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f-DIVERGENCE FUNCTIONAL OF OPERATOR LOG-CONVEX FUNCTIONS

MOHSEN KIAN AND S. S. DRAGOMIR

ABSTRACT. We investigate the non-commutative f-divergence functional Θ of operator log-convex functions. In particular, we show that a continuous function $f:(0,\infty)\to(0,\infty)$ is operator log-convex if and only if

$$\Theta\left(\widetilde{A}\nabla\widetilde{C},\widetilde{B}\nabla\widetilde{D}\right)\leq\left(\Theta\left(\widetilde{A},\widetilde{B}\right)\nabla\Theta\left(\widetilde{A},\widetilde{D}\right)\right)\sharp\left(\Theta\left(\widetilde{C},\widetilde{B}\right)\nabla\Theta\left(\widetilde{C},\widetilde{D}\right)\right)$$

for all continuous fields $\widetilde{A}=(A_t)_{t\in T}$, $\widetilde{B}=(B_t)_{t\in T}$, $\widetilde{C}=(C_t)_{t\in T}$ and $\widetilde{D}=(D_t)_{t\in T}$ of strictly positive operators in $\mathfrak{A}$. Moreover, We define operator φ-convex functions and investigate some of its properties. In particular, we show that this class contains operator convex functions and operator log-convex functions.

1. Introduction and Preliminaries

Throughout this paper, assume that $\mathbb{B}(\mathscr{H})$ is the C^*-algebra of all bounded linear operators on a Hilbert space $\mathscr{H}$ and I denotes the identity operator. An operator $A\in\mathbb{B}(\mathscr{H})$ is called positive (denoted by $A\geq 0$) if $\langle Ax,x\rangle\geq 0$ for every $x\in\mathscr{H}$. If A is positive and invertible, then it is called strictly positive (denoted by $A>0$). A linear map Φ on $\mathbb{B}(\mathscr{H})$ is said to be positive if $\Phi(A)\geq 0$ whenever $A\geq 0$ and is called unital if $\Phi(I)=I$.

A continuous real function f defined on an interval J is said to be operator convex if

$$f(\lambda A+(1-\lambda)B)\leq\lambda f(A)+(1-\lambda)f(B)$$

for all self-adjoint operators A,B with spectra contained in J and every $\lambda\in[0,1]$. If $-f$ is operator convex, then f is said to be operator concave. If $f:J\to\mathbb{R}$ is operator convex, then the celebrated Hansen–Pedersen–Jensen operator inequality (see [6]) $f(C^*AC)\leq C^*f(A)C$ holds true for every self-adjoint operator A with spectrum contained in J and every isometry C. Another variant of this inequality, the Choi–Davis–Jensen inequality asserts that f is operator convex if and only if

$$f(\Phi(A))\leq\Phi(f(A)) \qquad (1)$$

for all unital positive linear maps Φ on $\mathbb{B}(\mathscr{H})$ and all self-adjoint operators A with spectrum in J (see e.g. [6]). The reader is referred to [6, 7, 8, 10, 12] and references therein for more information about operator convex functions and the Jensen operator inequality.

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Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras of Hilbert space operators and T be a locally compact Hausdorff space with a bounded Radon measure μ. A field $(A_t)_{t\in T}$ of operators in $\mathfrak{A}$ is said to be continuous if the function $t \mapsto A_t$ is norm continuous on T. Moreover, if the function $t \mapsto A_t$ is integrable on T, then the Bochner integral $\int_T A_t d\mu(t)$ is defined to be the unique element of $\mathfrak{A}$ for which

$$\rho\left(\int_T A_t d\mu(t)\right) = \int_T \rho(A_t) d\mu(t),$$

for every linear functional ρ in the norm dual $\mathfrak{A}^*$ of $\mathfrak{A}$.

A field $(\Phi_t)_{t\in T} : \mathfrak{A} \to \mathfrak{B}$ of positive linear mappings is said to be continuous if the function $t \mapsto \Phi_t(A)$ is continuous on T for every $A \in \mathfrak{A}$. If the C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ are unital and the function $t \mapsto \Phi_t(I)$ is integrable on T with integral I, then we say that the field $(\Phi_t)_{t\in T}$ is unital.

By the well-known Kubo–Ando theory [9], an operator mean σ is a binary operation on the set of strictly positive operators which satisfies the following conditions:

(1) monotonicity: if $A \le C$ and $B \le D$, then $A\sigma B \le C\sigma D$;
(2) Transformer inequality: $C(A\sigma B)C \le (CAC)\sigma(CBC)$.
(3) Continuity: if A_n and B_n are two decreasing sequences of positive operators which are converging respectively to A and B in the strong operator topology, then $A_n\sigma B_n$ converges to $A\sigma B$.

Kubo and Ando [9] showed that for every operator mean σ there exists an operator monotone function $f : (0,\infty) \to (0,\infty)$ such that

$$A\sigma B = B^{\frac{1}{2}} f\left(B^{\frac{-1}{2}} A B^{\frac{-1}{2}}\right) B^{\frac{1}{2}} \tag{2}$$

for all strictly positive operators A, B. Conversely, they proved that if $f : (0,\infty) \to (0,\infty)$ is operator monotone, the binary operation defined by (2) is an operator mean. Some of the most familiar operator means are $A\nabla B = \frac{A+B}{2}$ (arithmetic mean), $A\sharp B = B^{\frac{1}{2}}\left(B^{\frac{-1}{2}} A B^{\frac{-1}{2}}\right)^{\frac{1}{2}} B^{\frac{1}{2}}$ (geometric mean) and $A!B = \left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}$ (harmonic mean).

A continuous real function $f : (0,\infty) \to (0,\infty)$ is called operator log-convex if

$$f(A\nabla B) \le f(A)\sharp f(B) \tag{3}$$

for all positive operators A and B. Inequality (3) was considered by Aujla et. all [3]. The operator arithmetic-geometric mean inequality $f(A)\sharp f(B) \le \frac{f(A)+f(B)}{2}$ shows that the class of operator log-convex functions contains the class of operator convex functions. The converse is not true. To see this consider $f(x) = x$ [3]. Ando and Hiai [1] established various characterizations for operator log-convex functions. They presented the following result.

Theorem A.[1, Theorem 2.1] Let $f : (0,\infty) \to (0,\infty)$ be continuous. The following conditions are equivalent:

(1) f is operator monotone decreasing;
(2) f is operator log-convex;

(3) $f(A\nabla B) \leq f(A)\sigma f(B)$ for all positive operators A and B and every operator mean σ;

(4) $f(A\nabla B) \leq f(A)\sigma f(B)$ for all positive operators A and B and some operator mean $\sigma \neq \nabla$.

If f is operator convex, then (2) defines [4, 5] the perspective of f denoted by g, i.e.,

$$g(A,B) = B^{\frac{1}{2}} f\left(B^{\frac{-1}{2}} A B^{\frac{-1}{2}}\right) B^{\frac{1}{2}}.$$

It is known that f is operator convex if and only if g is jointly operator convex [4, 5]. A more general version of g, the non-commutative f-divergence functional Θ was defined in [11] to be

$$\Theta(\widetilde{A},\widetilde{B}) = \int_T B_t^{\frac{1}{2}} f\left(B_t^{-\frac{1}{2}} A_t B_t^{-\frac{1}{2}}\right) B_t^{\frac{1}{2}} d\mu(t),$$

where $\widetilde{A} = (A_t)_{t\in T}$ and $\widetilde{B} = (B_t)_{t\in T}$ are continuous fields of strictly positive operators in $\mathfrak{A}$.

In Section 2, we study the non-commutative f-divergence functional Θ of operator log-convex functions. In particular, we prove that $f : (0,\infty) \to (0\,\infty)$ is operator log-convex if and only if Θ is operator log-convex in its first variable and operator convex in its second variable.

In Section 3, we define operator φ-convex functions, which contains operator convex and operator log-convex functions and give some of their properties. Various examples are also presented. .

2. THE NON-COMMUTATIVE f-DIVERGENCE FUNCTIONAL

Let $X_1,\cdots,X_n$ and $Y_1,\cdots,Y_n$ be n-tuples of positive operators on $\mathscr{H}$. If follows from the jointly operator concavity of the operator geometric mean that

$$\sum_{i=1}^n X_i \sharp Y_i \leq \left(\sum_{i=1}^n X_i\right) \sharp \left(\sum_{i=1}^n Y_i\right). \tag{4}$$

This inequality is known as the operator version of the Cauchy–Schwarz inequality.

To achieve our result, we need a more general version of (4). Assume that $(A_t)_{t\in T}$ and $(B_t)_{t\in T}$ are continuous fields of strictly positive operators in $\mathfrak{A}$. We can generalize (4) as follows.

Lemma 2.1. *If $(A_t)_{t\in T}$ and $(B_t)_{t\in T}$ are continuous fields of strictly positive operators in $\mathfrak{A}$, then*

$$\int_T (A_t \sharp B_t) d\mu(t) \leq \left(\int_T A_t d\mu(t)\right) \sharp \left(\int_T B_t d\mu(t)\right).$$

Proof. Put $A=\int_T A_t d\mu(t)$ and $B=\int_T B_t d\mu(t)$. we have

$$\begin{aligned}\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)^{\frac{1}{2}} &= \left(\left(\int_T B_s d\mu(s)\right)^{-\frac{1}{2}}\int_T A_t d\mu(t)\left(\int_T B_s d\mu(s)\right)^{-\frac{1}{2}}\right)^{\frac{1}{2}}\\ &= \left(\int_T \left(\int_T B_s d\mu(s)\right)^{-\frac{1}{2}} B_t^{\frac{1}{2}}(B_t^{-\frac{1}{2}}A_tB_t^{-\frac{1}{2}})B_t^{\frac{1}{2}}\left(\int_T B_s d\mu(s)\right)^{-\frac{1}{2}} d\mu(t)\right)^{\frac{1}{2}}\\ &= \left(\int_T C_t^*(B_t^{-\frac{1}{2}}A_tB_t^{-\frac{1}{2}})C_t d\mu(t)\right)^{\frac{1}{2}}, \qquad (5)\end{aligned}$$

where $C=B_t^{\frac{1}{2}}\left(\int_T B_s d\mu(s)\right)^{-\frac{1}{2}}$ so that $\int_T C_t^*C_t d\mu(t)=I$. It follows from the operator concavity of the function $t^{\frac{1}{2}}$ that

$$\begin{aligned}&\left(\int_T C_t^*(B_t^{-\frac{1}{2}}A_tB_t^{-\frac{1}{2}})C_t d\mu(t)\right)^{\frac{1}{2}}\\ &\geq \int_T C_t^*\left(B_t^{-\frac{1}{2}}A_tB_t^{-\frac{1}{2}}\right)^{\frac{1}{2}}C_t d\mu(t) \qquad \text{(by the operator Jensen inequality)}\\ &= \left(\int_T B_s d\mu(s)\right)^{-\frac{1}{2}}\int_T B_t^{\frac{1}{2}}\left(B_t^{-\frac{1}{2}}A_tB_t^{-\frac{1}{2}}\right)^{\frac{1}{2}}B_t^{\frac{1}{2}}d\mu(t)\left(\int_T B_s d\mu(s)\right)^{-\frac{1}{2}}. \qquad (6)\end{aligned}$$

It follows that from (5) and (6) that

$$\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)^{\frac{1}{2}} \geq B^{-\frac{1}{2}}\left(\int_T B_t^{\frac{1}{2}}\left(B_t^{-\frac{1}{2}}A_tB_t^{-\frac{1}{2}}\right)^{\frac{1}{2}}B_t^{\frac{1}{2}}d\mu(t)\right)B^{-\frac{1}{2}}$$

from which we get the desired result. □

Now we present a property of the non-commutative f-divergence functional of an operator log-convex function.

Theorem 2.2. *A continuous function $f:(0,\infty)\to(0,\infty)$ is operator log-convex function if and only if*

$$\Theta\left(\widetilde{A}\nabla\widetilde{C},\widetilde{B}\nabla\widetilde{D}\right) \leq \left(\Theta\left(\widetilde{A},\widetilde{B}\right)\nabla\Theta\left(\widetilde{A},\widetilde{D}\right)\right)\sharp\left(\Theta\left(\widetilde{C},\widetilde{B}\right)\nabla\Theta\left(\widetilde{C},\widetilde{D}\right)\right) \qquad (7)$$

for all continuous fields $\widetilde{A}=(A_t)_{t\in T},\widetilde{B}=(B_t)_{t\in T},\widetilde{C}=(C_t)_{t\in T}$ and $\widetilde{D}=(D_t)_{t\in T}$ of strictly positive operators in $\mathfrak{A}$.

Proof. First we show that if f is operator log-convex on $(0,\infty)$, then Θ is operator log-convex in its first variable. Assume that $\widetilde{A}=(A_t)_{t\in T},\widetilde{B}=(B_t)_{t\in T},\widetilde{C}=(C_t)_{t\in T}$ and $\widetilde{D}=(D_t)_{t\in T}$ are continuous fields of strictly positive operators in $\mathfrak{A}$. For every $t\in T$

$$\begin{aligned}f\left(B_t^{-\frac{1}{2}}(A_t\nabla C_t)B_t^{-\frac{1}{2}}\right) &= f\left(\left(B_t^{-\frac{1}{2}}A_tB_t^{-\frac{1}{2}}\right)\nabla\left(B_t^{-\frac{1}{2}}C_tB_t^{-\frac{1}{2}}\right)\right)\\ &\leq f\left(B_t^{-\frac{1}{2}}A_tB_t^{-\frac{1}{2}}\right)\sharp f\left(B_t^{-\frac{1}{2}}C_tB_t^{-\frac{1}{2}}\right), \qquad (8)\end{aligned}$$

where we use the operator log-convexity of f. Multiplying both sides of (8) by $B_t^{\frac{1}{2}}$ we get

$$\begin{aligned} B_t^{\frac{1}{2}} f\left(B_t^{-\frac{1}{2}}(A_t \nabla C_t) B_t^{-\frac{1}{2}}\right) B_t^{\frac{1}{2}} &\leq B_t^{\frac{1}{2}}\left(f\left(B_t^{-\frac{1}{2}} A_t B_t^{-\frac{1}{2}}\right) \sharp f\left(B_t^{-\frac{1}{2}} C_t B_t^{-\frac{1}{2}}\right)\right) B_t^{\frac{1}{2}} \\ &= B_t^{\frac{1}{2}} f\left(B_t^{-\frac{1}{2}} A_t B_t^{-\frac{1}{2}}\right) B_t^{\frac{1}{2}} \sharp B_t^{\frac{1}{2}} f\left(B_t^{-\frac{1}{2}} C_t B_t^{-\frac{1}{2}}\right) B_t^{\frac{1}{2}}. \end{aligned} \tag{9}$$

The last equality follows from the property of (geometric) means. Integrating (9) over T and using Lemma 2.1 we obtain

$$\begin{aligned} &\int_T B_t^{\frac{1}{2}} f\left(B_t^{-\frac{1}{2}}(A_t \nabla C_t) B_t^{-\frac{1}{2}}\right) B_t^{\frac{1}{2}} d\mu(t) \\ &\quad \leq \int_T \left(B_t^{\frac{1}{2}} f\left(B_t^{-\frac{1}{2}} A_t B_t^{-\frac{1}{2}}\right) B_t^{\frac{1}{2}} \sharp B_t^{\frac{1}{2}} f\left(B_t^{-\frac{1}{2}} C_t B_t^{-\frac{1}{2}}\right) B_t^{\frac{1}{2}}\right) d\mu(t) \qquad \text{(by (9))} \\ &\quad \leq \left(\int_T B_t^{\frac{1}{2}} f\left(B_t^{-\frac{1}{2}} A_t B_t^{-\frac{1}{2}}\right) B_t^{\frac{1}{2}} d\mu(t)\right) \sharp \left(\int_T B_t^{\frac{1}{2}} f\left(B_t^{-\frac{1}{2}} C_t B_t^{-\frac{1}{2}}\right) B_t^{\frac{1}{2}} d\mu(t)\right), \end{aligned}$$

i.e.,

$$\Theta\left(\widetilde{A} \nabla \widetilde{C}, \widetilde{B}\right) \leq \Theta\left(\widetilde{A}, \widetilde{B}\right) \sharp \Theta\left(\widetilde{C}, \widetilde{B}\right). \tag{10}$$

Therefore

$$\begin{aligned} \Theta\left(\widetilde{A} \nabla \widetilde{C}, \widetilde{B} \nabla \widetilde{D}\right) &\leq \Theta\left(\widetilde{A}, \widetilde{B} \nabla \widetilde{D}\right) \sharp \Theta\left(\widetilde{C}, \widetilde{B} \nabla \widetilde{D}\right) \qquad \text{(by (10))} \\ &\leq \left(\Theta\left(\widetilde{A}, \widetilde{B}\right) \nabla \Theta\left(\widetilde{A}, \widetilde{D}\right)\right) \sharp \left(\Theta\left(\widetilde{C}, \widetilde{B}\right) \nabla \Theta\left(\widetilde{C}, \widetilde{D}\right)\right). \end{aligned}$$

The last inequality follows from the joint operator convexity of Θ [11] and monotonicity of operator means.

Assume for the converse that Θ satisfies (7). Let $T = \{1\}$ and μ be the counting measure on T. Let A and C be strictly positive operators in $\mathfrak{A}$. Then

$$f(A \nabla C) = \Theta(A \nabla C, I) \leq \Theta(A, I) \sharp \Theta(C, I) = f(A) \sharp f(C),$$

which means that f is operator log-convex. □

Remark 2.3. In fact Theorem 2.2 assert that f is operator log-convex if and only if the non-commutative f-divergence functional Θ is operator log-convex in its first variable and operator convex in its second variable.

Corollary 2.4. *A continuous non-negative function f is operator log-convex function if and only if the associated perspective function g is operator log-convex function in its first variable and operator convex in its second variable.*

The next theorem provides a Choi–Davis–Jensen type inequality for perspectives of operator log-convex functions.

Theorem 2.5. *Let $f:(0,\infty)\to(0,\infty)$ be a continuous function and g be the associated perspective function. Then f is operator log-convex if and only if*

$$g\left(\int_T \Phi_t(A_t\nabla C_t)d\mu(t),\int_T \Phi_t(B_t)d\mu(t)\right)\leq\left(\int_T \Phi_t(g(A_t,B_t))d\mu(t)\right)\sharp\left(\int_T \Phi_t(g(C_t,B_t))d\mu(t)\right) \tag{11}$$

for all unital fields $\widetilde{\Phi}=(\Phi_t)_{t\in T}:\mathfrak{A}\to\mathfrak{B}$ of positive linear maps and all continuous fields $\widetilde{A}=(A_t)_{t\in T},\widetilde{B}=(B_t)_{t\in T}$ and $\widetilde{C}=(C_t)_{t\in T}$ of strictly positive operators in $\mathfrak{A}$.

Proof. Assume that f is operator log convex. Then f is operator convex and so g is jointly operator convex [5, 4]. Put $X=\int_T\Phi_s(B_s)d\mu(s)$ and let the continuous filed of positive linear mappings $(\Psi_t)_{t\in T}:\mathfrak{A}\to\mathfrak{B}$ be defined by

$$\Psi_t(Y)=X^{\frac{-1}{2}}\Phi_t\left(B_t^{\frac{1}{2}}YB_t^{\frac{1}{2}}\right)X^{\frac{-1}{2}}$$

so that $\int_T\Psi_t(I)d\mu(t)=I$. Therefore

$$\begin{aligned}
&g\left(\int_T \Phi_t(A_t\nabla C_t)d\mu(t),\int_T \Phi_t(B_t)d\mu(t)\right)=X^{\frac{1}{2}}f\left(X^{-\frac{1}{2}}\int_T \Phi_t(A_t\nabla C_t)d\mu(t)X^{-\frac{1}{2}}\right)X^{\frac{1}{2}}\\
&=X^{\frac{1}{2}}f\left(\int_T\Psi_t\left(B_t^{-\frac{1}{2}}(A_t\nabla C_t)B_t^{-\frac{1}{2}}\right)d\mu(t)\right)X^{\frac{1}{2}}\\
&\leq X^{\frac{1}{2}}\left(\int_T\Psi_t\left(f\left(B_t^{-\frac{1}{2}}(A_t\nabla C_t)B_t^{-\frac{1}{2}}\right)\right)d\mu(t)\right)X^{\frac{1}{2}}\quad\text{(by the Jensen operator inequality)}\\
&=\int_T\Phi_t\left(B_t^{\frac{1}{2}}f\left(B_t^{-\frac{1}{2}}(A_t\nabla C_t)B_t^{-\frac{1}{2}}\right)B_t^{\frac{1}{2}}\right)d\mu(t)\\
&=\int_T\Phi_t(g(A_t\nabla C_t,B_t))d\mu(t)\\
&\leq\int_T\Phi_t(g(A_t,B_t)\sharp g(C_t,B_t))d\mu(t)\qquad\text{(by Corollary 2.4)}\\
&\leq\int_T\Phi_t(g(A_t,B_t))\sharp\Phi_t(g(C_t,B_t))d\mu(t)\qquad\text{(by operator concavity of }\sharp)\\
&\leq\left(\int_T\Phi_t(g(A_t,B_t))d\mu(t)\right)\sharp\left(\int_T\Phi_t(g(C_t,B_t))d\mu(t)\right)\quad\text{(by Lemma 2.1).}
\end{aligned}$$

For the converse, put $T=\{1\}$ and let μ be the counting measure on T. If A and C are strictly positive, then with $\Phi(A)=A$ and $B=I$, inequality (11) implies the operator log-convexity of f. □

Example 2.6. Let the operator log-convex function $f:(0,\infty)\to(0,\infty)$ be defined by $f(t)=t^{-1}$. It follows from Theorem 2.5 that

$$g(\Phi(A\nabla C),\Phi(B))\leq\Phi(g(A,B))\sharp\Phi(g(C,B))$$

or equivalently

$$\Phi(B)\Phi(A\nabla C)^{-1}\Phi(B)\leq\Phi(BA^{-1}B)\sharp\Phi(BC^{-1}B).$$

Therefore

$$\begin{aligned}\Phi(A\nabla C)^{-1} &\le \Phi(B)^{-1}\left(\Phi(BA^{-1}B)\sharp\Phi(BC^{-1}B)\right)\Phi(B)^{-1} \\ &= \Phi(B)^{-1}\Phi(BA^{-1}B)\Phi(B)^{-1}\sharp\Phi(B)^{-1}\Phi(BC^{-1}B)\Phi(B)^{-1}.\end{aligned} \tag{12}$$

Note that it follows from the operator convexity of f that

$$\begin{aligned}\Phi(A\nabla C)^{-1} &= \left(\frac{\Phi(A)+\Phi(C)}{2}\right)^{-1} \\ &\le \frac{\Phi(A)^{-1}+\Phi(C)^{-1}}{2} \qquad \text{(by operator convexity of } f(t)=t^{-1}\text{)} \\ &\le \frac{\Phi\left(A^{-1}\right)+\Phi\left(C^{-1}\right)}{2} \qquad \text{(by operator convexity of } f(t)=t^{-1}\text{)},\end{aligned}$$

while with $B=I$, inequality (12) provides a sharper inequality:

$$\Phi(A\nabla C)^{-1} \le \Phi\left(A^{-1}\right)\sharp\Phi\left(C^{-1}\right) \le \frac{\Phi\left(A^{-1}\right)+\Phi\left(C^{-1}\right)}{2}.$$

Corollary 2.7. *Let A,B,C be strictly positive operators. If f is an operator log-convex function and g is its perspective function, then*

$$g\left(\left\langle\frac{A+C}{2}x,x\right\rangle,\langle Bx,x\rangle\right) \le \sqrt{\langle g(A,B)x,x\rangle\langle g(C,B)x,x\rangle}$$

for every unit vector x.

Example 2.8. Applying Corollary 2.7 to the operator log-convex function $f(t)=t^{-1}$ defined on $(0,\infty)$ we get

$$\begin{aligned}\langle Bx,x\rangle\left\langle\frac{A+C}{2}x,x\right\rangle^{-1}\langle Bx,x\rangle &\le \sqrt{\langle BA^{-1}Bx,x\rangle\langle BC^{-1}Bx,x\rangle} \\ &\le \left\langle B\left(\frac{A^{-1}+C^{-1}}{2}\right)Bx,x\right\rangle\end{aligned}$$

for every unit vector x.

3. Operator φ-convex functions

Let φ be a continuous one to one real function. Assume that

$$A\sigma_\varphi B = \varphi^{-1}\left(\varphi(A)\nabla\varphi(B)\right)$$

for all self-adjoint operators A,B with spectra in domain of φ.

Definition 3.1. Let $f: J\to\mathbb{R}$ be a continuous real function and φ be a continuous one to one real function defined on an interval containing $f(J)$. We say that f is operator φ-convex if

$$f(A\nabla B) \le f(A)\sigma_\varphi f(B) \tag{13}$$

for all self-adjoint operators A,B with spectra in J.

We remark that if σ_φ is an operator mean, then every operator φ-convex function belongs to the class of operator log-convex functions verified by Ando and Hiai [1]. If $\varphi(t)=t$, then Definition 3.1 gives operator convex functions.

Example 3.2. Let $\varphi(t)=\log t$. If $f:(0,\infty)\to(0,\infty)$ is operator φ-convex, then for all positive operators A and B we have

$$f\left(\frac{A+B}{2}\right)\leq f(A)\sigma_\varphi f(B)=exp\left(\frac{\log f(A)+\log f(B)}{2}\right).$$

Since $\log t$ is operator monotone, we get

$$\log f\left(\frac{A+B}{2}\right)\leq\frac{\log f(A)+\log f(B)}{2}.$$

Therefore, if $\varphi(t)=\log t$, then Definition 3.1 reduces to the notion of operator log-convexity.

Example 3.3. Let $\varphi(t)=\frac{1}{t}$. If $f:(0,\infty)\to(0,\infty)$ is operator φ-convex, then

$$f\left(\frac{A+B}{2}\right)\leq f(A)\sigma_\varphi f(B)=\left(\frac{f(A)^{-1}+f(B)^{-1}}{2}\right)^{-1}=f(A)!f(B)$$

for all strictly positive operators A,B. So, f is operator log-convex in this case too.

Lemma 3.4. *If φ^{-1} is operator decreasing, then every operator φ-convex function is operator log-convex.*

Proof. Let f be operator φ-convex. If φ^{-1} is operator decreasing, then $\frac{1}{\varphi^{-1}}$ is operator concave and so

$$\varphi^{-1}\left(\frac{\varphi(f(A))+\varphi(f(B))}{2}\right)^{-1}\geq\left(\frac{f(A)^{-1}+f(B)^{-1}}{2}\right),$$

which implies that f is operator log-convex. □

Example 3.5. Let $p\in\mathbb{R}$. The power mean of positive operators A and B is defined by $\left(\frac{A^p+B^p}{2}\right)^{\frac{1}{2}}$. Operator arithmetic mean and operator Harmonic mean are special case of power means with $p=1$ and $p=-1$, respectively. It is known that power mean is an operator mean only if $-1\leq p\leq1$, while power means with $p>1$ have many applications for example in mathematical physics and theory of operator spaces [2].

Assume that $\varphi:(0,\infty)\to(0,\infty)$ is defined by $\varphi(t)=t^{-p}$ $(p\geq1)$. Then $\varphi^{-1}(t)=t^{-\frac{1}{p}}$, which is operator decreasing. Therefore, if $f:(0,\infty)\to(0,\infty)$ is operator φ-convex, then it is operator log-convex. It means that if

$$f\left(\frac{A+B}{2}\right)\leq\left(\frac{f(A)^{-p}+f(B)^{-p}}{2}\right)^{\frac{-1}{p}}$$

for all strictly positive operator A,B, then f is operator log-convex and

$$f\left(\frac{A+B}{2}\right)\leq\left(\frac{f(A)^{-1}+f(B)^{-1}}{2}\right)^{-1}.$$

If f is operator φ-convex, then a variant of the Jensen operator inequality holds true. The proof of the next lemma is based on that of [6, Theorem 1.9].

Lemma 3.6. *Let φ be a continuous one-to-one function. If $f : J \to \mathbb{R}$ is an operator φ-convex function, then*

$$f\left(C^*AC\right) \le \varphi^{-1}\left(C^*\varphi(f(A))C\right)$$

for every self-adjoint operator A with spectrum in J and every isometry C.

Proof. If A and B are two self-adjoint operators in $\mathbb{B}(\mathscr{H})$, then $X = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ can be regarded as a self-adjoint operator in $\mathbb{B}(\mathscr{H} \oplus \mathscr{H})$. Set $D = \sqrt{I - CC^*}$ so that operators U and V defined by

$$U = \begin{pmatrix} C & D \\ 0 & -C^* \end{pmatrix}, \qquad V = \begin{pmatrix} C & -D \\ 0 & C^* \end{pmatrix},$$

are unitary operators in $\mathbb{B}(\mathscr{H} \oplus \mathscr{H})$ and

$$U^*XU = \begin{pmatrix} C^*AC & C^*AD \\ DAC & DAD + CBC^* \end{pmatrix}, \qquad V^*XV = \begin{pmatrix} C^*AC & -C^*AD \\ -DAC & DAD + CBC^* \end{pmatrix}.$$

Therefore

$$\begin{aligned}
&\begin{pmatrix} f(C^*AC) & 0 \\ 0 & f(DAD + CBC^*) \end{pmatrix} \\
&= f\begin{pmatrix} C^*AC & 0 \\ 0 & DAD + CBC^* \end{pmatrix} \\
&= f\left(\frac{U^*XU + V^*XV}{2}\right) \\
&\le \varphi^{-1}\left(\frac{\varphi(f(U^*XU)) + \varphi(f(V^*XV))}{2}\right) \qquad \text{(since } f \text{ is operator } \varphi\text{-convex)} \\
&= \varphi^{-1}\left(\frac{U^*\varphi(f(X))U + V^*\varphi(f(X))V}{2}\right) \\
&= \varphi^{-1}\begin{pmatrix} C^*\varphi(f(A))C & 0 \\ 0 & D\varphi(f(A))D + C\varphi(f(B))C^* \end{pmatrix}.
\end{aligned}$$

Hence $f(C^*AC) \le \varphi^{-1}(C^*\varphi(f(A))C)$. □

Remark 3.7. Note that if φ^{-1} is operator convex, then every operator φ-convex is operator convex:

$$f(C^*AC) \le \varphi^{-1}(C^*\varphi(f(A))C) \le C^*f(A)C.$$

Corollary 3.8. *If $f : J \to \mathbb{R}$ is an operator φ-convex function and $A_1, \cdots, A_n$ are self-adjoint operators with spectra in J, then*

$$f\left(\sum_{i=1}^{n} C_i^* A_i C_i\right) \le \varphi^{-1}\left(\sum_{i=1}^{n} C_i^* \varphi(f(A_i)) C_i\right)$$

for all operators C_i $(i=1,\cdots,n)$ *with* $\sum_{i=1}^n C_i^*C_i = I$.

Proof. Apply Lemma 3.6 to the operator $A = A_1 \oplus \cdots \oplus A_n$ and the isometry $C = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}$. □

We can present the following characterization of operator φ-convex functions using positive linear mappings.

Theorem 3.9. *A continuous function* $f : J \to \mathbb{R}$ *is operator* φ*-convex if and only if*

$$f(\Phi(A)) \leq \varphi^{-1}(\Phi\left(\varphi(f(A))\right)) \tag{14}$$

for every unital positive linear map Φ *and every self-adjoint operator* A *with spectrum in* J.

Proof. Suppose that A is a self-adjoint operator on a finite dimensional Hilbert space $\mathscr{H}$ with the spectral decomposition $A = \sum_{i=1}^n \lambda_i P_i$. If Φ is a unital positive linear map on $\mathbb{B}(\mathscr{H})$, then $\Phi(A) = \sum_{i=1}^n \lambda_i \Phi(P_i)$ and $\sum_{i=1}^n \Phi(P_i) = I$. Therefore

$$\begin{aligned} f(\Phi(A)) = f\left(\sum_{i=1}^n \lambda_i \Phi(P_i)\right) &= f\left(\sum_{i=1}^n \Phi(P_i)^{\frac{1}{2}} \lambda_i \Phi(P_i)^{\frac{1}{2}}\right) \\ &\leq \varphi^{-1}\left(\sum_{i=1}^n \Phi(P_i)^{\frac{1}{2}} \varphi(f(\lambda_i)) \Phi(P_i)^{\frac{1}{2}}\right) \quad \text{(by Corollary 3.8)} \\ &= \varphi^{-1}\left(\sum_{i=1}^n \varphi(f(\lambda_i)) \Phi(P_i)\right) \\ &= \varphi^{-1}(\Phi\left(\varphi(f(A))\right)). \end{aligned}$$

If A is a self-adjoint operator on an infinite dimensional Hilbert space, then (14) follows by using a continuity argument.

For the converse assume that (14) holds true. put

$$\mathfrak{D}(\mathscr{H} \oplus \mathscr{H}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}; \quad A, B \in \mathbb{B}(\mathscr{H}) \right\}.$$

Then $\mathfrak{D}(\mathscr{H} \oplus \mathscr{H})$ is a unital closed $*$-subalgebra of $\mathbb{B}(\mathscr{H} \oplus \mathscr{H})$. Let the unital positive linear map Ψ be defined on $\mathfrak{D}(\mathscr{H} \oplus \mathscr{H})$ by

$$\Psi\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right) = \frac{A+B}{2}.$$

Now if A and B are two self-adjoint operators on $\mathscr{H}$ with spectra in J, then it follows from (14) that

$$\begin{aligned} f(A\nabla B) &= f\left(\Psi\left(\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right)\right)\right) \leq \varphi^{-1}\left(\Psi\left(\varphi o f\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right)\right) \\ &= \varphi^{-1}\left(\Psi\begin{pmatrix} \varphi(f(A)) & 0 \\ 0 & \varphi(f((B)) \end{pmatrix}\right) = \varphi^{-1}\left(\frac{\varphi(f(A))+\varphi(f(B))}{2}\right) \\ &= f(A)\sigma_\varphi f(B), \end{aligned}$$

which implies that f is operator φ-convex. □

Example 3.10. Let $\varphi(t) = t^{-1}$. Then Theorem 3.9 implies that $f : (0, \infty) \to (0, \infty)$ is operator log-convex if and only if

$$f(\Phi(A)) \leq \Phi\left(f(A)^{-1}\right)^{-1} \tag{15}$$

for every strictly positive operator A and every unital positive linear map Φ. Inequality (15) implies that $\frac{1}{f}$ is operator concave and so f is operator decreasing. This ensures that f is operator log-convex if and only if it is operator decreasing.

Example 3.11. The function $f(x) = x^{-\frac{1}{2}}$ is operator log-convex on $(0, \infty)$. Assume that the unital positive linear map $\Phi : \mathcal{M}_3(\mathbb{C}) \to \mathcal{M}_2(\mathbb{C})$ is defined by

$$\Phi((a_{ij})) = (a_{ij})_{2\leq i,j\leq 3}.$$

If $A \in \mathcal{M}_3(\mathbb{C})$ is the positive matrix

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix},$$

then by a simple calculation we have

$$f(\Phi(A)) = \begin{pmatrix} 1.1945 & -0.2706 \\ -0.2706 & 0.6533 \end{pmatrix}, \qquad \Phi\left(f(A)^{-1}\right)^{-1} = \begin{pmatrix} 1.2192 & -0.2933 \\ -0.2933 & 0.6760 \end{pmatrix}$$

$$\Phi(f(A)) = \begin{pmatrix} 1.2420 & -0.3261 \\ -0.3261 & 0.7234 \end{pmatrix}$$

and so

$$f(\Phi(A)) \lneqq \Phi\left(f(A)^{-1}\right)^{-1} \lneqq \Phi(f(A)).$$

Corollary 3.12. *If Φ is a unital positive linear map and A is a strictly positive operator, then*

(1) $\Phi(A)^{-\alpha} \leq \Phi\left(A^{\alpha}\right)^{-1} \leq \Phi\left(A^{-\alpha}\right)$ *for all* $0 \leq \alpha \leq 1$.
(2) $\Phi\left(A^{\alpha}\right)^{\frac{1}{\alpha}} \leq \Phi\left(A^{-1}\right)^{-1} \leq \Phi(A)$ *for all* $\alpha \leq -1$.

Proof. (1): follows from the operator log-convexity of $t^{-\alpha}$.
(2): the function $t \to t^{\frac{1}{\alpha}}$ is operator log-convex. So it follows from Theorem 3.9 that $\Phi(A)^{\frac{1}{\alpha}} \leq \left(\Phi\left(A^{\frac{-1}{\alpha}}\right)\right)^{-1}$. Replacing A by A^{α} we get desired inequality. □

Corollary 3.13. *Let $\Phi_1, \cdots, \Phi_n$ be positive linear mappings on $\mathbb{B}(\mathscr{H})$ such that $\sum_{i=1}^n \Phi_i(I) = I$. If $f : (0,\infty) \to (0,\infty)$ is an operator log-convex function, then*

$$f\left(\sum_{i=1}^n \Phi_i(A_i)\right) \le \left(\sum_{i=1}^n \Phi_i\left(f(A_i)^{-1}\right)\right)^{-1}$$

for all strictly positive operators $A_1, \cdots, A_n$.

Proof. Apply Theorem 3.9 to the strictly positive operator $A = A_1 \oplus \cdots \oplus A_n$ and the unital positive linear map $\Phi : \mathbb{B}(\mathscr{H} \oplus \cdots \oplus \mathscr{H}) \to \mathbb{B}(\mathscr{H})$ defined by $\Phi(A_1 \oplus \cdots \oplus A_n) = \sum_{i=1}^n \Phi_i(A_i)$. □

The next result shows that every operator log-convex function is sub-additive.

Proposition 3.14. *If $f : (0,\infty) \to (0,\infty)$ is an operator log-convex function, then f is sub-additive. More precisely*

$$f(A+B) \le f(A)\sharp f(B) \le f(A) + f(B)$$

for all strictly positive operators A, B.

Proof. Assume that A and B are strictly positive operators. Then

$$\begin{aligned} f(A+B) = f((2A)\nabla(2B)) &\le f(2A)\sharp f(2B) \\ &\le f(A)\sharp f(B) \qquad \text{(by (1) of Theorem A)} \\ &\le f(A)\nabla f(B) \qquad \text{(by the A-G inequality)} \\ &\le f(A) + f(B). \end{aligned}$$

□

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Mohsen Kian: School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran, Iran

Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, P.O. Box 1339, Bojnord 94531, Iran.

E-mail address: kian@member.ams.org and kian@ub.ac.ir.

S. S. Dragomir: Mathematics, School of Engineering and Science, Victoria University, P.O. Box 14428, Melbourne City, MC 8001, Australia

E-mail address: sever.dragomir@vu.edu.au