{aligned}$$

*Proof.* We continue inequality (2.7) in the proof of Theorem 1. Now, by the quasi-convexity on the co-ordinates of $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$on $[a,b]\times[c,d]$ , we obtain

$$\begin{aligned}(2.19)\quad \left|\frac{\partial^2}{\partial s\partial t}f(U_1(t),U_2(s))\right|&\le\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|,\right.\\ &\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\},\end{aligned}$$

$$\left|\frac{\partial^2}{\partial s\partial t}f\left(L_1\left(t\right),U_2\left(s\right)\right)\right|\leq\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,d\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|,\right.$$
$$\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\},\tag{2.20}$$

$$\left|\frac{\partial^2}{\partial s\partial t}f\left(U_1\left(t\right),L_2\left(s\right)\right)\right|\leq\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,c\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|,\right.$$
$$\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\},\tag{2.21}$$

and

$$\left|\frac{\partial^2}{\partial s\partial t}f\left(L_1\left(t\right),L_2\left(s\right)\right)\right|\leq\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,c\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|,\right.$$
$$\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\},\tag{2.22}$$

for all $(t,s)\in[0,1]\times[0,1]$. A combination of (2.19)-(2.22) and (2.7)gives the required inequality (2.18). □

**Corollary 1.** *Suppose the assumptions of Theorem 8 are fulfilled and if* $p(x,y)=\frac{1}{(b-a)(d-c)}$ *for all* $(x,y)\in[a,b]\times[c,d]$*, then the following inequality holds valid*

$$\left|\frac{f\left(a,c\right)+f\left(a,d\right)+f\left(b,c\right)+f\left(b,d\right)}{4}\right.$$
$$-\frac{1}{2\left(b-a\right)}\int_a^b\left[f\left(x,c\right)+f\left(x,d\right)\right]dx$$
$$\left.-\frac{1}{2\left(d-c\right)}\int_c^d\left[f\left(a,y\right)+f\left(b,y\right)\right]dy+\frac{1}{\left(b-a\right)\left(d-c\right)}\int_c^d\int_a^b f\left(x,y\right)dxdy\right|$$
$$\leq\frac{\left(b-a\right)\left(d-c\right)}{16}\left[\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,d\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|,\right.\right.$$
$$\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}$$
$$+\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,d\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|,\right.$$
$$\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}$$
$$+\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,c\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|,\right.$$
$$\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}$$
$$+\max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,c\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|,\right.$$
$$\left.\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right\}\right].\tag{2.23}$$

**Corollary 2.** *Suppose the assumptions of Theorem 8 are satisfied and additionally*

(1) *If $f$ is non-decreasing on the co-ordinates on $[a,b]\times[c,d]$, then the following inequality holds true*

$$\begin{aligned}(2.24)\quad &\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_c^d\int_a^b p(x,y)\,dxdy\right.\\ &-\frac{1}{2}\int_c^d\int_a^b [f(x,c)+f(x,d)]\,p(x,y)\,dxdy\\ &\left.-\frac{1}{2}\int_c^d\int_a^b [f(a,y)+f(b,y)]\,p(x,y)\,dxdy+\int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy\right|\\ &\le \frac{(b-a)(d-c)}{16}\left[\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|\right.\\ &\left.+\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right]\int_0^1\int_0^1\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdydtds.\end{aligned}$$

(2) *If $f$ is non-increasing on the co-ordinates on $[a,b]\times[c,d]$, then the following inequality holds true*

$$\begin{aligned}(2.25)\quad &\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_c^d\int_a^b p(x,y)\,dxdy\right.\\ &-\frac{1}{2}\int_c^d\int_a^b [f(x,c)+f(x,d)]\,p(x,y)\,dxdy\\ &\left.-\frac{1}{2}\int_c^d\int_a^b [f(a,y)+f(b,y)]\,p(x,y)\,dxdy+\int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy\right|\\ &\le \frac{(b-a)(d-c)}{16}\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|\right.\\ &\left.+\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right]\int_0^1\int_0^1\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdydtds.\end{aligned}$$

**Corollary 3.** *If we take $p(x,y)=\frac{1}{(b-a)(d-c)}$ for all $(x,y)\in[a,b]\times[c,d]$ in Corollary 2 and additionally*

(1) *If $f$ is non-decreasing on the co-ordinates on $[a,b]\times[c,d]$, then the following inequality holds true*

$$\begin{aligned}(2.26)\quad &\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.\\ &-\frac{1}{2(b-a)}\int_a^b [f(x,c)+f(x,d)]\,dx-\frac{1}{2(d-c)}\int_c^d [f(a,y)+f(b,y)]\,dy\\ &\left.+\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,dxdy\right|\le\frac{(b-a)(d-c)}{16}\\ &\times\left[\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right].\end{aligned}$$

(2) *If $f$ is non-increasing on the co-ordinates on $[a,b]\times[c,d]$, then the following inequality holds true*

$$\begin{aligned}(2.27)\quad &\Bigg|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\\ &-\frac{1}{2(b-a)}\int_a^b [f(x,c)+f(x,d)]\,dx-\frac{1}{2(d-c)}\int_c^d [f(a,y)+f(b,y)]\,dy\\ &+\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,dxdy\Bigg|\le\frac{(b-a)(d-c)}{16}\\ &\times\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\right].\end{aligned}$$

**Theorem 9.** *Suppose the assumptions of Theorem 5 are satisfied. If the mapping $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is quasi-convex on the co-ordinates on $[a,b]\times[c,d]$ for $q\ge 1$, then the following inequality holds*

$$\begin{aligned}(2.28)\quad &\Bigg|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_c^d\int_a^b p(x,y)\,dxdy\\ &-\frac{1}{2}\int_c^d\int_a^b [f(x,c)+f(x,d)]\,p(x,y)\,dxdy\\ &-\frac{1}{2}\int_c^d\int_a^b [f(a,y)+f(b,y)]\,p(x,y)\,dxdy+\int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy\Bigg|\\ &\le\frac{(b-a)(d-c)}{16}\Bigg[\Bigg(\max\Bigg\{\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|^q,\\ &\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\Bigg\}\Bigg)^{\frac{1}{q}}\\ &+\Bigg(\max\Bigg\{\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|^q,\\ &\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\Bigg\}\Bigg)^{\frac{1}{q}}\\ &+\Bigg(\max\Bigg\{\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|^q,\\ &\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\Bigg\}\Bigg)^{\frac{1}{q}}\\ &+\Bigg(\max\Bigg\{\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|^q,\\ &\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\Bigg\}\Bigg)^{\frac{1}{q}}\Bigg]\\ &\times\int_0^1\int_0^1\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdydtds.\end{aligned}$$

*Proof.* We continue inequality (2.13) in the proof of Theorem **??**. Now, by the quasi-convexity on the co-ordinates of $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ on $[a,b]\times[c,d]$ for $q\geq 1$ and the power-mean inequality, we obtain

$$\left|\frac{\partial^2}{\partial s\partial t}f\left(U_1\left(t\right),U_2\left(s\right)\right)\right|^q \leq \max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,d\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|^q,\right.$$
$$\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\}, \tag{2.29}$$

$$\left|\frac{\partial^2}{\partial s\partial t}f\left(L_1\left(t\right),U_2\left(s\right)\right)\right|^q \leq \max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,d\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|^q,\right.$$
$$\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},d\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\}, \tag{2.30}$$

$$\left|\frac{\partial^2}{\partial s\partial t}f\left(U_1\left(t\right),L_2\left(s\right)\right)\right|^q \leq \max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,c\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(b,\frac{c+d}{2}\right)\right|^q,\right.$$
$$\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\}, \tag{2.31}$$

and

$$\left|\frac{\partial^2}{\partial s\partial t}f\left(L_1\left(t\right),L_2\left(s\right)\right)\right|^q \leq \max\left\{\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,c\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(a,\frac{c+d}{2}\right)\right|^q,\right.$$
$$\left.\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},c\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q\right\}, \tag{2.32}$$

for all $(t,s)\in[0,1]\times[0,1]$. Using (2.29)-(2.32) in (2.13) we get the desired result. □

**Corollary 4.** *Suppose the assumptions of Theorem 9 are fulfilled and if* $p(x,y) = \frac{1}{(b-a)(d-c)}$ *for all* $(x,y) \in [a,b] \times [c,d]$, *then the following inequality holds valid*

$$\begin{aligned}
(2.33)\quad &\left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \right. \\
&-\frac{1}{2}\int_a^b [f(x,c)+f(x,d)]\,dx - \frac{1}{2}\int_c^d [f(a,y)+f(b,y)]\,dy \\
&\left. + \int_c^d \int_a^b f(x,y)\,dxdy \right| \le \frac{(b-a)(d-c)}{16} \left[ \left( \max\left\{ \left|\frac{\partial^2 f}{\partial s \partial t}(b,d)\right|^q, \left|\frac{\partial^2 f}{\partial s \partial t}\left(b,\frac{c+d}{2}\right)\right|^q, \right.\right.\right. \\
&\left.\left. \left|\frac{\partial^2 f}{\partial s \partial t}\left(\frac{a+b}{2},d\right)\right|^q, \left|\frac{\partial^2 f}{\partial s \partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q \right\} \right)^{\frac{1}{q}} \\
&+ \left( \max\left\{ \left|\frac{\partial^2 f}{\partial s \partial t}(a,d)\right|^q, \left|\frac{\partial^2 f}{\partial s \partial t}\left(a,\frac{c+d}{2}\right)\right|^q, \right.\right. \\
&\left.\left. \left|\frac{\partial^2 f}{\partial s \partial t}\left(\frac{a+b}{2},d\right)\right|^q, \left|\frac{\partial^2 f}{\partial s \partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q \right\} \right)^{\frac{1}{q}} \\
&+ \left( \max\left\{ \left|\frac{\partial^2 f}{\partial s \partial t}(b,c)\right|^q, \left|\frac{\partial^2 f}{\partial s \partial t}\left(b,\frac{c+d}{2}\right)\right|^q, \right.\right. \\
&\left.\left. \left|\frac{\partial^2 f}{\partial s \partial t}\left(\frac{a+b}{2},c\right)\right|^q, \left|\frac{\partial^2 f}{\partial s \partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q \right\} \right)^{\frac{1}{q}} \\
&+ \left( \max\left\{ \left|\frac{\partial^2 f}{\partial s \partial t}(a,c)\right|^q, \left|\frac{\partial^2 f}{\partial s \partial t}\left(a,\frac{c+d}{2}\right)\right|^q, \right.\right. \\
&\left.\left.\left. \left|\frac{\partial^2 f}{\partial s \partial t}\left(\frac{a+b}{2},c\right)\right|^q, \left|\frac{\partial^2 f}{\partial s \partial t}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^q \right\} \right)^{\frac{1}{q}} \right].
\end{aligned}$$

**Remark 6.** *Suppose the assumptions of Theorem 9 are satisfied and additionally*

(1) *If* $f$ *is non-decreasing on the co-ordinates on* $[a,b] \times [c,d]$, *then (2.24) holds valid.*
(2) *If* $f$ *is non-increasing on the co-ordinates on* $[a,b] \times [c,d]$, *then (2.25) holds true.*

**Remark 7.** *In Corollary 4*

(1) *If* $f$ *is non-decreasing on the co-ordinates on* $[a,b] \times [c,d]$, *then (2.26) holds valid.*
(2) *If* $f$ *is non-increasing on the co-ordinates on* $[a,b] \times [c,d]$, *then (2.27) holds true.*

## 3. Applications to Random Variables

Let $0 < a < b$, $0 < c < d$, $\alpha, \beta \in \mathbb{R}$ and let $X$ and $Y$ be two independent continuous random variables having the bi-variate continuous probability density function $p : [a,b] \times [c,d] \to [0,\infty)$ which is symmetric to $\frac{a+b}{2}$ and $\frac{c+d}{2}$ the $\alpha$-moment of $X$ and the $\beta$-moment of $Y$ about the origin are respectively defined as follows

$$E_\alpha(X) = \int_c^b t^\alpha p_1(t)\,dt, E_\beta(Y) = \int_c^b s^\beta p_2(s)\,ds$$

which are assumed to be finite, here $p_1 : [a,b] \to [0,\infty)$ and $p_2 : [c,d] \to [0,\infty)$ are the marginal probability density functions of $X$ and $Y$. Since $X$ and $Y$ are independent random variables, we have

$$p(t,s) = p_1(t)\, p_2(s)$$

for all $(t,s) \in [a,b] \times [c,d]$.

Now we give some applications of our result to random variables.

**Theorem 10.** *The inequality*

$$\left|\left(E_\alpha(X) - \frac{a^\alpha + b^\alpha}{4}\right)\left(E_\beta(Y) - \frac{c^\beta + d^\beta}{2}\right)\right| \leq \frac{(b-a)(d-c)}{4}\alpha\beta\left(\frac{a^{\alpha-1}+b^{\alpha-1}}{2}\right)\left(\frac{c^{\beta-1}+d^{\beta-1}}{2}\right). \tag{3.1}$$

*holds holds for* $0 < a < b$, $0 < c < d$ *and* $\alpha, \beta \geq 2$.

*Proof.* Let $f(t,s) = t^\alpha s^\beta$ on $[a,b] \times [c,d]$ for $\alpha, \beta \geq 2$, we observe that $\left|\frac{\partial^2 f(t,s)}{\partial s \partial t}\right| = \alpha\beta t^{\alpha-1} s^{\beta-1}$ is convex on the co-ordinates on $[a,b] \times [c,d]$. Since

$$\begin{aligned}&\left|\frac{\partial^2 f}{\partial s \partial t}(a,c)\right| + \left|\frac{\partial^2 f}{\partial s \partial t}(a,d)\right| + \left|\frac{\partial^2 f}{\partial s \partial t}(b,c)\right| + \left|\frac{\partial^2 f}{\partial s \partial t}(b,d)\right| \\ &= \alpha\beta\left(a^{\alpha-1}+b^{\alpha-1}\right)\left(c^{\beta-1}+d^{\beta-1}\right),\end{aligned}$$

$$\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdy \leq \int_c^d\int_a^b p(x,y)\,dxdy = 1$$

and hence

$$\int_0^1\int_0^1\int_{L_2(s)}^{U_2(s)}\int_{L_1(t)}^{U_1(t)} p(x,y)\,dxdydtds \leq 1.$$

Also

$$\begin{aligned}&\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\int_c^d\int_a^b p(x,y)\,dxdy \\ &= \frac{a^\alpha c^\beta + a^\alpha d^\beta + b^\alpha c^\beta + b^\alpha d^\beta}{4} = \frac{(a^\alpha+b^\alpha)\left(c^\beta+d^\beta\right)}{4},\end{aligned}$$

$$\begin{aligned}&\frac{1}{2}\int_c^d\int_a^b [f(x,c)+f(x,d)]\,p(x,y)\,dxdy + \frac{1}{2}\int_c^d\int_a^b [f(a,y)+f(b,y)]\,p(x,y)\,dxdy \\ &= \left(\frac{c^\beta+d^\beta}{2}\right)E_\alpha(X) + \left(\frac{a^\alpha+b^\alpha}{2}\right)E_\beta(Y)\end{aligned}$$

and

$$\int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy = E_\alpha(X)\,E_\beta(Y).$$

The result follows immediately from the inequality (2.6). □

**Theorem 11.** *The inequality*

$$\left|\left(E_{\alpha}(X)-\frac{a^{\alpha}+b^{\alpha}}{4}\right)\left(E_{\beta}(Y)-\frac{c^{\beta}+d^{\beta}}{2}\right)\right| \leq \frac{(b-a)(d-c)}{16}\alpha\beta\left(b^{\alpha-1}+\left(\frac{a+b}{2}\right)^{\alpha-1}\right)\left(d^{\beta-1}+\left(\frac{c+d}{2}\right)^{\beta-1}\right). \tag{3.2}$$

*holds holds for* $0<a<b$, $0<c<d$ *and* $\alpha,\beta\geq 1$.

*Proof.* Let $f(t,s)=t^{\alpha}s^{\beta}$ on $[a,b]\times[c,d]$ for $\alpha,\beta\geq 1$, we observe that $\left|\frac{\partial^2 f(t,s)}{\partial s\partial t}\right|=\alpha\beta t^{\alpha-1}s^{\beta-1}$ is non-decreasing and quasi-convex on the co-ordinates on $[a,b]\times[c,d]$. The proof is similar to that of Theorem 10 by using the inequality (2.24) we obtain the required result. □

**Remark 8.** *For* $\alpha=\beta=1$, *we have from Theorem 11 that*

$$\left|\left(E(X)-\frac{a+b}{4}\right)\left(E(Y)-\frac{c+d}{2}\right)\right| \leq \frac{(b-a)(d-c)}{4}, \tag{3.3}$$

*where* $E_1(X)=E(X)$ *and* $E_1(Y)=E(Y)$ *are the expectation of the random variables* $X$ *and* $Y$ *respectively.*

## 4. Applications to 2D weighted trapezoidal formula

Let $[a,b]\times[c,d]$ be a rectangle from the plane $\mathbb{R}^2$. Suppose $d_1$ and $d_2$ are the divisions $a=x_0<x_1<\cdots<x_{n-1}<x_n=b$ and $c=y_0<y_1<\cdots<y_{m-1}<y_m=b$ of the intervals $[a,b]$ and $[c,d]$ respectively and let $d=\{[x_i,x_{i+1}]\times[y_j,y_{j+1}]:0\leq i\leq n-1, 0\leq j\leq m-1\}$ be a corresponding division of the rectangle $[a,b]\times[c,d]$ from the plane $\mathbb{R}^2$.

Consider the following $2D$ weighted quadrature formula

$$\int_c^d\int_a^b f(x,y)\,p(x,y)\,dxdy = T(f,p,d)+E(f,p,d), \tag{4.1}$$

where

$$\begin{aligned} T(f,p,d) &= -\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\Bigg[\frac{f(x_i,y_j)+f(x_i,y_{j+1})+f(x_{i+1},y_j)+f(x_{i+1},y_{j+1})}{4} \\ &\quad\times\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}} p(x,y)\,dxdy \\ &\quad+\frac{1}{2}\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}}\left[f(x,y_j)+f(x,y_{j+1})\right]p(x,y)\,dxdy \\ &\quad+\frac{1}{2}\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}}\left[f(x_i,y)+f(x_{i+1},y)\right]p(x,y)\,dxdy\Bigg] \end{aligned} \tag{4.2}$$

for the trapezoidal version and $E(f,p,d)$ denotes the associated approximation error.

The following results provide some estimates of the remainder term $E(f,p,d)$.

**Theorem 12.** *Suppose the assumptions of Theorem 6 are satisfied. If* $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$ *is convex on the co-ordinates on* $[a,b]\times[c,d]$ *for* $q \geq 1$*, then in (4.1), for every division* $d$ *of the rectangle* $[a,b]\times[c,d]$ *from the plane* $\mathbb{R}^2$*, the following holds*

$$
\begin{aligned}
(4.3)\quad & |E(f,p,d)| \\
&\leq \frac{1}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(x_{i+1}-x_i)(y_{j+1}-y_j) \\
&\times\left[\frac{\left|\frac{\partial^2 f}{\partial s\partial t}(x_i,y_j)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(x_i,y_{j+1})\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(x_{i+1},y_j)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(x_{i+1},y_{j+1})\right|^q}{4}\right] \\
&\times\int_0^1\int_0^1\int_{L_2(y_j,y_{j+1},s)}^{U_2(y_j,y_{j+1},s)}\int_{L_1(x_i,x_{i+1},t)}^{U_1(x_i,x_{i+1},t)}p(x,y)\,dxdydsdt,
\end{aligned}
$$

*where*

$$
\begin{aligned}
U_1(x_i,x_{i+1},t) &= \frac{1-t}{2}x_i+\frac{1+t}{2}x_{i+1}, \\
L_1(x_i,x_{i+1},t) &= \frac{1+t}{2}x_i+\frac{1-t}{2}x_{i+1}, \\
U_2(y_j,y_{j+1},s) &= \frac{1-s}{2}y_j+\frac{1+s}{2}y_{j+1}, \\
L_2(y_j,y_{j+1},s) &= \frac{1+s}{2}y_j+\frac{1-s}{2}y_{j+1}.
\end{aligned}
$$

*Proof.* Applying Theorem 6 on the rectangles $[x_i,x_{i+1}]\times[y_j,y_{j+1}]$ $(0\leq i\leq n-1, 0\leq j\leq m-1)$ of the division $d$ of the rectangle $[a,b]\times[c,d]$ from the plane $\mathbb{R}^2$, we get

$$
\begin{aligned}
(4.4)\quad & \left|\frac{f(x_i,y_j)+f(x_i,y_{j+1})+f(x_{i+1},y_j)+f(x_{i+1},y_{j+1})}{4}\right. \\
&\times\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}}p(x,y)\,dxdy-\frac{1}{2}\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}}\left[f(x,y_j)+f(x,y_{j+1})\right]p(x,y)\,dxdy \\
&-\frac{1}{2}\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}}\left[f(x_i,y)+f(x_{i+1},y)\right]p(x,y)\,dxdy \\
&\left.+\int_{y_j}^{y_{j+1}}\int_{x_i}^{x_{i+1}}f(x,y)\,p(x,y)\,dxdy\right|\leq\frac{(x_{i+1}-x_i)(y_{j+1}-y_j)}{4} \\
&\times\left[\frac{\left|\frac{\partial^2 f}{\partial s\partial t}(x_i,y_j)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(x_i,y_{j+1})\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(x_{i+1},y_j)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(x_{i+1},y_{j+1})\right|^q}{4}\right]^{\frac{1}{q}} \\
&\times\int_0^1\int_0^1\int_{L_2(y_j,y_{j+1},s)}^{U_2(y_j,y_{j+1},s)}\int_{L_1(x_i,x_{i+1},t)}^{U_1(x_i,x_{i+1},t)}p(x,y)\,dxdydsdt.
\end{aligned}
$$

Summing over $i$ from 0 to $n-1$ and $j$ over 0 to $m-1$, we deduce, by the triangle inequality, that (4.3) holds. ☐

**Remark 9.** *The inequality holds if the condition of convexity of* $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$ *on the co-ordinates on* $[a,b]\times[c,d]$ *is replaced with the condition of wright-convexity of* $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$ *on the co-ordinates on* $[a,b]\times[c,d]$ *for* $q\geq 1$.

**Theorem 13.** *Suppose the assumptions of Theorem 6 are satisfied. If* $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$ *is convex on the co-ordinates on* $[a,b]\times[c,d]$ *for* $q\geq 1$*, then in (4.1), for every division* $d$ *of the rectangle* $[a,b]\times[c,d]$ *from the plane* $\mathbb{R}^2$*, the following holds*

$$
\begin{aligned}
(4.5)\quad |E(f,p,d)| &\leq \frac{1}{16}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(x_{i+1}-x_i)(y_{j+1}-y_j) \\
&\times\Bigg[\Bigg(\max\Bigg\{\left|\frac{\partial^2 f}{\partial s\partial t}(x_{i+1},y_{j+1})\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(x_{i+1},\frac{y_j+y_{j+1}}{2}\right)\right|^q, \\
&\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{x_i+x_{i+1}}{2},y_{j+1}\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)\right|^q\Bigg\}\Bigg)^{\frac{1}{q}} \\
&+\Bigg(\max\Bigg\{\left|\frac{\partial^2 f}{\partial s\partial t}(x_i,y_{j+1})\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(x_i,\frac{y_j+y_{j+1}}{2}\right)\right|^q, \\
&\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{x_i+x_{i+1}}{2},y_{j+1}\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)\right|^q\Bigg\}\Bigg)^{\frac{1}{q}} \\
&+\Bigg(\max\Bigg\{\left|\frac{\partial^2 f}{\partial s\partial t}(x_{i+1},y_j)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(x_{i+1},\frac{y_j+y_{j+1}}{2}\right)\right|^q, \\
&\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{x_i+x_{i+1}}{2},y_j\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)\right|^q\Bigg\}\Bigg)^{\frac{1}{q}} \\
&+\Bigg(\max\Bigg\{\left|\frac{\partial^2 f}{\partial s\partial t}(x_i,y_j)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(x_i,\frac{y_j+y_{j+1}}{2}\right)\right|^q, \\
&\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{x_i+x_{i+1}}{2},y_j\right)\right|^q,\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)\right|^q\Bigg\}\Bigg)^{\frac{1}{q}}\Bigg] \\
&\times\int_0^1\int_0^1\int_{L_2(y_j,y_{j+1},s)}^{U_2(y_j,y_{j+1},s)}\int_{L_1(x_i,x_{i+1},t)}^{U_1(x_i,x_{i+1},t)} p(x,y)\,dxdydsdt.
\end{aligned}
$$

*Proof.* The proof follows from (2.28) by using the similar arguments as that of the proof of Theorem 12. ☐

**Remark 10.** *If* $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$ *is non-decreasing in Theorem 13, then the following inequality holds*

$$
\begin{aligned}
(4.6)\quad |E(f,p,d)| &\leq \frac{1}{16}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(x_{i+1}-x_i)(y_{j+1}-y_j) \\
\times\Bigg[&\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{x_i+x_{i+1}}{2},y_{j+1}\right)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(x_{i+1},\frac{y_j+y_{j+1}}{2}\right)\right| \\
&+\left|\frac{\partial^2 f}{\partial s\partial t}(x_{i+1},y_{j+1})\right|\Bigg]\int_0^1\int_0^1\int_{L_2(y_j,y_{j+1},s)}^{2(y_j,y_{j+1},s)}\int_{L_1(x_i,x_{i+1},t)}^{U_1(x_i,x_{i+1},t)} p(x,y)\,dxdydsdt.
\end{aligned}
$$

*and if* $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$ *is non-increasing in Theorem 13, then the following inequality holds*

$$|E(f,p,d)| \leq \frac{1}{16}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(x_{i+1}-x_i)(y_{j+1}-y_j) \tag{4.7}$$

$$\times\left[\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(x_i,\frac{y_j+y_{j+1}}{2}\right)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{x_i+x_{i+1}}{2},y_j\right)\right|\right.$$
$$\left.+\left|\frac{\partial^2 f}{\partial s\partial t}(x_i,y_j)\right|\right]\int_0^1\int_0^1\int_{L_2(y_j,y_{j+1},s)}^{2(y_j,y_{j+1},s)}\int_{L_1(x_i,x_{i+1},t)}^{U_1(x_i,x_{i+1},t)}p(x,y)\,dxdydsdt.$$

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