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REFINING RECURSIVELY THE HERMITE-HADAMARD INEQUALITY ON SIMPLEX

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Abstract. In the present paper, a coupled algorithm refining recursively the Hermite-Hadamard inequality on simplex is investigated. Our approach allows us to express the integral mean value M_f of a convex function f on a simplex as limit of sequences and sum of series involving iterative lower and upper bounds of M_f. At the end, two examples of interest are discussed.

1. Introduction

The following result is well known in the literature, see [1, 6, 7, 8] for instance.

Theorem 1.1. *Let Δ be a $(n+1)$-simplex of $\mathbb{R}^n$ and $f:\Delta\longrightarrow\mathbb{R}$ be a convex function. If $p_1,p_2,...,p_{n+1}$ denote the vertices of Δ then we have*

$$f\Big(\sum_{i=1}^{n+1}\frac{p_i}{n+1}\Big)\le\frac{1}{|\Delta|}\int_\Delta f(x)dx\le\frac{1}{n+1}\sum_{i=1}^{n+1}f(p_i), \tag{1.1}$$

where $|\Delta|=\int_\Delta dx$ stands for the Lebesgue volume of Δ in $\mathbb{R}^n$.

The double inequality (1.1), known in the literature as the Hermite-Hadamard inequality, in short (HHI), has a large context of applications in many mathematical areas. It has been proved, throughout a lot of works, that such inequality is very useful in theoretical point of view as well as in practical purposes. Refinement of (HHI) was discussed by Mitroi and Spiridon in [4]. A converse version of (HHI) was investigated by Mitroi and Symeonidis in [5]. An extension of (HHI), due to Choquet, for convex functions on a compact set can be found in [9].

For the sake of simplicity, the middle side of (1.1) will be denoted by $M_f(\Delta)$, that is,

$$M_f(\Delta)=\frac{1}{|\Delta|}\int_\Delta f(x)dx,$$

and is known in the literature as the (arithmetic) integral mean value of f on Δ. The left and right sides of (1.1) will be called initial lower and upper bounds of $M_f(\Delta)$, respectively. The computation of $M_f(\Delta)$, when f and Δ are given, is in general hard. The initial lower and upper bounds of $M_f(\Delta)$ in (1.1) can be considered as estimates of $M_f(\Delta)$, but of course with a not good precision in general.

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The fundamental goal of the present paper turns out of the next way: we will construct a coupled algorithm involving two recursive sequences, denoted by $L_k(\Delta)$ and $U_k(\Delta)$, such that

$$f\Big(\sum_{i=1}^{n+1}\frac{p_i}{n+1}\Big) := L_0(\Delta) \leq L_1(\Delta) \leq L_2(\Delta) \leq ... \leq L_k(\Delta)$$

$$\leq M_f(\Delta) \leq ... \leq U_k(\Delta) \leq ... \leq U_2(\Delta) \leq U_1(\Delta) \leq U_0(\Delta) := \frac{1}{n+1}\sum_{i=1}^{n+1} f(p_i),$$

with together the next approximation

$$\lim_{k\to\infty} L_k(\Delta) = \lim_{k\to\infty} U_k(\Delta) = M_f(\Delta).$$

Our approach allows us also to give an expression of M_f in terms of series as well:

$$M_f(\Delta) = U_0(\Delta) - \frac{1}{n+1}\sum_{k=0}^{\infty}\Big(U_k(\Delta) - L_k(\Delta)\Big).$$

In the mono-dimensional case $n = 1$, the (HHI) takes the next form

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.2}$$

provided that $f : [a, b] \longrightarrow \mathbb{R}, a < b$, is convex. It is well-known that, [3], the left inequality of (1.2) gives a better estimate of the integral mean value than the inequality of the right, that is,

$$\frac{1}{b-a}\int_a^b f(x)dx - f\Big(\frac{a+b}{2}\Big) \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx. \tag{1.3}$$

In [10], Wasowicz and Witkowski proved, via a counter-example, that (1.3) does not remain true for convex functions involving several variables. This means, by using our previous notation, that the next inequality

$$M_f(\Delta) - L_0(\Delta) \leq U_0(\Delta) - M_f(\Delta)$$

does not hold for convex functions with several arguments. After checking a particular example for bi-dimensional case, they guessed that

$$M_f(\Delta) - L_0(\Delta) \leq n\Big(U_0(\Delta) - M_f(\Delta)\Big), \tag{1.4}$$

and proved it later in an elementary but long way. Our approach, described above, will allow us to deduce (1.4) as a simple consequence from our theoretical results. In fact, we show more: inequality (1.4) is conserved for all iterates $L_k(\Delta)$ and $U_k(\Delta)$ previously constructed. That is, the following iterative inequality

$$M_f(\Delta) - L_k(\Delta) \leq n\Big(U_k(\Delta) - M_f(\Delta)\Big)$$

remains true for each integer $k \geq 0$. In the mono-dimensional case $n = 1$ (i.e. $\Delta = [a, b]$), this latter inequality means that the left iterate $L_k([a, b])$ gives a better estimate of the integral mean value than the right iterate $U_k([a,b])$. That is, for all $k \geq 0$, we have

$$\frac{1}{b-a}\int_a^b f(x)dx - L_k\big([a,b]\big) \leq U_k([a,b]) - \frac{1}{b-a}\int_a^b f(x)dx,$$

where the general iterates $L_k([a,b])$ and $U_k([a,b])$ can be explicitly computed in terms of a, b and f by a simple recursive manner, see [2] or Example 3.1 presented later.

2. **Basic Notions**

In this section, we state some basic notions that will be needed later. Let D be an arbitrary $(n+1)$-simplex of $\mathbb{R}^n$ of vertices $x_1, x_2, ..., x_{n+1}$, in short $D = \overline{co}\big(x_1, x_2, ..., x_{n+1}\big)$, where $\overline{co}$ refers to the closed convex hull. Let $b = \frac{\sum_{i=1}^{n+1} x_i}{n+1}$ be the barycenter of D and $\overline{co}\Big(b,\, a_1,\, a_2,\, ...,\, a_n\Big)$ be the sub-simplex of D, where the points $a_1,\, a_2,\, ...,\, a_n$ are distinct and belong to the set $\Big\{x_1, x_2, ..., x_{n+1}\Big\}$. There are $(n+1)$ choices of $a_1,\, a_2,\, ...,\, a_n$ and so we have $(n+1)$ sub-simplices of D which will be denoted by $D_i,\ 1 \leq i \leq n+1$.

Following the above construction, the sub-simplices $D_i,\ 1 \leq i \leq n+1$, form a quasi-partition of D in the sense that

$$D = \bigcup_{i=1}^{n+1} D_i \text{ and } |D_i \cap D_j| = 0 \text{ for all } i \neq j, \tag{2.1}$$

and satisfy the following relationship

$$|D_i| = |D_j| = \frac{|D|}{n+1} \text{ for all } i, j. \tag{2.2}$$

Let D and $D_j,\ 1 \leq j \leq n+1$, be as in the above. Recall that D_j, for fixed $j = 1, 2, ..., n+1$, has the same vertices as D except one vertex which is the barycenter b of D. Explicitly, we can write:

$$D_j = \overline{co}\{x_1, x_2, ..., x_{j-1}, b, x_{j+1}, ..., x_{n+1}\}, \tag{2.3}$$

where b figures in the $j^{'}$-place, with $D = \overline{co}\{x_1, x_2, ..., x_{n+1}\}$.

For the sake of clearness, we state the following definition.

Definition 2.1. For an arbitrary $(n+1)$-simplex $D \subset \mathbb{R}^n$ with vertices $x_1, x_2, ..., x_{n+1}$ and a convex function $f : D \longrightarrow \mathbb{R}$ we set

$$L_0(D) = f\Big(\sum_{i=1}^{n+1} \frac{x_i}{n+1}\Big) \text{ and } U_0(D) = \frac{1}{n+1}\sum_{i=1}^{n+1} f(x_i), \tag{2.4}$$

which are the lower and upper bounds of (HHI) for f on D. We also define

$$L_{k+1}(D) = \frac{1}{n+1}\sum_{j=1}^{n+1} L_k(D_j), \quad U_{k+1}(D) = \frac{1}{n+1}\sum_{j=1}^{n+1} U_k(D_j). \tag{2.5}$$

For example, for all fixed $j = 1, 2, ..., n+1$,

$$L_0(D_j) = f\Big(\sum_{i=1, i\neq j}^{n+1} \frac{x_i}{n+1} + \frac{b}{n+1}\Big) \text{ and} \tag{2.6}$$

$$U_0(D_j) = \frac{1}{n+1} \sum_{i=1, i\neq j}^{n+1} f(x_i) + \frac{1}{n+1} f(b).$$

3. Refinement of (HHI): The main Results

Let Δ and f be fixed as in Theorem 1.1. As previously defined, Δ_j for $j = 1, 2, ..., n+1$ are the $(n+1)$ sub-simplices of Δ. The barycenter of Δ will be denoted here by m. Applying (HHI) for f in $\Delta_j \subset \Delta$, for fixed $j = 1, 2, ..., n+1$, we obtain

$$L_0\big(\Delta_j\big) \leq \frac{1}{|\Delta_j|} \int_{\Delta_j} f(x)dx \leq U_0\big(\Delta_j\big), \tag{3.1}$$

where $L_0(\Delta_j)$ and $U_0(\Delta_j)$ are defined as in (2.6).

Replacing $|\Delta_j|$ by $\frac{|\Delta|}{n+1}$ (following (2.2)) and summing (3.1) side to side over $j = 1, 2, ..., n+1$, with (2.6), we obtain

$$\frac{1}{n+1} \sum_{j=1}^{n+1} f\Big(\sum_{i=1, i\neq j}^{n+1} \frac{p_i}{n+1} + \frac{m}{n+1}\Big) \leq \frac{1}{|\Delta|} \sum_{j=1}^{n+1} \int_{\Delta_j} f(x)dx$$

$$\leq \frac{1}{(n+1)^2} \sum_{j=1}^{n+1} \sum_{i=1, i\neq j}^{n+1} f(p_i) + \frac{1}{n+1} f(m),$$

which, with (2.1) and the fact that

$$\sum_{j=1}^{n+1} \int_{\Delta_j} f(x)dx = \int_{\bigcup_{j=1}^{n+1} \Delta_j} f(x)dx = \int_{\Delta} f(x)dx = |\Delta| M_f(\Delta),$$

yields

$$\frac{1}{n+1} \sum_{j=1}^{n+1} f\Big(\sum_{i=1, i\neq j}^{n+1} \frac{p_i}{n+1} + \frac{m}{n+1}\Big) \leq M_f(\Delta) \leq \frac{1}{(n+1)^2} \sum_{j=1}^{n+1} \sum_{i=1, i\neq j}^{n+1} f(p_i) + \frac{1}{n+1} f(m). \tag{3.2}$$

Summarizing the above, we have started from initial lower and upper bounds of (HHI), named $L_0(\Delta)$ and $U_0(\Delta)$, for obtaining new lower and upper bounds of $f : \Delta \longrightarrow \mathbb{R}$, respectively, given by

$$L_1(\Delta) = \frac{1}{n+1} \sum_{j=1}^{n+1} L_0\big(\Delta_j\big) = \frac{1}{n+1} \sum_{j=1}^{n+1} f\Big(\sum_{i=1, i\neq j}^{n+1} \frac{p_i}{n+1} + \frac{m}{n+1}\Big) \tag{3.3}$$

and

$$U_1(\Delta) = \frac{1}{n+1} \sum_{j=1}^{n+1} U_0\big(\Delta_j\big) = \frac{1}{(n+1)^2} \sum_{j=1}^{n+1} \sum_{i=1, i\neq j}^{n+1} f(p_i) + \frac{1}{n+1} f(m). \tag{3.4}$$

After this, the next result may be stated.

Proposition 3.1. *With the above, the following relationships are met:*

$$L_1(\Delta) = \frac{1}{n+1}\sum_{j=1}^{n+1} f\left(\frac{p_j + (n+2)\sum_{i=1, i\neq j}^{n+1} p_i}{(n+1)^2}\right), \tag{3.5}$$

$$U_1(\Delta) = \frac{n}{(n+1)^2}\sum_{i=1}^{n+1} f(p_i) + \frac{1}{n+1} f\left(\frac{\sum_{i=1}^{n+1} p_i}{n+1}\right). \tag{3.6}$$

Proof. It is straightforward: we use (3.3) and (3.4) with $m = \frac{\sum_{i=1}^{n+1} p_i}{n+1}$ and a classical manipulation on summation. Detail is simple and omitted here for the reader. □

Now, we are in position to state the following result.

Proposition 3.2. *With the above, (3.2) is a refinement of (1.1), that is,*

$$L_0(\Delta) \le L_1(\Delta) \le M_f(\Delta) \le U_1(\Delta) \le U_0(\Delta). \tag{3.7}$$

Proof. Inequalities (3.7) can be proved by using (3.5) and (3.6) with a help of the generalized Jensen inequality applied for the convex function f. See also [4] for a similar way. □

Example 3.1. Let $n = 1$ and $\Delta = [a, b]$ with $a < b$. Then we have

$$L_0(\Delta) = f\Big(\frac{a+b}{2}\Big),\ L_1(\Delta) = \frac{1}{2}\left(f\Big(\frac{a+3b}{4}\Big) + f\Big(\frac{3a+b}{4}\Big)\right),$$

$$U_0(\Delta) = \frac{f(a)+f(b)}{2},\ U_1(\Delta) = \frac{1}{2}\left(\frac{f(a)+f(b)}{2} + f\Big(\frac{a+b}{2}\Big)\right).$$

Substituting these relationships in (3.7) we obtain a well-known refinement of (HHI) for convex function $f : [a, b] \longrightarrow \mathbb{R}$, see [2, 3] for instance.

Example 3.2. Let $n = 2$ and Δ be the triangle of sides a, b and c. We have

$$L_0(\Delta) = f\Big(\frac{a+b+c}{3}\Big),\ \ U_0(\Delta) = \frac{f(a)+f(b)+f(c)}{3},$$

and using (3.5) and (3.6) we obtain, respectively,

$$L_1(\Delta) = \frac{1}{3}\left(f\Big(\frac{a+4b+4c}{9}\Big) + f\Big(\frac{4a+b+4c}{9}\Big) + f\Big(\frac{4a+4b+c}{9}\Big)\right),$$

$$U_1(\Delta) = \frac{2}{3}\left(\frac{f(a)+f(b)+f(c)}{3}\right) + \frac{1}{3} f\Big(\frac{a+b+c}{3}\Big).$$

We can repeat the same procedure as above by starting from the new lower and upper bounds $L_1(\Delta)$ and $U_1(\Delta)$ of $M_f(\Delta)$, respectively. We then construct, by a mathematical induction, two recursive sequences $\big(L_k\big)_k$ and $\big(U_k\big)_k$ such that

$$L_{k+1}(\Delta) = \frac{1}{n+1}\sum_{j=1}^{n+1} L_k(\Delta_j),\ \ U_{k+1}(\Delta) = \frac{1}{n+1}\sum_{j=1}^{n+1} U_k(\Delta_j), \tag{3.8}$$

where the initial data $L_0(\Delta_j)$ and $U_0(\Delta_j)$ are given following Definition 2.1.

The following result may immediately be deduced from the above.

Proposition 3.3. *With the above notations, $\big(L_k(\Delta)\big)_k$ is an increasing sequence while $\big(U_k(\Delta)\big)_k$ is a decreasing one. Further, the following chain of refinements for (HHI)*

$$L_0\big(\Delta\big) \le L_1\big(\Delta\big) \le L_2\big(\Delta\big) \le ... \le L_{k-1}\big(\Delta\big) \le L_k\big(\Delta\big) \le M_f(\Delta) \le U_k\big(\Delta\big) \le U_{k-1}\big(\Delta\big) \le ... \le U_1\big(\Delta\big) \le U_0\big(\Delta\big), \tag{3.9}$$

holds true for every integer $k \ge 0$.

We need to state the following result which will be a good tool for ensuring our desired claim.

Theorem 3.4. *With the above, we have for all integer $k \ge 0$*

$$U_{k+1}\big(\Delta\big) = \frac{n}{n+1}U_k\big(\Delta\big) + \frac{1}{n+1}L_k\big(\Delta\big), \tag{3.10}$$

where $L_k(\Delta)$ and $U_k(\Delta)$ are defined recursively as in (2.4) and (2.5) for $D = \Delta$.

Proof. We use a mathematical induction on $k \ge 0$. For $k = 0$, it follows from (3.6) with (2.4). Assume that (3.10) is true for $k = p$. We have, with (3.8),

$$(n+1)U_{p+1}\big(\Delta\big) = \sum_{j=1}^{n+1} U_p\big(\Delta_j\big) = \sum_{j=1}^{n+1}\left(\frac{n}{n+1}U_{p-1}\big(\Delta_j\big) + \frac{1}{n+1}L_{p-1}\big(\Delta_j\big)\right)$$
$$= \frac{n}{n+1}\sum_{j=1}^{n+1} U_{p-1}\big(\Delta_j\big) + \frac{1}{n+1}\sum_{j=1}^{n+1} L_{p-1}\big(\Delta_j\big).$$

By (3.8) again we deduce

$$U_{p+1}\big(\Delta\big) = \frac{n}{n+1}U_p\big(\Delta\big) + \frac{1}{n+1}L_p\big(\Delta\big),$$

that is, (3.10) is true for $p + 1$. This concludes the proof. □

Now, we are in position to state the following result which ensures our above claim.

Theorem 3.5. *The sequences $\big(L_k(\Delta)\big)_k$ and $\big(U_k(\Delta)\big)_k$ both converge with the same limit $M_f(\Delta)$:*

$$\lim_{k\to\infty} L_k\big(\Delta\big) = \sup_{k\ge 0} L_k\big(\Delta\big) = M_f\big(\Delta\big) = \inf_{k\ge 0} U_k\big(\Delta\big) = \lim_{k\to\infty} U_k\big(\Delta\big). \tag{3.11}$$

Further, the following estimation holds

$$\forall k \ge 0 \quad 0 \le U_k\big(\Delta\big) - M_f\big(\Delta\big) \le \left(\frac{n}{n+1}\right)^k \Big(U_0\big(\Delta\big) - L_0\big(\Delta\big)\Big). \tag{3.12}$$

Proof. Following Proposition 3.3, the sequence $\big(L_k(\Delta)\big)_k$ is monotonic increasing bounded above by $U_0(\Delta)$ while the sequence $\big(U_k(\Delta)\big)_k$ is monotonic decreasing bounded below by $L_0(\Delta)$, then they both converge. According to (3.10) we obtain, by letting $k \to \infty$:

$$\lim_k U_{k+1}(\Delta) = \frac{n}{n+1}\lim_k U_k(\Delta) + \frac{1}{n+1}\lim_k L_k(\Delta).$$

Reducing this latter equality we get $lim_k L_k(\Delta) = lim_k U_k(\Delta) := L(\Delta)$. Now, letting $k \to \infty$ in (3.9) we immediately deduce that $L(\Delta) = M_f(\Delta)$.

Now, we will prove (3.12). According to (3.10) again, we can write

$$U_{k+1}(\Delta) - M_f(\Delta) = \frac{n}{n+1}\Big(U_k(\Delta) - M_f(\Delta)\Big) + \frac{1}{n+1}\Big(L_k(\Delta) - M_f(\Delta)\Big), \tag{3.13}$$

which, with a help of Proposition 3.3, yields

$$0 \le U_{k+1}(\Delta) - M_f(\Delta) \le \frac{n}{n+1}\Big(U_k(\Delta) - M_f(\Delta)\Big).$$

The desired estimation follows by a simple mathematical induction on k, with the fact that $L_0(\Delta) \le M_f(\Delta)$. The proof of the theorem is completed. □

The relationship (3.10) is very useful: it can be used for showing again that the limits of $\big(L_k(\Delta)\big)_k$ and $\big(U_k(\Delta)\big)_k$ coincide. Such relation is also interesting in the practical context for computing recursively the iterate terms of $\big(U_k(\Delta)\big)_k$, see section below for some examples. Further, (3.10) will be a good tool for deducing more interesting results as discussed below.

Corollary 3.6. *The following inequalities*

$$0 \le M_f(\Delta) - L_k(\Delta) \le n\Big(U_k(\Delta) - M_f(\Delta)\Big) \tag{3.14}$$

hold true for every integer $k \ge 0$.

In what follows and for the sake of simplicity, we can omit the Δ in the iterative lower and upper bounds of $M_f(\Delta)$ and we briefly write L_k, M_f, U_k.

Proof. By Proposition 3.3 with (3.10), we have (for all $k \ge 0$)

$$M_f \le U_{k+1} = \frac{nU_k + L_k}{n+1},$$

or again,

$$0 \le M_f - L_k \le nU_k - nM_f = n\big(U_k - M_f\big),$$

which is the desired result. □

As already pointed in the introduction, the particular case $k = 0$ in the above corollary was proved (in a different and long way) in [10] (pages 595-596).

From the above we can also deduce the next result.

Corollary 3.7. *The two numerical series $\sum_{k=0}^{\infty}\big(U_k - M_f\big)$ and $\sum_{k=0}^{\infty}\big(M_f - L_k\big)$ both converge with the following estimations*

$$\sum_{k=0}^{\infty}\big(M_f - L_k\big) \le n\sum_{k=0}^{\infty}\big(U_k - M_f\big) \le n(n+1)\big(U_0 - L_0\big), \tag{3.15}$$

and the next relationships

$$\sum_{k=0}^{\infty}\big(M_f - L_k\big) + \sum_{k=0}^{\infty}\big(U_k - M_f\big) = (n+1)\big(U_0 - M_f\big), \tag{3.16}$$

$$\sum_{k=0}^{\infty}\big(U_k-L_k\big)=(n+1)\big(U_0-M_f\big). \tag{3.17}$$

Proof. Since $0<\frac{n}{n+1}<1$ we deduce from (3.12) that the series $\sum_{k=0}^{\infty}\big(U_k-M_f\big)$ converges with

$$\sum_{k=0}^{\infty}\big(U_k-M_f\big)\le(U_0-L_0)\sum_{k=0}^{\infty}\left(\frac{n}{n+1}\right)^k=(n+1)(U_0-L_0).$$

This, with (3.14), implies that the series $\sum_{k=0}^{\infty}\big(M_f-L_k\big)$ converges with

$$\sum_{k=0}^{\infty}\big(M_f-L_k\big)\le n\sum_{k=0}^{\infty}\big(U_k-M_f\big).$$

Summarizing the above, inequalities (3.15) are completely proved.

Now, by virtue of (3.10) we can write

$$\sum_{k=0}^{\infty}\big(U_{k+1}-M_f\big)=\frac{n}{n+1}\sum_{k=0}^{\infty}\big(U_k-M_f\big)-\frac{1}{n+1}\sum_{k=0}^{\infty}\big(M_f-L_k\big),$$

or equivalently,

$$\sum_{k=0}^{\infty}\big(U_k-M_f\big)-\big(U_0-M_f\big)=\frac{n}{n+1}\sum_{k=0}^{\infty}\big(U_k-M_f\big)-\frac{1}{n+1}\sum_{k=0}^{\infty}\big(M_f-L_k\big).$$

The desired relationship (3.16) follows from this latter equality with a simple reduction.

Since the two above series converge then we can write

$$\sum_{k=0}^{\infty}\big(M_f-L_k\big)+\sum_{k=0}^{\infty}\big(U_k-M_f\big)=\sum_{k=0}^{\infty}\Big(\big(M_f-L_k\big)+\big(U_k-M_f\big)\Big)=\sum_{k=0}^{\infty}\big(U_k-L_k\big).$$

Relationship (3.17) follows by combining this latter equality with (3.16). The proof of the corollary is complete. □

Remark 3.1. Relationship (3.11) states that M_f can be expressed as common limit of its iterate sequences $(L_k)_k$ and $(U_k)_k$ while (3.17) states that

$$M_f=U_0-\frac{1}{n+1}\sum_{k=0}^{\infty}\big(U_k-L_k\big),$$

i.e. M_f is expressed in sum of the series $\sum_{k=0}^{\infty}(U_k-L_k)$ whose general term is the difference between the above iterate estimates of M_f.

4. **Two Examples**

Let E be the canonical $(n+1)$-simplex of $\mathbb{R}^n$ of vertices $0,e_1,e_2,...,e_n$ where $(e_1,e_2,...,e_n)$ denotes the canonical basis of $\mathbb{R}^n$, that is,

$$E=\Big\{(x_1,x_2,...,x_n)\in\mathbb{R}^n,\ \sum_{i=1}^{n}x_i\le 1 \text{ and } x_i\ge 0 \text{ for all } i\Big\}.$$

Recalling that $|E| = (n!)^{-1}$, (HHI) yields

$$f\Big(\frac{e}{n+1}\Big) \leq n! \int_E f(x)dx \leq \frac{1}{n+1}\Big(f(0) + \sum_{i=1}^{n} f(e_i)\Big), \tag{4.1}$$

where $e := \sum_{i=1}^n e_i = (1,1,1,...,1)$. We will consider some typical situations by choosing appropriate convex functions.

4.1. **Case where f is a power-norm.** Let $p \geq 1$ be a real number and take $f(x) = \|x\|^p$ in (4.1) for obtaining:

$$L_0(E) := \frac{\|e\|^p}{(n+1)^p} \leq n! \int_E \|x\|^p dx \leq \frac{1}{n+1}\sum_{i=1}^{n} \|e_i\|^p := U_0(E).$$

Following (3.11) or (3.7) we have (after simple reduction)

$$U_1(E) = \frac{n}{(n+1)^2}\sum_{i=1}^{n} \|e_i\|^p + \frac{\|e\|^p}{(n+1)^{p+1}},$$

and by (3.6) one has (after long but elementary computation and reduction)

$$L_1(E) = \frac{(n+2)^p}{(n+1)^{2p+1}}\|e\|^p + \frac{1}{(n+1)^{2p+1}}\sum_{i=1}^{n} \|(n+2)e - (n+1)e_i\|^p.$$

If the norm $\|.\|$ is symmetric in $x_1, x_2, ..., x_n$, as the three classical norms of $\mathbb{R}^n$, then the above expressions are reduced to

$$U_0(E) = \frac{n}{n+1}\|e_1\|^p, \;\; U_1(E) = \frac{n^2}{(n+1)^2}\|e_1\|^p + \frac{\|e\|^p}{(n+1)^{p+1}},$$
$$L_1(E) = \frac{(n+2)^p}{(n+1)^{2p+1}}\|e\|^p + \frac{n}{(n+1)^{2p+1}}\|(n+2)e - (n+1)e_1\|^p.$$

4.2. **Case where f is power-quadratic.** Let $A = (a_{ij})$ be a real or complex (self-adjoint) positive matrix of size n. For $\alpha > 0$ fixed real number, we set

$$f_\alpha(x) = \big(< Ax, x > \big)^\alpha = \Big(\sum_{i,j=1}^{n} a_{ij}x_ix_j\Big)^\alpha$$

for all $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, where $< .,. >$ is the standard inner product of $\mathbb{R}^n$.

The next result may be stated.

Proposition 4.1. *With the above, assume that $\alpha \geq 1/2$. Then f_α is convex on $\mathbb{R}^n$.*

Proof. Since A is (self-adjoint) positive then $< Ax, x >= \|A^{1/2}x\|^2$, where $A^{1/2}$ denotes the matrix root of A and $\|.\|$ the euclidian norm of $\mathbb{R}^n$. The desired result follows after an elementary manipulation. Detail is simple and omitted here. □

Assuming that $\alpha \geq 1/2$ in what follows, we can then apply (HHI) for f_α on $E \subset \mathbb{R}^n$ and (4.1) yields

$$0 \leq \frac{\big(< Ae, e > \big)^\alpha}{(n+1)^{2\alpha}} \leq (n!) \int_E \big(< Ax, x > \big)^\alpha dx \leq \frac{\sum_{i=1}^n \big(< Ae_i, e_i > \big)^\alpha}{n+1}.$$

It is easy to see that $< Ae_i, e_i >= a_{ii}$ and $< Ae, e >= \sum_{i,j=1}^n a_{ij}$, and so the above double inequality becomes

$$0 \leq L_0(E) := \frac{\left(\sum_{i,j=1}^n a_{ij}\right)^\alpha}{(n+1)^{2\alpha}} \leq (n!)\int_E \Big(\sum_{i,j=1}^n a_{ij}x_ix_j\Big)^\alpha dx \leq \frac{\sum_{i=1}^n (a_{ii})^\alpha}{n+1} := U_0(E).$$

We left to the reader the routine task for computing the corresponding $L_1(E)$ and $U_1(E)$ via (3.5) and (3.6), respectively.

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