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Approximation by Interpolating Neural Network Operators

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Abstract

Here we introduce some general interpolating neural network operators in the univariate and multivariate cases. Initially we establish the interpolation property of the operators on functions. Then we derive the approximation properties of these operators on functions. We prove first the ordinary real quantitative pointwise and uniform convergences of these operators to the unit. Smoothness of functions is taken into consideration and speed of convergence improves dramatically. As extensions we consider also the fractional, fuzzy, fuzzy-fractional, fuzzy-random, complex and iterated cases. Furthermore we give Voronovskaya type asymptotic-expansions at all studied settings for the errors of related approximations.

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1 Introduction

This article is mainly inspired by the great article of D. Costarelli [27], where he establishes interpolation and approximation properties of very specific neural network operators.

We present here the general related theory of similar general neural network operators. We expand to all possible directions.

The featured interpolation and approximation properties of our approximations is something very rare.

We mention next in very brief the initial D. Costarelli ([27]) theory.

We consider $C([a,b])$ the space of all continuous functions $f:[a,b]\to\mathbb{R}$, $a,b\in\mathbb{R}$, $a<b$. Let now $\sigma_R:\mathbb{R}\to[0,1]$ the ramp function defined by

$$\sigma_R(x):=\begin{cases}0, \ x\leq-\frac{1}{2},\\ 1, \ x\geq\frac{1}{2},\\ x+\frac{1}{2}, \ \ -\frac{1}{2}<x<\frac{1}{2}.\end{cases} \tag{1}$$

The ramp function is a sigmoidal function $\sigma:\mathbb{R}\to\mathbb{R}$ which is measurable with $\lim_{x\to-\infty}\sigma(x)=0$ and $\lim_{x\to+\infty}\sigma(x)=1$. The last features arise in the theory of neural networks, where sigmoidal functions play the role of activation functions in the networks, see [38].

In [27], the author introduces

$$\Phi_R(x):=\sigma_R\left(x+\frac{1}{2}\right)-\sigma_R\left(x-\frac{1}{2}\right), \quad x\in\mathbb{R}. \tag{2}$$

The function $\Phi_R(x)$ has the properties: it is even, non-decreasing for $x<0$ and non-increasing for $x\geq0$, $\sup p(\Phi_R)\subseteq[-1,1]$. Notice that $\Phi_R(\pm1)=0$.

Thus for $f:[a,b]\to\mathbb{R}$ a bounded and measurable function D. Costarelli [27], defines the neural network interpolation operator

$$F_n(f,x):=\frac{\sum_{k=0}^{n}f(x_k)\Phi_R\left(\frac{n(x-x_k)}{b-a}\right)}{\sum_{k=0}^{n}\Phi_R\left(\frac{n(x-x_k)}{b-a}\right)}, \quad x\in[a,b], \tag{3}$$

where the x_k's are the uniform spaced nodes defined by $x_k:=a+kh$, $k=0,1,...,n$, with $h:=\frac{b-a}{n}$.

For a bounded measurable function f he proves

$$\|F_n(f)\|_\infty\leq\|f\|_\infty<+\infty, \tag{4}$$

where $\|f\|_\infty:=\sup_{x\in[a,b]}|f(x)|$.

He also proves

Theorem 1 *([27]) Let $f:[a,b]\to\mathbb{R}$ a bounded measurable function and $n\in\mathbb{N}$. Then*

$$F_n(f,x_i)=f(x_i), \quad i=0,1,...,n. \tag{5}$$

Theorem 2 *([27]) Let $f\in C([a,b])$. Then*

$$\|F_n(f)-f\|_\infty\leq4\omega_1\left(f,\frac{b-a}{n}\right), \quad \forall\ n\in\mathbb{N}. \tag{6}$$

Above he uses

$$\omega_1(f,\delta) := \sup_{\substack{x,y:\\ |x-y|\leq\delta}} |f(x)-f(y)|, \quad 0<\delta\leq b-a, \tag{7}$$

and if $\delta > b-a$, $\omega_1(f,\delta) := \omega_1(f,b-a)$, the first modulus of continuity.

D. Costarelli ([27]) gives also another specific example of interpolation neural network operators with the same properties as the F_n operators.

Denote by

$$M_s(x) := \frac{1}{(s-1)!}\sum_{i=0}^{s}(-1)^i\binom{s}{i}\left(\frac{s}{2}+x-i\right)_+^{s-1}, \quad x\in\mathbb{R}, \tag{8}$$

the B-spline of order $s\in\mathbb{N}$ ([25]), where $(x)_+ = \max\{x,0\}$, and $\sup p(M_S) \subseteq \left[-\frac{s}{2},\frac{s}{2}\right]$.

He defines ([27]) the sigmoidal functions

$$\sigma_{M_s}(x) := \int_{-\infty}^{x} M_s(t)\,dt, \quad x\in\mathbb{R}, \tag{9}$$

and the non-negative density functions:

$$\Phi_s(x) := \sigma_{M_s}\left(x+\frac{1}{2}\right) - \sigma_{M_s}\left(x-\frac{1}{2}\right), \quad x\in\mathbb{R},\ \forall\ s\in\mathbb{N}. \tag{10}$$

The functions Φ_s have the properties: even, non-decreasing for $x<0$ and non-increasing for $x\geq 0$, $\sup p(\Phi_s) \subseteq [-K_s,K_s] := \left[-\frac{(s+1)}{2},\frac{(s+1)}{2}\right]$ and $\Phi_s\left(\frac{K_s}{2}\right) > 0$. Notice that $\Phi_s(\pm K_s) = 0$.

He ([27]) defines similarly the neural network operators

$$F_n^s(f,x) := \frac{\sum_{k=0}^{n} f(x_k)\,\Phi_s\left(K_s\frac{n(x-x_k)}{b-a}\right)}{\sum_{k=0}^{n}\Phi_s\left(K_s\frac{n(x-x_k)}{b-a}\right)}, \quad \forall\ x\in[a,b], \tag{11}$$

where $x_k := a+kh$, $k=0,1,...,n$, and $h := \frac{b-a}{n}$.

Theorem 3 *([27]) Let $f:[a,b]\to\mathbb{R}$ a bounded and measurable function, $n\in\mathbb{N}$. Then*

$$F_n^s(f,x_k) = f(x_k), \quad k=0,1,...,n, \ s\in\mathbb{N}, \tag{12}$$

the interpolation property.

In addition, for $f\in C([a,b])$ we have

$$\|F_n^s(f)-f\|_\infty \leq \frac{2}{\Phi_s\left(\frac{K_s}{2}\right)}\omega_1\left(f,\frac{b-a}{n}\right), \quad \forall\ n,s\in\mathbb{N}. \tag{13}$$

Above the samples $f(x_k)$ can be viewed as the elements of the training set that can be used to train the normalized neural networks F_n, F_n^s. According to [27], the interpolation results show that the representation errors made by F_n, F_n^s on the elements of the training set are zero.

Furthermore the uniform approximation results, show the closeness property of neural network operators to well estimate elements outside the training set.

So our general theory presented in this article is the natural and complete outgrowth of [27] in very general diverse settings.

Other books and articles that inspired our work are: [12], [16], [17], [18], [19], [20], [21], [22], [23], [26], [36], [37].

The author was the first in 1997 to establish quantitative neural network approximations, see [1], [2], [3], [5], etc.

2 Main Results

2.1 Neural Networks: Univariate theory of Interpolation and Approximation

We need

Definition 4 *Let $B : \mathbb{R} \to \mathbb{R}_+$, be a bell-shaped function of compact support $[-T, T]$, $T > 0$. We assume it is even, non-decreasing for $x < 0$ and non-increasing for $x \geq 0$. Suppose also that $B(0) =: B^* > 0$ is the global maximum of B. The function B may have jump discontinuities and it is measurable. Assume further that $B(\pm T) = 0$.*

Examples for B can be the hat function

$$\beta(x) := \begin{cases} 1 + x, & -1 \leq x \leq 0, \\ 1 - x, & 0 < x \leq 1, \\ 0, & \text{elsewhere}, \end{cases}$$

the function Φ_R, see (2), and the function Φ_s, see (10). Etc.

Definition 5 *Let $f : [a, b] \to \mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$, a bounded and measurable function, $n \in \mathbb{N}$, $h := \frac{b-a}{n}$, $x_k := a + kh$, $k = 0, 1, ..., n$, $x \in [a, b]$.*

We define the interpolation neural network operator

$$H_n(f, x) := \frac{\sum_{k=0}^{n} f(x_k) B\left(\frac{Tn(x-x_k)}{b-a}\right)}{\sum_{k=0}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}. \tag{14}$$

We make

Remark 6 *(on $H_n(f,x)$) We observe that*

$$|H_n(f,x)| \le \frac{\sum_{k=0}^{n} |f(x_k)| B\left(\frac{Tn(x-x_k)}{b-a}\right)}{\sum_{k=0}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)} \le \|f\|_\infty < +\infty. \quad (15)$$

That is

$$\|H_n(f)\|_\infty \le \|f\|_\infty. \quad (16)$$

We make

Remark 7 *Let $x \in [a,b]$, then $x_k \le x \le x_{k+1}$, for some $k \in \{0,1,...,n-1\}$, and $|x-x_k| \le h$, $|x-x_{k+1}| \le h$.*

Notice that $B\left(\frac{Tn(x-x_k)}{b-a}\right) \ne 0$

$$\begin{gathered}\Leftrightarrow -T < \frac{Tn(x-x_k)}{b-a} < T \\ \Leftrightarrow -1 < \frac{n(x-x_k)}{b-a} < 1 \\ \Leftrightarrow -h < x-x_k < h \\ \Leftrightarrow |x-x_k| < h.\end{gathered} \quad (17)$$

So when $x \in (x_k, x_{k+1})$, for some $k \in \{0,1,...,n-1\}$, we get both

$$B\left(\frac{Tn(x-x_k)}{b-a}\right),\ B\left(\frac{Tn(x-x_{k+1})}{b-a}\right) \ne 0.$$

When $x = x_k$, then

$$B\left(\frac{Tn(x_k-x_k)}{b-a}\right) = B(0) = B^* > 0,$$

and

$$B\left(\frac{Tn(x_k-x_{k+1})}{b-a}\right) = B(-T) = 0.$$

When $x = x_{k+1}$, then

$$B\left(\frac{Tn(x_{k+1}-x_k)}{b-a}\right) = B(T) = 0,$$

and

$$B\left(\frac{Tn(x_{k+1}-x_{k+1})}{b-a}\right) = B(0) = B^* > 0.$$

Clearly for any $x \in [x_k, x_{k+1}]$ we get that

$$B\left(\frac{Tn(x-x_i)}{b-a}\right) = 0, \ \text{for all } i \ne k, k+1. \quad (18)$$

We make

Remark 8 *For $x \in [a,b]$ we notice that*

$$V(x) := \sum_{k=0}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right) = \sum_{k=0}^{n} B\left(\frac{Tn|x-x_k|}{b-a}\right) \geq B\left(\frac{Tn|x-x_i|}{b-a}\right), \tag{19}$$

where $i \in \{0,1,...,n\}$ is such that $|x-x_i| \leq \frac{h}{2}$. Thus

$$\frac{Tn|x-x_i|}{b-a} \leq \frac{Tnh}{2(b-a)} = \frac{T}{2}. \tag{20}$$

Therefore

$$B\left(\frac{Tn|x-x_i|}{b-a}\right) \geq B\left(\frac{T}{2}\right), \tag{21}$$

where $B\left(\frac{T}{2}\right) > 0$.

Thus $V(x) \geq B\left(\frac{T}{2}\right)$.

Consequently it holds

$$\frac{1}{V(x)} = \frac{1}{\sum_{k=0}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)} \leq \frac{1}{B\left(\frac{T}{2}\right)}. \tag{22}$$

We state the interpolation result

Theorem 9 *Let $f : [a,b] \to \mathbb{R}$ be a bounded and measurable function. Then*

$$H_n(f, x_i) = f(x_i), \quad i = 0,1,...,n, \tag{23}$$

where $x_i := a + ih$, $h := \frac{b-a}{n}$, $n \in \mathbb{N}$.

Proof. Let $i \in \{0,1,...,n\}$ be fixed. When $k = i$, we have that

$$B\left(\frac{Tn(x_i-x_k)}{b-a}\right) = B(0) = B^* > 0. \tag{24}$$

But when $k \neq i$ we have

$$\frac{Tn|x_i-x_k|}{b-a} \geq \frac{Tnh}{b-a} = T, \tag{25}$$

hence

$$0 \leq B\left(\frac{Tn(x_i-x_k)}{b-a}\right) = B\left(\frac{Tn|x_i-x_k|}{b-a}\right) \leq B(T) = 0. \tag{26}$$

So we conclude that

$$B\left(\frac{Tn(x_i-x_k)}{b-a}\right) = \left\{\begin{array}{l} B^*, \ \ i = k, \\ 0, \ \ i \neq k \end{array}\right\}, \tag{27}$$

for any $i, k = 0, 1, ..., n$.

By (27) we derive that

$$H_n(f, x_i) = \frac{f(x_i) B\left(\frac{Tn(x_i - x_i)}{b-a}\right)}{B\left(\frac{Tn(x_i - x_i)}{b-a}\right)} = \frac{f(x_i) B^*}{B^*} = f(x_i), \quad i = 0, 1, ..., n, \quad (28)$$

proving the claim. ■

We state our first approximation result at Jackson speed of convergence $\frac{1}{n}$.

Theorem 10 *Let $f \in C([a, b])$. Then*

$$\|H_n(f) - f\|_\infty \leq \frac{2B^*}{B\left(\frac{T}{2}\right)} \omega_1\left(f, \frac{b-a}{n}\right), \quad \forall\, n \in \mathbb{N}. \quad (29)$$

Proof. Let $x \in [a, b]$, we can write

$$H_n(f, x) - f(x) = \frac{\sum_{k=0}^{n} f(x_k) B\left(\frac{Tn(x-x_k)}{b-a}\right)}{\sum_{k=0}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)} - f(x) =$$

$$\frac{\sum_{k=0}^{n} f(x_k) B\left(\frac{Tn(x-x_k)}{b-a}\right) - f(x)\left(\sum_{k=0}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)\right)}{V(x)} =$$

$$\frac{\sum_{k=0}^{n} \left(f(x_k) - f(x)\right) B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}. \quad (30)$$

Therefore it holds

$$|H_n(f, x) - f(x)| \leq \frac{\sum_{k=0}^{n} |f(x_k) - f(x)| B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \overset{(22)}{\leq}$$

$$\frac{1}{B\left(\frac{T}{2}\right)} \left\{ \sum_{k=0}^{n} |f(x_k) - f(x)| B\left(\frac{Tn(x - x_k)}{b-a}\right) \right\} =: (*). \quad (31)$$

Let now $i \in \{0, 1, ..., n-1\}$ such that $x_i \leq x \leq x_{i+1}$. Hence

$$(*) = \frac{1}{B\left(\frac{T}{2}\right)} \left\{ \sum_{\substack{k=0 \\ k \neq i, i+1}}^{n} |f(x_k) - f(x)| B\left(\frac{Tn(x - x_k)}{b-a}\right) + \right.$$

$$\left. |f(x_i) - f(x)| B\left(\frac{Tn(x - x_i)}{b-a}\right) + |f(x_{i+1}) - f(x)| B\left(\frac{Tn(x - x_{i+1})}{b-a}\right) \right\} \leq$$

$$\frac{1}{B\left(\frac{T}{2}\right)}\left\{0+\omega_1\left(f,h\right)B^*+\omega_1\left(f,h\right)B^*\right\}=\frac{2B^*}{B\left(\frac{T}{2}\right)}\omega_1\left(f,h\right). \tag{32}$$

We derive for $f \in C\left([a,b]\right)$ that it holds

$$\left|H_n\left(f,x\right)-f\left(x\right)\right| \leq \frac{2B^*}{B\left(\frac{T}{2}\right)}\omega_1\left(f,\frac{b-a}{n}\right), \quad \forall\, x \in [a,b]. \tag{33}$$

The theorem now is proved. ■

Taking into account the smoothness of f, we present the following high order approximation result.

Theorem 11 *Let $f \in C^N\left([a,b]\right)$, $N \in \mathbb{N}$, $x \in [a,b]$. Then*

i)

$$\left|H_n\left(f,x\right)-f\left(x\right)\right| \leq$$

$$\frac{2B^*}{B\left(\frac{T}{2}\right)}\left[\sum_{j=1}^{N}\frac{\left|f^{(j)}\left(x\right)\right|}{j!}\frac{\left(b-a\right)^j}{n^j}+\omega_1\left(f^{(N)},\frac{b-a}{n}\right)\frac{\left(b-a\right)^N}{n^N N!}\right], \tag{34}$$

ii)

$$\left\|H_n\left(f\right)-f\right\|_\infty \leq$$

$$\frac{2B^*}{B\left(\frac{T}{2}\right)}\left[\sum_{j=1}^{N}\frac{\left\|f^{(j)}\right\|_\infty}{j!}\frac{\left(b-a\right)^j}{n^j}+\omega_1\left(f^{(N)},\frac{b-a}{n}\right)\frac{\left(b-a\right)^N}{n^N N!}\right], \tag{35}$$

iii) Assume more that $f^{(j)}\left(x\right)=0$, $j=1,...,N$, where $x \in [a,b]$ is fixed, we get

$$\left|H_n\left(f,x\right)-f\left(x\right)\right| \leq \frac{2B^*}{B\left(\frac{T}{2}\right)}\omega_1\left(f^{(N)},\frac{b-a}{n}\right)\frac{\left(b-a\right)^N}{n^N N!}, \tag{36}$$

a high speed $\frac{1}{n^{N+1}}$ pointwise convergence, and

iv)

$$\left|H_n\left(f,x\right)-f\left(x\right)-\sum_{j=1}^{N}\frac{f^{(j)}\left(x\right)}{j!}H_n\left(\left(\cdot-x\right)^j,x\right)\right| \leq$$

$$\frac{2B^*}{B\left(\frac{T}{2}\right)}\omega_1\left(f^{(N)},\frac{b-a}{n}\right)\frac{\left(b-a\right)^N}{n^N N!}. \tag{37}$$

Proof. Let $f \in C^N\left([a,b]\right)$, $N \in \mathbb{N}$. Then

$$f\left(x_k\right)=\sum_{j=0}^{N}\frac{f^{(j)}\left(x\right)}{j!}\left(x_k-x\right)^j+\int_x^{x_k}\left(f^{(N)}\left(t\right)-f^{(N)}\left(x\right)\right)\frac{\left(x_k-t\right)^{N-1}}{\left(N-1\right)!}dt. \tag{38}$$

Hence it holds

$$\frac{f(x_k) B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} = \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} (x_k - x)^j \frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} +$$

$$\frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \int_x^{x_k} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{(x_k - t)^{N-1}}{(N-1)!} dt. \tag{39}$$

Thus we can write

$$H_n(f,x) - f(x) = \frac{\sum_{k=0}^{n} f(x_k) B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} - f(x) =$$

$$\sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} \frac{\sum_{k=0}^{n} (x_k - x)^j B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} +$$

$$\frac{\sum_{k=0}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \int_x^{x_k} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{(x_k - t)^{N-1}}{(N-1)!} dt. \tag{40}$$

Call

$$R_n(x) := \frac{\sum_{k=0}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \int_x^{x_k} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{(x_k - t)^{N-1}}{(N-1)!} dt. \tag{41}$$

Also call

$$\gamma(x, x_k) := \left| \int_x^{x_k} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{(x_k - t)^{N-1}}{(N-1)!} dt \right|. \tag{42}$$

We distinguish the cases:

(i) Let $x \leq x_k$, then

$$\gamma(x, x_k) \leq \int_x^{x_k} \left| f^{(N)}(t) - f^{(N)}(x) \right| \frac{(x_k - t)^{N-1}}{(N-1)!} dt$$

$$\leq \omega_1\left(f^{(N)}, x_k - x\right) \frac{(x_k - x)^N}{N!}. \tag{43}$$

(ii) Let $x \geq x_k$, then

$$\gamma(x, x_k) = \left| \int_{x_k}^{x} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{(t - x_k)^{N-1}}{(N-1)!} dt \right|$$

$$\leq \int_{x_k}^{x} \left| f^{(N)}(t) - f^{(N)}(x) \right| \frac{(t-x_k)^{N-1}}{(N-1)!} dt \leq \omega_1 \left(f^{(N)}, x - x_k \right) \frac{(x-x_k)^N}{N!}. \tag{44}$$

We have found that

$$\gamma(x, x_k) \leq \omega_1 \left(f^{(N)}, |x - x_k| \right) \frac{|x-x_k|^N}{N!}. \tag{45}$$

Therefore it holds

$$|R_n(x)| \leq \frac{\sum_{k=0}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \omega_1 \left(f^{(N)}, |x - x_k| \right) \frac{|x-x_k|^N}{N!} =: (*). \tag{46}$$

Given that $x_k \leq x \leq x_{k+1}$, for some $k \in \{0, 1, ..., n-1\}$, we get

$$(*) = \frac{B\left(\frac{Tn(x-x_k)}{b-a}\right) \omega_1 \left(f^{(N)}, |x - x_k| \right) \frac{|x-x_k|^N}{N!}}{V(x)} +$$

$$\frac{B\left(\frac{Tn(x-x_{k+1})}{b-a}\right) \omega_1 \left(f^{(N)}, |x - x_{k+1}| \right) \frac{|x-x_{k+1}|^N}{N!}}{V(x)}$$

$$\leq \frac{2B^* \omega_1 \left(f^{(N)}, \frac{b-a}{n} \right) \frac{(b-a)^N}{n^N N!}}{B\left(\frac{T}{2}\right)}. \tag{47}$$

We have proved that

$$|R_n(x)| \leq \frac{2B^*}{B\left(\frac{T}{2}\right)} \omega_1 \left(f^{(N)}, \frac{b-a}{n} \right) \frac{(b-a)^N}{n^N N!}. \tag{48}$$

Next we observe

$$\frac{\left| \sum_{k=0}^{n} (x_k - x)^j B\left(\frac{Tn(x-x_k)}{b-a}\right) \right|}{V(x)} \leq \frac{\sum_{k=0}^{n} |x_k - x|^j B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}$$

$$\leq \frac{1}{B\left(\frac{T}{2}\right)} \left\{ |x_k - x|^j B\left(\frac{Tn(x-x_k)}{b-a}\right) + |x_{k+1} - x|^j B\left(\frac{Tn(x-x_{k+1})}{b-a}\right) \right\}$$

$$\leq \frac{2B^* \frac{(b-a)^j}{n^j}}{B\left(\frac{T}{2}\right)}. \tag{49}$$

Therefore we derive

$$\left| \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} \frac{\sum_{k=0}^{n} (x_k - x)^j B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \right| \leq$$

$$\frac{2B^*}{B\left(\frac{T}{2}\right)}\left(\sum_{j=1}^{N}\frac{\left|f^{(j)}(x)\right|}{j!}\frac{(b-a)^j}{n^j}\right). \tag{50}$$

Using (48) and (50) we derive (34)-(36).

Noticing that

$$\frac{\sum_{k=0}^{n}(x_k-x)^j B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} = H_n\left((\cdot - x)^j, x\right), \tag{51}$$

we derive (37).

The theorem is proved. ■

We present a related Voronovskaya type asymptotic expansion for the error of approximation.

Theorem 12 *Let $f \in C^N([a,b])$, $N \in \mathbb{N}$. Then*

$$H_n(f,x) - f(x) - \sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}H_n\left((\cdot - x)^j, x\right) = o\left(\frac{1}{n^{N-\varepsilon}}\right), \tag{52}$$

where $0 < \varepsilon \le N$, $n \in \mathbb{N}$.

If $N = 1$, the sum above disappears.

Asymptotic expansion (52) implies

$$n^{N-\varepsilon}\left[H_n(f,x) - f(x) - \sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}H_n\left((\cdot - x)^j, x\right)\right] \to 0, \text{ as } n \to \infty, \tag{53}$$

$0 < \varepsilon \le N$.

When $N = 1$, or $f^{(j)}(x) = 0$, $j = 1, ..., N-1$, then

$$n^{N-\varepsilon}\left[H_n(f,x) - f(x)\right] \to 0, \text{ as } n \to \infty, \ 0 < \varepsilon \le N. \tag{54}$$

Proof. Let $x \in [a,b]$, then

$$f(x_k) = \sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}(x_k - x)^j + \int_x^{x_k} f^{(N)}(t)\frac{(x_k - t)^{N-1}}{(N-1)!}dt. \tag{55}$$

Let here $i \in \{0, 1, ..., n-1\}$ such that $x_i \le x \le x_{i+1}$.

Hence we have

$$\frac{f(x_k)B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} = \sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}(x_k - x)^j\frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} + \tag{56}$$

$$\frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}\int_x^{x_k} f^{(N)}(t)\frac{(x_k-t)^{N-1}}{(N-1)!}dt.$$

Thus it holds

$$H_n(f,x)-f(x)=\frac{\sum_{k=0}^{n} f(x_k)\,B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}-f(x)=$$

$$\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}\frac{\sum_{k=0}^{n}(x_k-x)^j B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}+ \tag{57}$$

$$\frac{\sum_{k=0}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}\int_x^{x_k} f^{(N)}(t)\frac{(x_k-t)^{N-1}}{(N-1)!}dt.$$

Call

$$R(x):=\frac{\sum_{k=0}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}\int_x^{x_k} f^{(N)}(t)\frac{(x_k-t)^{N-1}}{(N-1)!}dt. \tag{58}$$

So that

$$H_n(f,x)-f(x)-\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}H_n\left((\cdot-x)^j,x\right)=R(x). \tag{59}$$

Hence it holds

$$|R(x)|\leq\frac{\sum_{k=0}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{B\left(\frac{T}{2}\right)}\left|\int_x^{x_k} f^{(N)}(t)\frac{(x_k-t)^{N-1}}{(N-1)!}dt\right|\leq(*). \tag{60}$$

But we find:

i) Let $x_k\geq x$. Then

$$\left|\int_x^{x_k} f^{(N)}(t)\frac{(x_k-t)^{N-1}}{(N-1)!}dt\right|\leq$$

$$\int_x^{x_k}\left|f^{(N)}(t)\right|\frac{(x_k-t)^{N-1}}{(N-1)!}dt\leq\left\|f^{(N)}\right\|_\infty\frac{(x_k-x)^N}{N!}. \tag{61}$$

ii) Let $x_k\leq x$. Then

$$\left|\int_x^{x_k} f^{(N)}(t)\frac{(x_k-t)^{N-1}}{(N-1)!}dt\right|=\left|\int_{x_k}^{x} f^{(N)}(t)\frac{(t-x_k)^{N-1}}{(N-1)!}dt\right|\leq$$

$$\leq\int_{x_k}^{x}\left|f^{(N)}(t)\right|\frac{(t-x_k)^{N-1}}{(N-1)!}dt\leq\left\|f^{(N)}\right\|_\infty\frac{(x-x_k)^N}{N!}. \tag{62}$$

So in either case we have proved

$$\left|\int_{x}^{x_k} f^{(N)}(t)\frac{(x_k-t)^{N-1}}{(N-1)!}dt\right| \leq \left\|f^{(N)}\right\|_{\infty}\frac{|x-x_k|^N}{N!}. \tag{63}$$

Therefore we find

$$(*) \leq \frac{\sum_{k=0}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{B\left(\frac{T}{2}\right)}\left\|f^{(N)}\right\|_{\infty}\frac{|x-x_k|^N}{N!}$$

$$\leq \frac{2B^*}{B\left(\frac{T}{2}\right)}\left\|f^{(N)}\right\|_{\infty}\frac{(b-a)^N}{N!n^N}. \tag{64}$$

We have proved that

$$|R(x)| \leq \frac{\psi}{n^N}, \tag{65}$$

where

$$\psi := \frac{2B^*}{B\left(\frac{T}{2}\right)}\frac{\left\|f^{(N)}\right\|_{\infty}(b-a)^N}{N!}. \tag{66}$$

Hence we derive

$$|R(x)| = O\left(\frac{1}{n^N}\right), \tag{67}$$

and

$$|R(x)| = o(1). \tag{68}$$

Letting $0 < \varepsilon \leq N$, we obtain

$$\frac{|R(x)|}{\left(\frac{1}{n^{N-\varepsilon}}\right)} \leq \frac{\psi}{n^{\varepsilon}} \to 0, \tag{69}$$

as $n \to \infty$. So that

$$|R(x)| = o\left(\frac{1}{n^{N-\varepsilon}}\right), \quad n \in \mathbb{N}, \tag{70}$$

proving the claim. ■

We need

Definition 13 *Let $\nu > 0$, $m = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^m([a,b])$ (space of functions f with $f^{(m-1)} \in AC([a,b])$, absolutely continuous functions). We call left Caputo fractional derivative (see [28], pp. 49-52, [31], [39]) the function*

$$D_{*a}^{\nu} f(x) := \frac{1}{\Gamma(m-\nu)}\int_a^x (x-t)^{m-\nu-1} f^{(m)}(t)\,dt, \tag{71}$$

$\forall\, x \in [a,b]$, where Γ is the gamma function $\Gamma(\nu) := \int_0^{\infty} e^{-t}t^{\nu-1}dt$, $\nu > 0$.

We set $D_{*a}^{0} f(x) = f(x)$, $\forall\, x \in [a,b]$.

Lemma 14 *([8]) Let $\nu > 0$, $\nu \notin \mathbb{N}$, $m = \lceil \nu \rceil$, $f \in C^{m-1}([a,b])$ and $f^{(m)} \in L_\infty([a,b])$. Then $D_{*a}^{\nu} f(a) = 0$.*

Definition 15 *(see also [9], [30], [31]) Let $f \in AC^m([a,b])$, $m = \lceil \nu \rceil$, $\nu > 0$. The right Caputo fractional derivative of order $\nu > 0$ is given by*

$$D_{b-}^{\nu} f(x) := \frac{(-1)^m}{\Gamma(m-\nu)} \int_x^b (z-x)^{m-\nu-1} f^{(m)}(z)\, dz, \tag{72}$$

$\forall\ x \in [a,b]$. We set $D_{b-}^{0} f(x) = f(x)$.

Lemma 16 *([8]) Let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \nu \rceil$, $\nu > 0$. Then $D_{b-}^{\nu} f(b) = 0$.*

Convention 17 *([8]) We assume that*

$$\begin{aligned} &D_{*x_0}^{\nu} f(x) = 0, \ \text{ for } x < x_0, \\ &\text{and} \\ &D_{x_0-}^{\nu} f(x) = 0, \ \text{ for } x > x_0, \end{aligned} \tag{73}$$

for all $x, x_0 \in [a,b]$.

We present the related fractional approximation result

Theorem 18 *Let $\beta > 0$, $N = \lceil \beta \rceil$, $\beta \notin \mathbb{N}$, $f \in AC^N([a,b])$, $f^{(N)} \in L_\infty([a,b])$. Then*

i)

$$|H_n(f,x) - f(x)| \leq \frac{B^*}{B\left(\frac{T}{2}\right)} \left[2 \sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right|}{j!} \frac{(b-a)^j}{n^j} + \right. \tag{74}$$

$$\left. \frac{(b-a)^\beta}{\Gamma(\beta+1)\, n^\beta} \left[\omega_1 \left(D_{x-}^{\beta} f, \frac{b-a}{n} \right) + \omega_1 \left(D_{*x}^{\beta} f, \frac{b-a}{n} \right) \right] \right],$$

and

ii)

$$\|H_n(f) - f\|_\infty \leq \frac{B^*}{B\left(\frac{T}{2}\right)} \left[2 \sum_{j=1}^{N-1} \frac{\left\|f^{(j)}\right\|_\infty}{j!} \frac{(b-a)^j}{n^j} + \right. \tag{75}$$

$$\left. \frac{(b-a)^\beta}{\Gamma(\beta+1)\, n^\beta} \left[\sup_{x\in[a,b]} \omega_1 \left(D_{x-}^{\beta} f, \frac{b-a}{n} \right) + \sup_{x\in[a,b]} \omega_1 \left(D_{*x}^{\beta} f, \frac{b-a}{n} \right) \right] \right] < \infty.$$

Proof. Let fixed $x \in [a,b]$ with $x_i \leq x \leq x_{i+1}$, for some $i \in \{0,1,...,n-1\}$. We have that

$$D_{x-}^{\beta} f(x) = D_{*x}^{\beta} = 0. \tag{76}$$

By Convention 17, $D_{*x}^{\beta} f(z) = 0$, for $z < x$; $D_{x-}^{\beta} f(z) = 0$, for $z > x$, all $x, z \in [a, b]$.

From [28], p. 54, we get by the left Caputo fractional Taylor formula that

$$f(x_k) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} (x_k - x)^j + \tag{77}$$

$$\frac{1}{\Gamma(\beta)} \int_x^{x_k} (x_k - J)^{\beta-1} \left(D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x) \right) dJ,$$

for all $x \leq x_k \leq b$.

Also from [9], using the right Caputo fractional Taylor formula we get

$$f(x_k) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} (x_k - x)^j + \tag{78}$$

$$\frac{1}{\Gamma(\beta)} \int_{x_k}^{x} (J - x_k)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x) \right) dJ,$$

for all $a \leq x_k \leq x$.

Hence it holds

$$\frac{f(x_k) B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} (x_k - x)^j \frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} + \tag{79}$$

$$\frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \frac{1}{\Gamma(\beta)} \int_x^{x_k} (x_k - J)^{\beta-1} \left(D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x) \right) dJ,$$

all $x \leq x_k \leq b$.

Also we have

$$\frac{f(x_k) B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} (x_k - x)^j \frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} +$$

$$\frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \frac{1}{\Gamma(\beta)} \int_{x_k}^{x} (J - x_k)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x) \right) dJ, \tag{80}$$

all $a \leq x_k \leq x$.

Hence we derive

$$\frac{\sum_{k=i+1}^{n} f(x_k) B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \frac{\sum_{k=i+1}^{n} (x_k - x)^j B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} + R_1, \tag{81}$$

where

$$R_1 := \frac{\sum_{k=i+1}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \frac{1}{\Gamma(\beta)} \int_x^{x_k} (x_k - J)^{\beta-1} \left(D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x)\right) dJ, \tag{82}$$

all $x \leq x_k \leq b$.

Also it holds

$$\frac{\sum_{k=0}^{i} f(x_k) B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \frac{\sum_{k=0}^{i} (x_k - x)^j B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} + R_2, \tag{83}$$

where

$$R_2 := \frac{\sum_{k=0}^{i} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \frac{1}{\Gamma(\beta)} \int_{x_k}^{x} (J - x_k)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\right) dJ, \tag{84}$$

all $a \leq x_k \leq x$.

Consequently, by adding (81) and (83), we obtain

$$H_n(f,x) - f(x) = \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{\sum_{k=0}^{n} (x_k - x)^j B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}\right) + R_1 + R_2. \tag{85}$$

Hence we find

$$|H_n(f,x) - f(x)| \leq \sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right|}{j!} \left(\frac{\sum_{k=0}^{n} |x_k - x|^j B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}\right) + |R_1| + |R_2|$$

$$\leq \sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right|}{j!} \frac{2(b-a)^j B^*}{n^j B\left(\frac{T}{2}\right)} + |R_1| + |R_2|. \tag{86}$$

Next we estimate $|R_1|, |R_2|$.

We have that

$$|R_1| \leq \frac{\sum_{k=i+1}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{B\left(\frac{T}{2}\right)} \frac{1}{\Gamma(\beta)} \int_x^{x_k} (x_k - J)^{\beta-1} \left|D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x)\right| dJ \tag{87}$$

$$\leq \frac{\sum_{k=i+1}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{B\left(\frac{T}{2}\right)} \frac{1}{\Gamma(\beta)} \omega_1\left(D_{*x}^{\beta} f, (x_k - x)\right) \left(\int_x^{x_k} (x_k - J)^{\beta-1} dJ\right)$$

$$= \frac{\sum_{k=i+1}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{B\left(\frac{T}{2}\right)} \frac{1}{\Gamma(\beta)} \omega_1\left(D_{*x}^{\beta} f, x_k - x\right) \frac{(x_k - x)^{\beta}}{\beta}$$

$$= \frac{\sum_{k=i+1}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{B\left(\frac{T}{2}\right)\Gamma(\beta+1)} (x_k - x)^{\beta} \omega_1\left(D_{*x}^{\beta} f, x_k - x\right) \tag{88}$$

$$\leq \frac{B\left(\frac{Tn(x-x_{i+1})}{b-a}\right)}{B\left(\frac{T}{2}\right)\Gamma(\beta+1)} \frac{(b-a)^{\beta}}{n^{\beta}} \omega_1\left(D_{*x}^{\beta} f, \frac{b-a}{n}\right). \tag{89}$$

We have proved that

$$|R_1| \leq \frac{B^*}{B\left(\frac{T}{2}\right)\Gamma(\beta+1)} \frac{(b-a)^{\beta}}{n^{\beta}} \omega_1\left(D_{*x}^{\beta} f, \frac{b-a}{n}\right). \tag{90}$$

Furthermore we observe that

$$|R_2| \leq \frac{\sum_{k=0}^{i} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{B\left(\frac{T}{2}\right)} \frac{1}{\Gamma(\beta)} \left(\int_{x_k}^{x} (J - x_k)^{\beta-1} \left| D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x) \right| dJ\right) \tag{91}$$

$$\leq \frac{\sum_{k=0}^{i} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{B\left(\frac{T}{2}\right)} \frac{1}{\Gamma(\beta)} \omega_1\left(D_{x-}^{\beta} f, x - x_k\right) \frac{(x - x_k)^{\beta}}{\beta}$$

$$= \frac{B\left(\frac{Tn(x-x_i)}{b-a}\right)}{B\left(\frac{T}{2}\right)} \frac{1}{\Gamma(\beta+1)} (x - x_i)^{\beta} \omega_1\left(D_{x-}^{\beta} f, x - x_i\right) \tag{92}$$

$$\leq \frac{B^*}{B\left(\frac{T}{2}\right)} \frac{1}{\Gamma(\beta+1)} \frac{(b-a)^{\beta}}{n^{\beta}} \omega_1\left(D_{x-}^{\beta} f, \frac{b-a}{n}\right).$$

That is we have proved

$$|R_2| \leq \frac{B^*}{B\left(\frac{T}{2}\right)\Gamma(\beta+1)} \frac{(b-a)^{\beta}}{n^{\beta}} \omega_1\left(D_{x-}^{\beta} f, \frac{b-a}{n}\right). \tag{93}$$

Thus

$$|R_1| + |R_2| \leq \frac{B^*(b-a)^{\beta}}{B\left(\frac{T}{2}\right)\Gamma(\beta+1) n^{\beta}} \left[\omega_1\left(D_{x-}^{\beta} f, \frac{b-a}{n}\right) + \omega_1\left(D_{*x}^{\beta} f, \frac{b-a}{n}\right)\right]. \tag{94}$$

So by using (86) and (94) we obtain (74), which implies (75).

Next we justify that the right hand side of (75) is finite.

We have

$$\left(D_{*x}^{\beta} f\right)(t) = \frac{1}{\Gamma(N-\beta)} \int_{x}^{t} (t-z)^{N-\beta-1} f^{(N)}(z)\, dz, \quad x \leq t \leq b. \tag{95}$$

Hence

$$\left|D_{*x}^{\beta} f(t)\right| \leq \frac{\left\|f^{(N)}\right\|_{\infty}}{\Gamma(N-\beta+1)}(b-a)^{N-\beta}, \quad x \leq t \leq b. \tag{96}$$

Thus

$$\left\|D_{*x}^{\beta} f\right\|_{\infty} \leq \frac{\left\|f^{(N)}\right\|_{\infty}}{\Gamma(N-\beta+1)}(b-a)^{N-\beta}. \tag{97}$$

Similarly

$$D_{x-}^{\beta} f(t)=\frac{(-1)^{N}}{\Gamma(N-\beta)} \int_{t}^{x}(z-t)^{N-\beta-1} f^{(N)}(z)\, dz, \quad \text{all } a \leq t \leq x. \tag{98}$$

Hence

$$\left|D_{x-}^{\beta} f(t)\right| \leq \frac{\left\|f^{(N)}\right\|_{\infty}}{\Gamma(N-\beta+1)}(b-a)^{N-\beta}, \quad a \leq t \leq x. \tag{99}$$

Thus

$$\left\|D_{x-}^{\beta} f\right\|_{\infty} \leq \frac{\left\|f^{(N)}\right\|_{\infty}}{\Gamma(N-\beta+1)}(b-a)^{N-\beta}. \tag{100}$$

Consequently (for $\delta>0$)

$$\begin{gathered}
\omega_{1}\left(D_{x-}^{\beta} f, \delta\right)=\sup_{\substack{z_{1}, z_{2} \\ \left|z_{1}-z_{2}\right| \leq \delta}}\left|D_{x-}^{\beta} f\left(z_{1}\right)-D_{x-}^{\beta} f\left(z_{2}\right)\right| \leq \\
\sup_{\substack{z_{1}, z_{2} \\ \left|z_{1}-z_{2}\right| \leq \delta}}\left\{\left|D_{x-}^{\beta} f\left(z_{1}\right)\right|+\left|D_{x-}^{\beta} f\left(z_{2}\right)\right|\right\} \leq 2\left\|D_{x-}^{\beta} f\right\|_{\infty} \\
\leq \frac{2\left\|f^{(N)}\right\|_{\infty}}{\Gamma(N-\beta+1)}(b-a)^{N-\beta}<+\infty.
\end{gathered} \tag{101}$$

Hence it holds

$$\omega_{1}\left(D_{x-}^{\beta} f, \delta\right) \leq \frac{2\left\|f^{(N)}\right\|_{\infty}}{\Gamma(N-\beta+1)}(b-a)^{N-\beta}<+\infty. \tag{102}$$

Therefore

$$\sup_{x \in[a, b]} \omega_{1}\left(D_{x-}^{\beta} f, \delta\right) \leq \frac{2\left\|f^{(N)}\right\|_{\infty}}{\Gamma(N-\beta+1)}(b-a)^{N-\beta}<+\infty, \tag{103}$$

and, similarly, we get

$$\sup_{x \in[a, b]} \omega_{1}\left(D_{*x}^{\beta} f, \delta\right) \leq \frac{2\left\|f^{(N)}\right\|_{\infty}}{\Gamma(N-\beta+1)}(b-a)^{N-\beta}<+\infty. \tag{104}$$

The proof of the theorem now is complete. ■

Corollary 19 *(to Theorem 18) All as in Theorem 18. Additionally assume that* $f^{(j)}(x) = 0$, $j = 1, ..., N-1$. *Then*

$$|H_n(f,x) - f(x)| \leq \frac{B^*}{B\left(\frac{T}{2}\right)} \frac{(b-a)^\beta}{\Gamma(\beta+1) n^\beta} \cdot \tag{105}$$

$$\left[\omega_1\left(D^\beta_{x-}f, \frac{b-a}{n}\right) + \omega_1\left(D^\beta_{*x}f, \frac{b-a}{n}\right)\right].$$

In the last we have the high speed of pointwise convergence at $\frac{1}{n^{\beta+1}}$.

A fractional Voronovskaya type asymptotic expansion follows.

Theorem 20 *Let* $\beta > 0$, $N = \lceil \beta \rceil$, $\beta \notin \mathbb{N}$, $f \in AC^N([a,b])$, $f^{(N)} \in L_\infty([a,b])$. *Then*

$$H_n(f,x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} H_n\left((\cdot - x)^j, x\right) = o\left(\frac{1}{n^{\beta-\varepsilon}}\right), \tag{106}$$

where $0 < \varepsilon \leq \beta$, $n \in \mathbb{N}$.

If $N = 1$, *the sum above disappears.*

Asymptotic expansion (106) implies

$$n^{\beta-\varepsilon}\left[H_n(f,x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} H_n\left((\cdot - x)^j, x\right)\right] \to 0, \tag{107}$$

as $n \to \infty$, $0 < \varepsilon \leq \beta$.

When $N = 1$, *or* $f^{(j)}(x) = 0$, $j = 1, ..., N-1$, *then*

$$n^{\beta-\varepsilon}\left[H_n(f,x) - f(x)\right] \to 0, \tag{108}$$

as $n \to \infty$, $0 < \varepsilon \leq \beta$.

Of great interest is the case $\beta = \frac{1}{2}$.

Proof. From [28], p. 54, we get by the left Caputo fractional Taylor formula that

$$f(x_k) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!}(x_k - x)^j + \frac{1}{\Gamma(\beta)} \int_x^{x_k} (x_k - J)^{\beta-1} D^\beta_{*x} f(J)\, dJ, \tag{109}$$

for all $x \leq x_k \leq b$.

Also from [9], using the right Caputo fractional Taylor formula we get

$$f(x_k) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!}(x_k - x)^j + \frac{1}{\Gamma(\beta)} \int_{x_k}^{x} (J - x_k)^{\beta-1} D^\beta_{x-} f(J)\, dJ, \tag{110}$$

for all $a \leq x_k \leq x$.

Hence

$$\frac{f(x_k) B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} (x_k - x)^j \frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} + \tag{111}$$

$$\frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \frac{1}{\Gamma(\beta)} \int_x^{x_k} (x_k - J)^{\beta-1} D_{*x}^{\beta} f(J)\, dJ,$$

all $x \leq x_k \leq b$.

Also we have

$$\frac{f(x_k) B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} (x_k - x)^j \frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} +$$

$$\frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \frac{1}{\Gamma(\beta)} \int_{x_k}^{x} (J - x_k)^{\beta-1} D_{x-}^{\beta} f(J)\, dJ, \tag{112}$$

all $a \leq x_k \leq x$.

Hence $x \in [a, b]$ is fixed such that $x_i \leq x \leq x_{i+1}$, for some $i \in \{0, 1, ..., n-1\}$.

Hence it holds

$$\frac{\sum_{k=i+1}^{n} f(x_k) B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \frac{\sum_{k=i+1}^{n} (x_k - x)^j B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} + R_1, \tag{113}$$

where

$$R_1 := \frac{\sum_{k=i+1}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \frac{1}{\Gamma(\beta)} \int_x^{x_k} (x_k - J)^{\beta-1} D_{*x}^{\beta} f(J)\, dJ, \tag{114}$$

all $x \leq x_k \leq b$.

Also it holds

$$\frac{\sum_{k=0}^{i} f(x_k) B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \frac{\sum_{k=0}^{i} (x_k - x)^j B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} + R_2, \tag{115}$$

where

$$R_2 := \frac{\sum_{k=0}^{i} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \frac{1}{\Gamma(\beta)} \int_{x_k}^{x} (J - x_k)^{\beta-1} D_{x-}^{\beta} f(J)\, dJ, \tag{116}$$

all $a \leq x_k \leq x$.

Hence we get

$$H_n(f,x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{\sum_{k=0}^{n} (x_k - x)^j B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \right) = R_1 + R_2. \tag{117}$$

Notice also that for any $x \in [a,b]$, by (97) and (100), we have

$$\left\{ \left\| D_{*x}^{\beta} f \right\|_{\infty}, \left\| D_{x-}^{\beta} f \right\|_{\infty} \right\} \leq \frac{\left\| f^{(N)} \right\|_{\infty}}{\Gamma(N-\beta+1)} (b-a)^{N-\beta} =: M, \tag{118}$$

with $M > 0$.

That is we find

$$H_n(f,x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} H_n\left((\cdot - x)^j, x \right) = R_1 + R_2. \tag{119}$$

Notice that

$$|R_1| \leq M \frac{\sum_{k=i+1}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \frac{1}{\Gamma(\beta+1)} (x_k - x)^{\beta} \tag{120}$$

$$\leq \frac{MB^*}{B\left(\frac{T}{2}\right)\Gamma(\beta+1)} \frac{(b-a)^{\beta}}{n^{\beta}},$$

that is

$$|R_1| \leq \frac{MB^*(b-a)^{\beta}}{B\left(\frac{T}{2}\right)\Gamma(\beta+1)\, n^{\beta}}. \tag{121}$$

Similarly we have

$$|R_2| \leq M \frac{\sum_{k=0}^{i} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{B\left(\frac{T}{2}\right)} \frac{1}{\Gamma(\beta+1)} (x - x_k)^{\beta}$$

$$\leq \frac{MB^*}{B\left(\frac{T}{2}\right)\Gamma(\beta+1)} \frac{(b-a)^{\beta}}{n^{\beta}}. \tag{122}$$

Hence

$$|R_2| \leq \frac{MB^*(b-a)^{\beta}}{B\left(\frac{T}{2}\right)\Gamma(\beta+1)\, n^{\beta}}. \tag{123}$$

Therefore it holds

$$|R_1 + R_2| \leq |R_1| + |R_2| \leq \frac{\Phi}{n^{\beta}}, \tag{124}$$

where

$$\Phi := \frac{2MB^{*}(b-a)^{\beta}}{B\left(\frac{T}{2}\right)\Gamma(\beta+1)}. \tag{125}$$

Thus

$$|R_1+R_2| = O\left(\frac{1}{n^{\beta}}\right), \tag{126}$$

and

$$|R_1+R_2| = o(1).$$

Letting $0<\varepsilon\leq\beta$, we derive

$$\frac{|R_1+R_2|}{\left(\frac{1}{n^{\beta-\varepsilon}}\right)} \leq \frac{\Phi}{n^{\varepsilon}} \to 0, \tag{127}$$

as $n\to\infty$. So that

$$|R_1+R_2| = o\left(\frac{1}{n^{\beta-\varepsilon}}\right), \ \ n\in\mathbb{N}, \tag{128}$$

proving the claim. ■

2.2 Neural Networks: Multivariate theory of Interpolation and Approximation

We need

Definition 21 *Consider the d-dimensional bell-shaped function $E:\mathbb{R}^d\to\mathbb{R}_+$ ($d\in\mathbb{N}$) with the property for all $i=1,...,d$, $\mathbb{R}\ni t\longrightarrow E(x_1,...,t,...,x_d)$ is a bell-shaped function, as in Definition 4, where $x=(x_1,...,x_d)\in\mathbb{R}^d$ is arbitrary.*

More precisely here E is of compact support $K:=\prod_{i=1}^{d}[-T_i,T_i]$, $T_i>0$ and it may have jump discontinuities there, also it holds

$$E(x_1,...,\pm T_i,...,x_d)=0, \tag{129}$$

for any $i=1,...,d$, all $(x_1,...,x_d)\in\mathbb{R}^d$.

Furthermore assume that $E(0,...,0)=:E^{*}>0$ is the global maximum of E, also E is assumed to be measurable. That is $E(x_1,...,t,...,x_d)$ in t is even, non-decreasing for $t<0$ and non-increasing for $t\geq 0$.

Clearly it holds

$$E(\pm x_1,...,\pm x_d)=E(|x_1|,...,|x_d|). \tag{130}$$

Also it is $E(x_1,...,0,...,x_d)=:E^{*}(x_1,...,x_{i-1},x_{i+1},...,x_d)>0$, for all $i=1,...,d$, for any $(x_1,...,x_d)\in\prod_{i=1}^{d}(-T_i,T_i)$.

Examples: $\prod_{i=1}^{d} \beta(x_i)$, $\prod_{i=1}^{d} \Phi_R(x_i)$, $\prod_{i=1}^{d} \Phi_s(x_i)$, etc.

Definition 22 *Let* $f : \prod_{i=1}^{d} [a_i, b_i] \to \mathbb{R}$ *be a bounded and measurable function,* $a_i < b_i$, $n \in \mathbb{N}$, $h_i := \frac{b_i - a_i}{n}$, $x_{k_i i} := a_i + k_i h_i$, $k_i = 0, 1, ..., n$, $i = 1, ..., d$, $x = (x_1, ..., x_d) \in \prod_{i=1}^{d} [a_i, b_i]$.

Next we define the multivariate interpolation neural network operator:

$$M_n(f, x) := M_n(f, x_1, ..., x_d) :=$$

$$\frac{\sum_{k_1=0}^{n} ... \sum_{k_d=0}^{n} f(x_{k_1 1}, ..., x_{k_d d}) E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1 - a_1}, ..., \frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d - a_d}\right)}{\sum_{k_1=0}^{n} ... \sum_{k_d=0}^{n} E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1 - a_1}, ..., \frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d - a_d}\right)}. \quad (131)$$

Remark 23 *Trivially we get that*

$$|M_n(f, x)| \leq \|f\|_\infty < +\infty, \quad (132)$$

and

$$\|M_n(f)\|_\infty \leq \|f\|_\infty < +\infty. \quad (133)$$

Remark 24 *Let now* $x_{k_i i} < x_i < x_{(k_i+1)i}$, *for all* $i = 1, ..., d$, *for some* $(k_1, ..., k_d) \in \{0, 1, ..., n-1\}^d$. *Thus* $|x_i - x_{k_i i}| < h_i$ *and* $\left|x_i - x_{(k_i+1)i}\right| < h_i$, *for all* $i = 1, ..., d$, *for some* $(k_1, ..., k_d) \in \{0, 1, ..., n-1\}^d$.

Remark 25 *Notice next that be given* $(x_1, ..., x_d) \in \mathbb{R}^d$ *and*

$$E\left(\frac{T_1 n(x_1 - x_{k_1 1})}{b_1 - a_1}, ..., \frac{T_d n(x_d - x_{k_d d})}{b_d - a_d}\right) > 0, \quad (134)$$

for some $(k_1, ..., k_d) \in \{0, 1, ..., n\}^d$, $\Leftrightarrow$ *simultaneously it holds*

$$-T_i < \frac{T_i n(x_i - x_{k_i i})}{b_i - a_i} < T_i,$$

for all $i = 1, ..., d$, *for some* $(k_1, ..., k_d) \in \{0, 1, ..., n\}^d$, $\Leftrightarrow$

$$-1 < \frac{n(x_i - x_{k_i i})}{b_i - a_i} < 1, \quad (135)$$

for all $i = 1, ..., d$, *for some* $(k_1, ..., k_d) \in \{0, 1, ..., n\}^d$, $\Leftrightarrow$

$$-h_i < x_i - x_{k_i i} < h_i,$$

for all $i = 1, ..., d$, *for some* $(k_1, ..., k_d) \in \{0, 1, ..., n\}^d$, $\Leftrightarrow$

$$|x_i - x_{k_i i}| < h_i, \quad (136)$$

for all $i = 1, ..., d$, *for some* $(k_1, ..., k_d) \in \{0, 1, ..., n\}^d$.

Thus, when $x \in \prod_{i=1}^{d} \left[x_{k_i i}, x_{(k_i+1)i}\right)$, for some $(k_1, ..., k_d) \in \{0, 1, ..., n-1\}^d$, we get that

$$E\left(\frac{T_1 n (x_1 - x_{k_1 1})}{b_1 - a_1}, ..., \frac{T_d n (x_d - x_{k_d d})}{b_d - a_d}\right) > 0. \tag{137}$$

Remark 26 *Notice that* $\left(x \in \prod_{i=1}^{d} [a_i, b_i]\right)$

$$W := \sum_{k_1=0}^{n} ... \sum_{k_d=0}^{n} E\left(\frac{T_1 n (x_1 - x_{k_1 1})}{b_1 - a_1}, ..., \frac{T_d n (x_d - x_{k_d d})}{b_d - a_d}\right) =$$

$$\sum_{k_1=0}^{n} ... \sum_{k_d=0}^{n} E\left(\frac{T_1 n |x_1 - x_{k_1 1}|}{b_1 - a_1}, ..., \frac{T_d n |x_d - x_{k_d d}|}{b_d - a_d}\right) \geq$$

$$E\left(\frac{T_1 n |x_1 - x_{k_1 1}|}{b_1 - a_1}, ..., \frac{T_d n |x_d - x_{k_d d}|}{b_d - a_d}\right), \tag{138}$$

the last inequality is chosen for suitable x_i and $x_{k_i i}$, for all $i = 1, ..., d$, and for some $(k_1, ..., k_d) \in \{0, 1, ..., n\}^d$, such that $|x_i - x_{k_i i}| \leq \frac{h_i}{2}$.

Thus

$$\frac{T_i n |x_i - x_{k_i i}|}{b_i - a_i} \leq \frac{T_i n h_i}{2 (b_i - a_i)} = \frac{T_i}{2}, \tag{139}$$

all $i = 1, ..., d$.

Therefore it holds

$$E\left(\frac{T_1 n |x_1 - x_{k_1 1}|}{b_1 - a_1}, \frac{T_2 n |x_2 - x_{k_2 2}|}{b_2 - a_2}, ..., \frac{T_d n |x_d - x_{k_d d}|}{b_d - a_d}\right) \geq$$

$$E\left(\frac{T_1}{2}, \frac{T_2 n |x_2 - x_{k_2 2}|}{b_2 - a_2}, ..., \frac{T_d n |x_d - x_{k_d d}|}{b_d - a_d}\right) \geq$$

$$E\left(\frac{T_1}{2}, \frac{T_2}{2}, ..., \frac{T_d n |x_d - x_{k_d d}|}{b_d - a_d}\right) \geq ... \geq E\left(\frac{T_1}{2}, \frac{T_2}{2}, ..., \frac{T_d}{2}\right) > 0. \tag{140}$$

Hence we have

$$\frac{1}{W} = \frac{1}{\sum_{k_1=0}^{n} ... \sum_{k_d=0}^{n} E\left(\frac{T_1 n (x_1 - x_{k_1 1})}{b_1 - a_1}, ..., \frac{T_d n (x_d - x_{k_d d})}{b_d - a_d}\right)} \leq \frac{1}{E\left(\frac{T_1}{2}, ..., \frac{T_d}{2}\right)}. \tag{141}$$

Remark 27 *Let all $x_i = x_{k_i i}$, $i = 1, ..., d$, for some $(k_1, ..., k_d) \in \{0, 1, ..., n\}^d$. Then*

$$E\left(\frac{T_1 n (x_1 - x_{k_1 1})}{b_1 - a_1}, ..., \frac{T_d n (x_d - x_{k_d d})}{b_d - a_d}\right) = E(0, ..., 0) = E^* > 0. \tag{142}$$

Let next $\left|x_{k_i i} - x_{(k_i + j_i) i}\right| \geq h_i$, for some $i \in \{1, ..., d\}$, where $j_i \geq 1$ integer, and $k_i, k_i + j_i \in \{0, 1, ..., n\}$.

Then

$$\frac{T_i n \left| x_{k_i i} - x_{(k_i+j_i)i} \right|}{b_i - a_i} \geq \frac{T_i n h_i}{b_i - a_i} = T_i, \quad \text{for some } i = 1, ..., d. \tag{143}$$

Hence

$$0 \leq E\left(\frac{T_1 n \left| x_1 - x_{k_1 1} \right|}{b_1 - a_1}, ..., \frac{T_i n \left| x_{(k_i+j_i)i} - x_{k_i i} \right|}{b_i - a_i}, ..., \frac{T_d n \left| x_d - x_{k_d d} \right|}{b_d - a_d} \right) \leq$$

$$E\left(\frac{T_1 n \left| x_1 - x_{k_1 1} \right|}{b_1 - a_1}, ..., T_i, ..., \frac{T_d n \left| x_d - x_{k_d d} \right|}{b_d - a_d} \right) = 0. \tag{144}$$

Therefore it holds

$$E\left(\frac{T_1 n \left(x_1 - x_{k_1 1} \right)}{b_1 - a_1}, ..., \frac{T_i n \left(x_{(k_i+j_i)i} - x_{k_i i} \right)}{b_i - a_i}, ..., \frac{T_d n \left(x_d - x_{k_d d} \right)}{b_d - a_d} \right) = 0, \tag{145}$$

for any arbitrary $(x_1, ..., x_{i-1}, x_{i+1}, ..., x_d) \in \mathbb{R}^{d-1}$.

Let now $x_i = x_{k_i i}$, *for all* $i = 1, ..., d$, *for some* $(k_1, ..., k_d) \in \{0, 1, ..., n\}^d$.

Then

$$M_n\left(f, x_{k_1 1}, ..., x_{k_d d}\right) = \frac{f\left(x_{k_1 1}, ..., x_{k_d d}\right) E^*}{E^*} = f\left(x_{k_1 1}, ..., x_{k_d d}\right), \tag{146}$$

proving the interpolation property of operators M_n.

Theorem 28 *Operators* M_n *possess the interpolation property over* $x_{k_i i}$, $i = 1, ..., d$, $k_i = 0, 1, ..., n$.

Definition 29 *Let* $f \in C\left(\prod_{i=1}^{d} [a_i, b_i] \right)$. *We call*

$$\omega_1(f, h) := \sup_{\substack{all\ x, y \in \prod_{i=1}^{d} [a_i, b_i]: \\ \|x - y\|_\infty \leq h,}} |f(x) - f(y)| \tag{147}$$

$h > 0$, *the first multivariate modulus of continuity of* f, *above* $\|\cdot\|_\infty$ *is the max-norm.*

Approximation result follows

Theorem 30 *For* $f \in C\left(\prod_{i=1}^{d} [a_i, b_i] \right)$ *we have*

$$\|M_n(f) - f\|_\infty \leq \frac{2^d E^*}{E\left(\frac{T_1}{2}, ..., \frac{T_d}{2}\right)} \omega_1\left(f, \frac{\|b - a\|_\infty}{n}\right) =: \varphi_1(n), \tag{148}$$

where $\|b - a\|_\infty := \max_{i=1,...,d} \{b_i - a_i\}$.

Proof. Let $x \in \prod_{i=1}^{d}[a_i, b_i]$, we can write

$$M_n(f,x) - f(x) =$$

$$\frac{\sum_{k_1=0}^{n} \dots \sum_{k_d=0}^{n} f(x_{k_1 1}, \dots, x_{k_d d}) E\left(\frac{T_1 n(x_1 - x_{k_1 1})}{b_1 - a_1}, \dots, \frac{T_d n(x_d - x_{k_d d})}{b_d - a_d}\right)}{W} - \frac{f(x) W}{W} = \tag{149}$$

$$\frac{\sum_{k_1=0}^{n} \dots \sum_{k_d=0}^{n} \left(f(x_{k_1 1}, \dots, x_{k_d d}) - f(x_1, \dots, x_d)\right) E\left(\frac{T_1 n(x_1 - x_{k_1 1})}{b_1 - a_1}, \dots, \frac{T_d n(x_d - x_{k_d d})}{b_d - a_d}\right)}{W}. \tag{150}$$

Therefore

$$|M_n(f,x) - f(x)| \leq \frac{1}{E\left(\frac{T_1}{2}, \dots, \frac{T_d}{2}\right)} \cdot$$

$$\left\{\sum_{k_1=0}^{n} \dots \sum_{k_d=0}^{n} |f(x_{k_1 1}, \dots, x_{k_d d}) - f(x_1, \dots, x_d)| \cdot \right. \tag{151}$$

$$\left. E\left(\frac{T_1 n(x_1 - x_{k_1 1})}{b_1 - a_1}, \dots, \frac{T_d n(x_d - x_{k_d d})}{b_d - a_d}\right)\right\} \leq \frac{1}{E\left(\frac{T_1}{2}, \dots, \frac{T_d}{2}\right)} \{0 +$$

$$\sum_{\left\{\substack{\text{all } (k_1,\dots,k_d) \in \{0,1,\dots,n\}^d \\ |x_i - x_{k_i i}| < h_i,\ i=1,\dots,d}\right\}} |f(x_{k_1 1}, \dots, x_{k_d d}) - f(x_1, \dots, x_d)| \cdot \tag{152}$$

$$\left. E\left(\frac{T_1 n(x_1 - x_{k_1 1})}{b_1 - a_1}, \dots, \frac{T_d n(x_d - x_{k_d d})}{b_d - a_d}\right)\right\} \leq$$

(indeed x belongs to a specific box $\prod_{i=1}^{d}\left[x_{k_i i}, x_{(k_i+1)i}\right]$)

$$\frac{2^d E^*}{E\left(\frac{T_1}{2}, \dots, \frac{T_d}{2}\right)} \omega_1\left(f, \frac{\|b-a\|_\infty}{n}\right), \tag{153}$$

proving the claim. ■

Next we denote by $f_{\widetilde{\alpha}} := \frac{\partial^{\widetilde{\alpha}} f}{\partial x^{\widetilde{\alpha}}}$, where $\widetilde{\alpha} := (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, d$, such that $|\widetilde{\alpha}| := \sum_{i=1}^{d} \alpha_i = j$, $j = 1, \dots, N$.

High speed approximation using smoothness follows.

Theorem 31 *Let* $f \in C^N\left(\prod_{i=1}^{d}[a_i, b_i]\right)$, $N \in \mathbb{N}$, *and* $x \in \prod_{i=1}^{d}[a_i, b_i]$. *Then* *i)*

$$\left|M_n(f,x)-f(x)-\sum_{j=1}^{N}\left(\sum_{|\widetilde{\alpha}|=j}\left(\frac{f_{\widetilde{\alpha}}(x)}{\prod_{i=1}^{d}\alpha_i!}\right)M_n\left(\prod_{i=1}^{d}(\cdot-x_i)^{\alpha_i},x\right)\right)\right|\leq$$

$$\frac{2^d E^*}{E\left(\frac{T_1}{2},...,\frac{T_d}{2}\right)}\frac{\|b-a\|_\infty^N d^N}{N!n^N}\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\frac{\|b-a\|_\infty}{n}\right), \tag{154}$$

ii) assume more that $f_{\widetilde{\alpha}}(x)=0$, *for all* $\widetilde{\alpha}:|\widetilde{\alpha}|=1,...,N$; *where* $x\in\prod_{i=1}^{d}[a_i,b_i]$ *is fixed, we obtain*

$$|M_n(f,x)-f(x)|\leq$$

$$\frac{2^d E^*}{E\left(\frac{T_1}{2},...,\frac{T_d}{2}\right)}\frac{\|b-a\|_\infty^N d^N}{N!n^N}\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\frac{\|b-a\|_\infty}{n}\right), \tag{155}$$

with high speed of pointwise convergence at $\frac{1}{n^{N+1}}$,

iii)

$$|M_n(f,x)-f(x)|\leq\frac{2^d E^*}{E\left(\frac{T_1}{2},...,\frac{T_d}{2}\right)}\cdot$$

$$\left[\sum_{j=1}^{N}\left(\frac{\|b-a\|_\infty^j}{n^j}\right)\left(\sum_{|\widetilde{\alpha}|=j}\left(\frac{|f_{\widetilde{\alpha}}(x)|}{\prod_{i=1}^{d}\alpha_i!}\right)\right)+\right. \tag{156}$$

$$\left.\frac{\|b-a\|_\infty^N d^N}{N!n^N}\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\frac{\|b-a\|_\infty}{n}\right)\right],$$

iv)

$$\|M_n(f)-f\|_\infty\leq\frac{2^d E^*}{E\left(\frac{T_1}{2},...,\frac{T_d}{2}\right)}\cdot$$

$$\left[\sum_{j=1}^{N}\left(\frac{\|b-a\|_\infty^j}{n^j}\right)\left(\sum_{|\widetilde{\alpha}|=j}\left(\frac{\|f_{\widetilde{\alpha}}\|_\infty}{\prod_{i=1}^{d}\alpha_i!}\right)\right)+\right. \tag{157}$$

$$\left.\frac{\|b-a\|_\infty^N d^N}{N!n^N}\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\frac{\|b-a\|_\infty}{n}\right)\right]=:\varphi_2(n).$$

Proof. Here $f \in C^N\left(\prod_{i=1}^{d}[a_i,b_i]\right)$, $N \in \mathbb{N}$. We call $x_k = (x_{k_1 1}, ..., x_{k_d d})$. Set

$$g_{x_k}(t) := f(x + t(x_k - x)), \quad 0 \leq t \leq 1, \tag{158}$$

$x \in \prod_{i=1}^{d}[a_i,b_i]$, $x = (x_1, ..., x_d)$. Then

$$g_{x_k}^{(j)}(t) = \left[\left(\sum_{i=1}^{d}(x_{k_i i} - x_i)\frac{\partial}{\partial x_i}\right)^j f\right](x_1 + t(x_{k_1 1} - x_1), ..., x_d + t(x_{k_d d} - x_d)), \tag{159}$$

$$g_{x_k}^{(j)}(0) = \left[\left(\sum_{i=1}^{d}(x_{k_i i} - x_i)\frac{\partial}{\partial x_i}\right)^j f\right](x), \tag{160}$$

and

$$g_{x_k}(0) = f(x).$$

By Taylor's formula, we get

$$f(x_{k_1 1}, ..., x_{k_d d}) = g_{x_k}(1) = \sum_{j=0}^{N}\frac{g_{x_k}^{(j)}(0)}{j!} + R_N(x_k, 0), \tag{161}$$

where

$$R_N(x_k, 0) := \int_0^1\left(\int_0^{t_1} ... \left(\int_0^{t_{N-1}}\left(g_{x_k}^{(N)}(t_N) - g_{x_k}^{(N)}(0)\right)dt_N\right)...\right)dt_1. \tag{162}$$

Thus,

$$\frac{f(x_{k_1 1}, ..., x_{k_d d})\, E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1 - a_1}, ..., \frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d - a_d}\right)}{W} = \tag{163}$$

$$\sum_{j=0}^{N}\frac{g_{x_k}^{(j)}(0)}{j!}\frac{E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1 - a_1}, ..., \frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d - a_d}\right)}{W} +$$

$$\frac{E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1 - a_1}, ..., \frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d - a_d}\right)}{W} R_N(x_k, 0). \tag{164}$$

Therefore

$$M_n(f, x) - f(x) =$$

$$\sum_{j=1}^{N}\frac{1}{j!}\left(\frac{\sum_{k_1=0}^{n} \cdots \sum_{k_d=0}^{n} g_{x_k}^{(j)}(0)\, E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1 - a_1}, ..., \frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d - a_d}\right)}{W}\right) + R^*, \tag{165}$$

where

$$R^{*} := \frac{\sum_{k_1=0}^{n} \dots \sum_{k_d=0}^{n} E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1 - a_1}, \dots, \frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d - a_d}\right)}{W} R_N\left(x_k, 0\right). \tag{166}$$

Consequently, we obtain

$$\left|M_n\left(f, x\right) - f\left(x\right)\right| \leq$$

$$\sum_{j=1}^{N} \frac{1}{j!} \frac{\sum_{k_1=0}^{n} \dots \sum_{k_d=0}^{n} \left|g_{x_k}^{(j)}\left(0\right)\right| E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1 - a_1}, \dots, \frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d - a_d}\right)}{E\left(\frac{T_1}{2}, \dots, \frac{T_d}{2}\right)} + \left|R^{*}\right|$$

$$\leq \sum_{j=1}^{N} \frac{1}{j!} \frac{2^d \left(\frac{\|b-a\|_{\infty}^{j}}{n^j}\right) \left(\left(\sum_{i=1}^{d} \left|\frac{\partial}{\partial x_i}\right|\right)^{j} f\left(x\right)\right) E^{*}}{E\left(\frac{T_1}{2}, \dots, \frac{T_d}{2}\right)} + \left|R^{*}\right| = \tag{167}$$

$$\frac{2^d E^{*}}{E\left(\frac{T_1}{2}, \dots, \frac{T_d}{2}\right)} \left[\sum_{j=1}^{N} \frac{1}{j!} \left(\left(\sum_{i=1}^{d} \left|\frac{\partial}{\partial x_i}\right|\right)^{j} f\left(x\right)\right) \left(\frac{\|b-a\|_{\infty}^{j}}{n^j}\right)\right] + \left|R^{*}\right|. \tag{168}$$

Next, we estimate $\left|R^{*}\right|$.

For that, we observe

$$\left|R^{*}\right| \leq \frac{\sum_{k_1=0}^{n} \dots \sum_{k_d=0}^{n} E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1 - a_1}, \dots, \frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d - a_d}\right)}{E\left(\frac{T_1}{2}, \dots, \frac{T_d}{2}\right)} \cdot \tag{169}$$

$$\left(\int_0^1 \left(\int_0^{t_1} \dots \left(\int_0^{t_{N-1}} \left|g_{x_k}^{(N)}\left(t_N\right) - g_{x_k}^{(N)}\left(0\right)\right| dt_N\right) \dots\right) dt_1\right) =$$

$$\frac{\sum_{k_1=0}^{n} \dots \sum_{k_d=0}^{n} E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1 - a_1}, \dots, \frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d - a_d}\right)}{E\left(\frac{T_1}{2}, \dots, \frac{T_d}{2}\right)} \cdot$$

$$\left(\int_0^1 \left(\int_0^{t_1} \dots \left(\int_0^{t_{N-1}} \left|\left[\left(\sum_{i=1}^{d} \left(x_{k_i i} - x_i\right) \frac{\partial}{\partial x_i}\right)^{N} f\right]\right.\right.\right.\right.$$

$$\left(x_1 + t_N\left(x_{k_1 1} - x_1\right), \dots, x_d + t_N\left(x_{k_d d} - x_d\right)\right) -$$

$$\left.\left.\left.\left.\left[\left(\sum_{i=1}^{d} \left(x_{k_i i} - x_i\right) \frac{\partial}{\partial x_i}\right)^{N} f\right]\left(x_1, \dots, x_d\right)\right| dt_N\right) \dots\right) dt_1\right) \leq \tag{170}$$

$$\frac{2^d E^{*}}{E\left(\frac{T_1}{2}, \dots, \frac{T_d}{2}\right)} \left(\int_0^1 \left(\int_0^{t_1} \dots \left(\int_0^{t_{N-1}} \left\{\left(\frac{\|b-a\|_{\infty}^{N}}{n^N}\right) d^N\right.\right.\right.\right.$$

$$\max_{\widetilde{\alpha}:|\alpha|=N}\omega_1\left(f_{\widetilde{\alpha}},\frac{\|b-a\|_\infty}{n}\right)\Bigg\}dt_N\Bigg)...\Bigg)dt_1\Bigg)=$$

$$\frac{2^dE^*}{N!E\left(\frac{T_1}{2},...,\frac{T_d}{2}\right)}\frac{\|b-a\|_\infty^N d^N}{n^N}\max_{\widetilde{\alpha}:|\alpha|=N}\omega_1\left(f_{\widetilde{\alpha}},\frac{\|b-a\|_\infty}{n}\right). \quad (171)$$

That is

$$|R^*|\leq\frac{2^dE^*}{N!E\left(\frac{T_1}{2},...,\frac{T_d}{2}\right)}\frac{\|b-a\|_\infty^N d^N}{n^N}\max_{\widetilde{\alpha}:|\alpha|=N}\omega_1\left(f_{\widetilde{\alpha}},\frac{\|b-a\|_\infty}{n}\right). \quad (172)$$

The proof of the Theorem now is complete. ■

About Multivariate Taylor formula and estimates (see [15], pp. 284-286)

Let $\prod_{i=1}^d[a_i,b_i]$; $d\geq 2$; $z:=(z_1,...,z_d)$, $x_0:=(x_{01},...,x_{0d})\in\prod_{i=1}^d[a_i,b_i]$. We consider the space of functions $AC^N\left(\prod_{i=1}^d[a_i,b_i]\right)$ with $f:\prod_{i=1}^d[a_i,b_i]\to\mathbb{R}$ be such that all partial derivatives of order $(N-1)$ are coordinatewise absolutely continuous functions on $\prod_{i=1}^d[a_i,b_i]$, $N\in\mathbb{N}$. Also $f\in C^{N-1}\left(\prod_{i=1}^d[a_i,b_i]\right)$. Each N^{th} order partial derivative is denoted by $f_{\widetilde{\alpha}}:=\frac{\partial^{\widetilde{\alpha}}f}{\partial x^{\widetilde{\alpha}}}$, where $\widetilde{\alpha}:=(\alpha_1,...,\alpha_d)$, $\alpha_i\in\mathbb{Z}^+$, $i=1,...,d$ and $|\widetilde{\alpha}|:=\sum_{i=1}^d\alpha_i=N$. Consider $g_z(t):=f(x_0+t(z-x_0))$, $t\geq 0$. Then

$$g_z^{(j)}(t)=\left[\left(\sum_{i=1}^d(z_i-x_{0i})\frac{\partial}{\partial x_i}\right)^j f\right](x_{01}+t(z_1-x_{01}),...,x_{0d}+t(z_N-x_{0d})), \quad (173)$$

for all $j=0,1,2,...,N$.

We mention the following multivariate Taylor theorem.

Theorem 32 *Under the above assumptions we have*

$$f(z_1,...,z_d)=g_z(1)=\sum_{j=0}^{N-1}\frac{g_z^{(j)}(0)}{j!}+R_N(z,0), \quad (174)$$

where

$$R_N(z,0):=\int_0^1\left(\int_0^{t_1}...\left(\int_0^{t_{N-1}}g_z^{(N)}(t_N)\,dt_N\right)...\right)dt_1, \quad (175)$$

or

$$R_N(z,0)=\frac{1}{(N-1)!}\int_0^1(1-\theta)^{N-1}g_z^{(N)}(\theta)\,d\theta. \quad (176)$$

Notice that $g_z(0)=f(x_0)$.

We make

Remark 33 *Assume here that*

$$\|f_{\widetilde{\alpha}}\|_{\infty,N}^{\max} := \max_{|\widetilde{\alpha}|=N} \|f_{\widetilde{\alpha}}\|_{\infty} < \infty. \tag{177}$$

Then

$$\left\|g_z^{(N)}\right\|_{\infty,[0,1]} = \left\| \left[\left(\sum_{i=1}^{d} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^N f \right] (x_0 + t(z - x_0)) \right\|_{\infty,[0,1]} \leq$$

$$\left(\sum_{i=1}^{d} |z_i - x_{0i}| \right)^N \|f_{\widetilde{\alpha}}\|_{\infty,N}^{\max}, \tag{178}$$

that is

$$\left\|g_z^{(N)}\right\|_{\infty,[0,1]} \leq \left(\|z - x_0\|_{l_1}\right)^N \|f_{\widetilde{\alpha}}\|_{\infty,N}^{\max} < \infty. \tag{179}$$

Hence we get by (176) that

$$|R_N(z,0)| \leq \frac{\left\|g_z^{(N)}\right\|_{\infty,[0,1]}}{N!} < \infty. \tag{180}$$

And it holds

$$|R_N(z,0)| \leq \frac{\left(\|z - x_0\|_{l_1}\right)^N}{N!} \|f_{\widetilde{\alpha}}\|_{\infty,N}^{\max}, \tag{181}$$

$\forall\ z, x_0 \in \prod_{i=1}^{d} [a_i, b_i]$.

We will use decisively (181).

Next follows a multivariate Voronovskaya type asymptotic expansion

Theorem 34 *Let* $f \in AC^N \left(\prod_{i=1}^{d} [a_i, b_i] \right)$, $d \in \mathbb{N} - \{1\}$, $N \in \mathbb{N}$, *with*

$$\|f_{\widetilde{\alpha}}\|_{\infty,N}^{\max} := \max_{|\widetilde{\alpha}|=N} \|f_{\widetilde{\alpha}}\|_{\infty} < \infty. \tag{182}$$

Then

$$M_n(f,x) - f(x) - \sum_{j=1}^{N-1} \left(\sum_{\substack{\widetilde{\alpha} := (\alpha_1, \ldots, \alpha_d),\ \alpha_i \in \mathbb{Z}^+ \\ i=1,\ldots,d,\ |\widetilde{\alpha}| := \sum_{i=1}^{d} \alpha_i = j}} \left(\frac{f_{\widetilde{\alpha}}(x)}{\prod_{i=1}^{d} \alpha_i!} \right) M_n \left(\prod_{i=1}^{d} (\cdot - x_i)^{\alpha_i}, x \right) \right)$$

$$= o\left(\frac{1}{n^{N-\varepsilon}}\right), \quad 0 < \varepsilon \leq N. \tag{183}$$

If $N = 1$, the sum collapses.

The last (183) implies

$$n^{N-\varepsilon}\left[M_n(f,x) - f(x) - \sum_{j=1}^{N-1}\left(\sum_{\substack{\widetilde{\alpha}:=(\alpha_1,...,\alpha_d),\ \alpha_i\in\mathbb{Z}^+ \\ i=1,...,d,\ |\widetilde{\alpha}|:=\sum_{i=1}^{d}\alpha_i=j}}\left(\frac{f_{\widetilde{\alpha}}(x)}{\prod_{i=1}^{d}\alpha_i!}\right)M_n\left(\prod_{i=1}^{d}(\cdot - x_i)^{\alpha_i}, x\right)\right)\right] \to 0, \tag{184}$$

as $n \to \infty$, $0 < \varepsilon \leq N$.

When $N = 1$ or $f_{\widetilde{\alpha}}(x) = 0$, all $\widetilde{\alpha} : |\widetilde{\alpha}| = j = 1, ..., N-1$, then

$$n^{N-\varepsilon}\left[(M_n(f))(x) - f(x)\right] \to 0, \tag{185}$$

as $n \to \infty$, $0 < \varepsilon \leq N$.

Proof. We call $x_k = (x_{k_1 1}, ..., x_{k_d d})$. Set

$$g_{x_k}(t) := f(x + t(x_k - x)), \quad 0 \leq t \leq 1, \tag{186}$$

$x \in \prod_{i=1}^{d}[a_i, b_i]$. Then

$$g_{x_k}^{(j)}(t) = \left[\left(\sum_{i=1}^{d}(x_{k_i i} - x_i)\frac{\partial}{\partial x_i}\right)^j f\right](x_1 + t(x_{k_1 1} - x_1), ..., x_d + t(x_{k_d d} - x_d)), \tag{187}$$

and

$$g_{x_k}(0) = f(x).$$

By Taylor's formula, we get

$$f(x_k) = g_{x_k}(1) = \sum_{j=0}^{N-1}\frac{g_{x_k}^{(j)}(0)}{j!} + R_N(x_k, 0), \tag{188}$$

where

$$R_N(x_k, 0) := \frac{1}{(N-1)!}\int_0^1 (1-\theta)^{N-1} g_{x_k}^{(N)}(\theta)\, d\theta. \tag{189}$$

Here we denote by $f_{\widetilde{\alpha}} := \frac{\partial^{\widetilde{\alpha}} f}{\partial x^{\widetilde{\alpha}}}$, $\widetilde{\alpha} := (\alpha_1, ..., \alpha_d)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., d$, such that $|\widetilde{\alpha}| := \sum_{i=1}^{d}\alpha_i = N$. Thus

$$\frac{f(x_k)\, E\left(\frac{T_1 n(x_1 - x_{k_1 1})}{b_1 - a_1}, ..., \frac{T_d n(x_d - x_{k_d d})}{b_d - a_d}\right)}{W} = \tag{190}$$

$$\sum_{j=0}^{N-1} \frac{g_{x_k}^{(j)}(0)}{j!} \frac{E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1 - a_1}, ..., \frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d - a_d}\right)}{W} +$$

$$\frac{E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1 - a_1}, ..., \frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d - a_d}\right)}{W} R_N\left(x_k, 0\right). \tag{191}$$

Therefore it holds

$$M_n\left(f, x\right) - f\left(x\right) -$$

$$\sum_{j=1}^{N-1} \frac{1}{j!} \frac{\left(\sum_{k_1=0}^{n} \cdots \sum_{k_d=0}^{n} g_{x_k}^{(j)}(0)\, E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1 - a_1}, ..., \frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d - a_d}\right)\right)}{W} = R^*, \tag{192}$$

where

$$R^* := \frac{\sum_{k_1=0}^{n} \cdots \sum_{k_d=0}^{n} E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1 - a_1}, ..., \frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d - a_d}\right)}{W} R_N\left(x_k, 0\right). \tag{193}$$

Hence

$$M_n\left(f, x\right) - f\left(x\right) - \sum_{j=1}^{N-1} \left(\sum_{\substack{\widetilde{\alpha} := (\alpha_1, ..., \alpha_d),\ \alpha_i \in \mathbb{Z}^+ \\ i = 1, ..., d,\ |\widetilde{\alpha}| := \sum_{i=1}^{d} \alpha_i = j}} \left(\frac{f_{\widetilde{\alpha}}\left(x\right)}{\prod_{i=1}^{d} \alpha_i!} \right) M_n\left(\prod_{i=1}^{d} \left(\cdot - x_i\right)^{\alpha_i}, x \right) \right)$$

$$= R^*. \tag{194}$$

Notice that

$$R^* = \frac{\sum_{k_1=0}^{n} \cdots \sum_{k_d=0}^{n} E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1 - a_1}, ..., \frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d - a_d}\right)}{W}$$

$$N \int_0^1 \left(1 - \theta\right)^{N-1} \sum_{\substack{\widetilde{\alpha} := (\alpha_1, ..., \alpha_d),\ \alpha_i \in \mathbb{Z}^+ \\ i = 1, ..., d,\ |\widetilde{\alpha}| := \sum_{i=1}^{d} \alpha_i = N}} \left(\frac{1}{\prod_{i=1}^{d} \alpha_i!} \right)$$

$$\left(\prod_{i=1}^{d} \left(x_{k_i i} - x_i\right)^{\alpha_i} \right) f_{\widetilde{\alpha}}\left(x + \theta\left(x_k - x\right)\right) d\theta. \tag{195}$$

Hence it holds

$$|R^*| \overset{(181)}{\leq} \frac{\sum_{k_1=0}^{n} \cdots \sum_{k_d=0}^{n} E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1 - a_1}, \ldots, \frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d - a_d}\right)}{E\left(\frac{T_1}{2}, \ldots, \frac{T_d}{2}\right)} \tag{196}$$

$$\frac{\left(\|x_k - x\|_{l_1}\right)^N}{N!} \|f_{\widetilde{\alpha}}\|_{\infty,N}^{\max} \leq$$

$$\frac{2^d E^*}{E\left(\frac{T_1}{2}, \ldots, \frac{T_d}{2}\right)} \left(d \frac{\|b-a\|_\infty}{n}\right)^N \frac{\|f_{\widetilde{\alpha}}\|_{\infty,N}^{\max}}{N!}. \tag{197}$$

Thst is

$$|R^*| \leq \frac{\delta}{n^N}, \tag{198}$$

where

$$\delta := \frac{2^d E^* d^N \|b-a\|_\infty^N \|f_{\widetilde{\alpha}}\|_{\infty,N}^{\max}}{E\left(\frac{T_1}{2}, \ldots, \frac{T_d}{2}\right) N!} < +\infty. \tag{199}$$

That is

$$|R^*| = O\left(\frac{1}{n^N}\right), \tag{200}$$

and

$$|R^*| = o(1). \tag{201}$$

And letting $0 < \varepsilon \leq N$, we derive

$$\frac{|R^*|}{\left(\frac{1}{n^{N-\varepsilon}}\right)} \leq \frac{\delta}{n^\varepsilon} \to 0, \tag{202}$$

as $n \to \infty$.

I.e.

$$|R^*| = o\left(\frac{1}{n^{N-\varepsilon}}\right). \tag{203}$$

The proof is completed. ■

2.3 Neural Networks Iterated Approximation and Interpolation

We make

Remark 35 *Here E is assumed additionally to be continuous.*

Let $f \in C\left(\prod_{i=1}^{d} [a_i, b_i]\right)$. We (see (138), (140)) proved that $W > 0$. Hence $M_n(f) \in C\left(\prod_{i=1}^{d} [a_i, b_i]\right)$. Furthermore $M_n(f) - f \in C\left(\prod_{i=1}^{d} [a_i, b_i]\right)$.

We proved earlier (133) that

$$\|M_n(f)\|_\infty \leq \|f\|_\infty < +\infty. \tag{204}$$

Clearly then

$$\left\|M_n^2(f)\right\|_\infty = \|M_n(M_n(f))\|_\infty \leq \|M_n(f)\|_\infty \leq \|f\|_\infty. \tag{205}$$

Therefore

$$\left\|M_n^k(f)\right\|_\infty \leq \|f\|_\infty, \quad \forall\ k \in \mathbb{N}. \tag{206}$$

Also we see that

$$\left\|M_n^k(f)\right\|_\infty \leq \left\|M_n^{k-1}(f)\right\|_\infty \leq ... \leq \|M_n(f)\|_\infty \leq \|f\|_\infty. \tag{207}$$

Also it holds

$$M_n(1) = 1, \quad M_n^k(1) = 1, \quad \forall\ k \in \mathbb{N}. \tag{208}$$

Here M_n^k are positive linear operators.

Call $x_k = (x_{k_1 1}, ..., x_{k_d d})$, we proved (146), that

$$(M_n(f))(x_k) = f(x_k), \tag{209}$$

the interpolation property of M_n.

Hence we get

$$\left(M_n^2(f)\right)(x_k) = (M_n(M_n(f)))(x_k)$$

(by Theorem 28)

$$= (M_n(f))(x_k) = f(x_k), \tag{210}$$

In general it holds

$$\left(M_n^k(f)\right)(x_k) = f(x_k), \quad \forall\ k \in \mathbb{N}, \tag{211}$$

proving interpolation of the operators M_n^k.

Remark 36 *Let $r \in \mathbb{N}$ and M_n as above. We observe that*

$$M_n^r f - f = \left(M_n^r f - M_n^{r-1} f\right) + \left(M_n^{r-1} f - M_n^{r-2} f\right) + \tag{212}$$
$$\left(M_n^{r-2} f - M_n^{r-3} f\right) + ... + \left(M_n^2 f - M_n f\right) + (M_n f - f).$$

Then

$$\|M_n^r f - f\|_\infty \leq \left\|M_n^r f - M_n^{r-1} f\right\|_\infty + \left\|M_n^{r-1} f - M_n^{r-2} f\right\|_\infty +$$
$$\left\|M_n^{r-2} f - M_n^{r-3} f\right\|_\infty + ... + \left\|M_n^2 f - M_n f\right\|_\infty + \|M_n f - f\|_\infty =$$
$$\left\|M_n^{r-1}(M_n f - f)\right\|_\infty + \left\|M_n^{r-2}(M_n f - f)\right\|_\infty + \left\|M_n^{r-3}(M_n f - f)\right\|_\infty$$
$$+ ... + \|M_n(M_n f - f)\|_\infty + \|M_n f - f\|_\infty \leq r\|M_n f - f\|_\infty. \tag{213}$$

That is

$$\|M_n^r f - f\|_\infty \leq r\|M_n f - f\|_\infty. \tag{214}$$

Conclusion 37 *Thus, the speed of convergence to the unit operator of M_n^r is not worse than of M_n.*

Remark 38 *Let $m_1, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$, $r \in \mathbb{N}$.*
Let M_{m_i} as above, $i = 1, ..., r$.

Then it holds

$$\begin{gathered} M_{m_r}\left(M_{m_{r-1}}\left(...M_{m_2}\left(M_{m_1}\left(f\right)\right)\right)\right) - f = \\ \left[M_{m_r}\left(M_{m_{r-1}}\left(...M_{m_2}\left(M_{m_1}\left(f\right)\right)\right)\right) - M_{m_r}\left(M_{m_{r-1}}\left(...M_{m_2}\left(f\right)\right)\right)\right] + \\ \left[M_{m_r}\left(M_{m_{r-1}}\left(...M_{m_3}\left(M_{m_2}\left(f\right)\right)\right)\right) - M_{m_r}\left(M_{m_{r-1}}\left(...M_{m_3}\left(f\right)\right)\right)\right] + \\ \left[M_{m_r}\left(M_{m_{r-1}}\left(...M_{m_4}\left(M_{m_3}\left(f\right)\right)\right)\right) - M_{m_r}\left(M_{m_{r-1}}\left(...M_{m_4}\left(f\right)\right)\right)\right] + \\ ... + \left[M_{m_r}\left(M_{m_{r-1}}f\right) - M_{m_r}f\right] + \left[M_{m_r}f - f\right] = \end{gathered} \tag{215}$$

$$\begin{gathered} \left[M_{m_r}\left(M_{m_{r-1}}\left(...M_{m_2}\right)\right)\left(M_{m_1}f - f\right)\right] + \left[M_{m_r}\left(M_{m_{r-1}}\left(...M_{m_3}\right)\right)\left(M_{m_2}f - f\right)\right] + \\ \left[M_{m_r}\left(M_{m_{r-1}}\left(...M_{m_4}\right)\right)\left(M_{m_3}f - f\right)\right] + ... + \\ \left[M_{m_r}\left(M_{m_{r-1}}f - f\right)\right] + \left[M_{m_r}f - f\right]. \end{gathered} \tag{216}$$

Therefore

$$\begin{gathered} \left\|M_{m_r}\left(M_{m_{r-1}}\left(...M_{m_2}\left(M_{m_1}\left(f\right)\right)\right)\right) - f\right\|_\infty \leq \\ \left\|M_{m_r}\left(M_{m_{r-1}}\left(...M_{m_2}\right)\right)\left(M_{m_1}f - f\right)\right\|_\infty + \\ \left\|M_{m_r}\left(M_{m_{r-1}}\left(...M_{m_3}\right)\right)\left(M_{m_2}f - f\right)\right\|_\infty + \\ \left\|M_{m_r}\left(M_{m_{r-1}}\left(...M_{m_4}\right)\right)\left(M_{m_3}f - f\right)\right\|_\infty + ... + \\ \left\|M_{m_r}\left(M_{m_{r-1}}f - f\right)\right\|_\infty + \left\|M_{m_r}f - f\right\|_\infty \leq \end{gathered} \tag{217}$$

$$\left\|M_{m_1}f - f\right\|_\infty + \left\|M_{m_2}f - f\right\|_\infty + \left\|M_{m_3}f - f\right\|_\infty + \tag{218}$$

$$... + \left\|M_{m_{r-1}}f - f\right\|_\infty + \left\|M_{m_r}f - f\right\|_\infty = \sum_{i=1}^{r} \left\|M_{m_i}f - f\right\|_\infty. \tag{219}$$

We have proved that

$$\left\|M_{m_r}\left(M_{m_{r-1}}\left(...M_{m_2}\left(M_{m_1}\left(f\right)\right)\right)\right) - f\right\|_\infty \leq \sum_{i=1}^{r} \left\|M_{m_i}f - f\right\|_\infty. \tag{220}$$

Using (214) we derive

Theorem 39 *Let $f \in C\left(\prod_{i=1}^{d}\left[a_i, b_i\right]\right)$, $r \in \mathbb{N}$. Then*

$$\left\|M_n^r f - f\right\|_\infty \leq \frac{r 2^d E^*}{E\left(\frac{T_1}{2}, ..., \frac{T_d}{2}\right)} \omega_1\left(f, \frac{\|b - a\|_\infty}{n}\right). \tag{221}$$

Proof. Also use of (148). ■

Theorem 40 *Let* $f \in C^N\left(\prod_{i=1}^{d}[a_i,b_i]\right)$, $N \in \mathbb{N}$, $r \in \mathbb{N}$. *Then*

$$\|M_n^r f - f\|_\infty \leq r\varphi_2(n), \tag{222}$$

where $\varphi_2(n)$ *is as in (157).*

Proof. Use also of (157). ■

Next we use (220).

Theorem 41 *Let* $m_1,...,m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$, $r \in \mathbb{N}$, $f \in C\left(\prod_{i=1}^{d}[a_i,b_i]\right)$. *Then*

$$\left\|M_{m_r}\left(M_{m_{r-1}}(...M_{m_2}(M_{m_1}(f)))\right) - f\right\|_\infty \leq \sum_{i=1}^{r}\varphi_1(m_i) \tag{223}$$

$$\leq \frac{r2^d E^*}{E\left(\frac{T_1}{2},...,\frac{T_d}{2}\right)}\omega_1\left(f, \frac{\|b-a\|_\infty}{m_1}\right),$$

where φ_1 *as in (148).*

Proof. Use also of (148). ■

Theorem 42 *Let* $m_1,...,m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$, $r \in \mathbb{N}$, $f \in C^N\left(\prod_{i=1}^{d}[a_i,b_i]\right)$, $N \in \mathbb{N}$. *Then*

$$\left\|M_{m_r}\left(M_{m_{r-1}}(...M_{m_2}(M_{m_1}(f)))\right) - f\right\|_\infty \leq \sum_{i=1}^{r}\varphi_2(m_i)$$

$$\leq \frac{r2^d E^*}{E\left(\frac{T_1}{2},...,\frac{T_d}{2}\right)}\left[\sum_{j=1}^{N}\left(\frac{\|b-a\|_\infty^j}{m_1^j}\right)\left(\sum_{|\widetilde{\alpha}|=j}\frac{\|f_{\widetilde{\alpha}}\|_\infty}{\prod_{i=1}^{d}\alpha_i!}\right)+\right.$$

$$\left.\frac{\|b-a\|_\infty^N d^N}{N! m_1^N}\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}}, \frac{\|b-a\|_\infty}{m_1}\right)\right], \tag{224}$$

where φ_2 *as in (157).*

Proof. Also use of (157). ■

2.4 Complex Multivariate Neural Network Approximation and Interpolation

We make

Remark 43 *Let* $f : \prod_{i=1}^{d} [a_i, b_i] \rightarrow \mathbb{C}$ *with real and imaginary parts* $f_1, f_2 : f = f_1 + i f_2$, $i = \sqrt{-i}$. *Clearly* f *is continuous iff* f_1 *and* f_2 *are continuous.*

Given that $f_1, f_2 \in C^N \left(\prod_{i=1}^{d} [a_i, b_i] \right)$, $N \in \mathbb{N}$, *it holds*

$$f_{\widetilde{\alpha}} (x) = f_{1,\widetilde{\alpha}} (x) + i f_{2,\widetilde{\alpha}} (x) , \tag{225}$$

where $\widetilde{\alpha}$ *indicates a partial derivative of any order and arrangement.*

Let $f \in C \left(\prod_{i=1}^{d} [a_i, b_i] , \mathbb{C} \right)$ *the space of continuous functions* $f : \prod_{i=1}^{d} [a_i, b_i] \rightarrow \mathbb{C}$. *Then* $f_1, f_2 \in C \left(\prod_{i=1}^{d} [a_i, b_i] \right)$, *and thus both are bounded, implying that* f *is bounded.*

We define

$$M_n^{\mathbb{C}} (f, x) := M_n (f_1, x) + i M_n (f_2, x) , \quad \forall \ x \in \prod_{i=1}^{d} [a_i, b_i] . \tag{226}$$

We observe that

$$\left| M_n^{\mathbb{C}} (f, x) - f (x) \right| \leq | M_n (f_1, x) - f_1 (x) | + | M_n (f_2, x) - f_2 (x) | , \tag{227}$$

and

$$\left\| M_n^{\mathbb{C}} (f) - f \right\|_{\infty} \leq \| M_n (f_1) - f_1 \|_{\infty} + \| M_n (f_2) - f_2 \|_{\infty} . \tag{228}$$

If f *is bounded then* f_1, f_2 *are also bounded.*

For the interpolation property we assume that f *is bounded and measurable. Thus* f_1, f_2 *are measurable.*

We have (for any $(k_1, ..., k_d) \in \{0, 1, ..., n\}^d$*)*

$$\begin{aligned} M_n^{\mathbb{C}} (f, x_{k_1 1}, ..., x_{k_d d}) &= M_n (f_1, x_{k_1 1}, ..., x_{k_d d}) + i M_n (f_2, x_{k_1 1}, ..., x_{k_d d}) \\ &= f_1 (x_{k_1 1}, ..., x_{k_d d}) + i f_2 (x_{k_1 1}, ..., x_{k_d d}) = f (x_{k_1 1}, ..., x_{k_d d}) , \end{aligned} \tag{229}$$

proving interpolation of $M_n^{\mathbb{C}}$.

Theorem 44 *Let* $f \in C \left(\prod_{i=1}^{d} [a_i, b_i] , \mathbb{C} \right)$, *such that* $f = f_1 + i f_2$, $n \in \mathbb{N}$. *Then*

$$\left\| M_n^{\mathbb{C}} (f) - f \right\|_{\infty} \leq \frac{2^d E^*}{E \left(\frac{T_1}{2}, ..., \frac{T_d}{2} \right)} \cdot \left[\omega_1 \left(f_1, \frac{\| b - a \|_{\infty}}{n} \right) + \omega_1 \left(f_2, \frac{\| b - a \|_{\infty}}{n} \right) \right] . \tag{230}$$

Proof. By Theorem 30. ■

Theorem 45 *Let* $f : \prod_{i=1}^{d} [a_i, b_i] \to \mathbb{C}$*, such that* $f = f_1 + i f_2$*. Assume* $f_1, f_2 \in C^N \left(\prod_{i=1}^{d} [a_i, b_i] \right)$*,* $N \in \mathbb{N}$*,* $n \in \mathbb{N}$*. Then*

$$\left| M_n^{\mathbb{C}} (f, x) - f (x) \right| \leq \frac{2^d E^*}{E \left(\frac{T_1}{2}, ..., \frac{T_d}{2} \right)} \cdot$$

$$\left[\sum_{j=1}^{N} \frac{1}{j!} \left(\frac{\|b-a\|_{\infty}^{j}}{n^j} \right) \left[\left(\left(\sum_{i=1}^{d} \left| \frac{\partial}{\partial x_i} \right| \right)^j f_1 (x) \right) + \right.\right.$$

$$\left.\left(\left(\sum_{i=1}^{d} \left| \frac{\partial}{\partial x_i} \right| \right)^j f_2 (x) \right) \right] + \frac{\|b-a\|_{\infty}^{N} d^N}{N! n^N} \cdot$$

$$\left.\left[\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1 \left(f_{1,\widetilde{\alpha}}, \frac{\|b-a\|_{\infty}}{n} \right) + \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1 \left(f_{2,\widetilde{\alpha}}, \frac{\|b-a\|_{\infty}}{n} \right) \right] \right] = \tag{231}$$

$$\frac{2^d E^*}{E \left(\frac{T_1}{2}, ..., \frac{T_d}{2} \right)} \left[\sum_{j=1}^{N} \left(\frac{\|b-a\|_{\infty}^{j}}{n^j} \right) \left(\sum_{|\widetilde{\alpha}|=j} \left(\frac{|f_{1,\widetilde{\alpha}} (x)| + |f_{2,\widetilde{\alpha}} (x)|}{\prod_{i=1}^{d} \alpha_i!} \right) \right) + \right.$$

$$\left. \frac{\|b-a\|_{\infty}^{N} d^N}{N! n^N} \left[\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1 \left(f_{1,\widetilde{\alpha}}, \frac{\|b-a\|_{\infty}}{n} \right) + \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1 \left(f_{2,\widetilde{\alpha}}, \frac{\|b-a\|_{\infty}}{n} \right) \right] \right]. \tag{232}$$

Proof. By (156). ■

2.5 Fuzzy Fractional Mathematical Analysis Background

We need the following basic background

Definition 46 *(see [41]) Let* $\mu : \mathbb{R} \to [0, 1]$ *with the following properties:*

(i) is normal, i.e., $\exists\ x_0 \in \mathbb{R};\ \mu (x_0) = 1$*.*

(ii) $\mu (\lambda x + (1 - \lambda) y) \geq \min\{\mu (x), \mu (y)\}$*,* $\forall\ x, y \in \mathbb{R}$*,* $\forall\ \lambda \in [0, 1]$ *(μ is called a convex fuzzy subset).*

(iii) μ is upper semicontinuous on $\mathbb{R}$*, i.e.* $\forall\ x_0 \in \mathbb{R}$ *and* $\forall\ \varepsilon > 0$*,* $\exists$ *neighborhood* $V (x_0) : \mu (x) \leq \mu (x_0) + \varepsilon$*,* $\forall\ x \in V (x_0)$*.*

(iv) The set $\overline{supp (\mu)}$ *is compact in* $\mathbb{R}$ *(where* $supp(\mu) := \{x \in \mathbb{R} : \mu (x) > 0\}$*).*

We call μ a fuzzy real number. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$*.*

E.g. $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$*, for any* $x_0 \in \mathbb{R}$*, where* $\chi_{\{x_0\}}$ *is the characteristic function at* x_0*.*

For $0<r\leq 1$ *and* $\mu\in\mathbb{R}_{\mathcal{F}}$ *define*

$$[\mu]^{r}:=\{x\in\mathbb{R}:\mu(x)\geq r\}$$

and

$$[\mu]^{0}:=\overline{\{x\in\mathbb{R}:\mu(x)\geq 0\}}.$$

Then it is well known that for each $r\in[0,1]$*,* $[\mu]^{r}$ *is a closed and bounded interval on* $\mathbb{R}$ *([33]).*

For $u,v\in\mathbb{R}_{\mathcal{F}}$ and $\lambda\in\mathbb{R}$, we define uniquely the sum $u\oplus v$ and the product $\lambda\odot u$ by

$$[u\oplus v]^{r}=[u]^{r}+[v]^{r},\quad [\lambda\odot u]^{r}=\lambda[u]^{r},\ \ \forall\, r\in[0,1],$$

where

$[u]^{r}+[v]^{r}$ means the usual addition of two intervals (as substes of $\mathbb{R}$) and

$\lambda[u]^{r}$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g. [41]).

Notice $1\odot u=u$ and it holds

$$u\oplus v=v\oplus u,\ \lambda\odot u=u\odot\lambda.$$

If $0\leq r_{1}\leq r_{2}\leq 1$ then

$$[u]^{r_{2}}\subseteq[u]^{r_{1}}.$$

Actually $[u]^{r}=\left[u_{-}^{(r)},u_{+}^{(r)}\right]$, where $u_{-}^{(r)}\leq u_{+}^{(r)}$, $u_{-}^{(r)}$, $u_{+}^{(r)}\in\mathbb{R}$, $\forall\, r\in[0,1]$.

For $\lambda>0$ one has $\lambda u_{\pm}^{(r)}=(\lambda\odot u)_{\pm}^{(r)}$, respectively.

Define $D:\mathbb{R}_{\mathcal{F}}\times\mathbb{R}_{\mathcal{F}}\to\mathbb{R}_{\mathcal{F}}$ by

$$D(u,v):=\sup_{r\in[0,1]}\max\left\{\left|u_{-}^{(r)}-v_{-}^{(r)}\right|,\left|u_{+}^{(r)}-v_{+}^{(r)}\right|\right\},$$

where

$$[v]^{r}=\left[v_{-}^{(r)},v_{+}^{(r)}\right];\ \ u,v\in\mathbb{R}_{\mathcal{F}}.$$

We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$.

Then $(\mathbb{R}_{\mathcal{F}},D)$ is a complete metric space, see [41], [42].

Here $\sum^{*}$ stands for fuzzy summation and $\widetilde{o}:=\chi_{\{0\}}\in\mathbb{R}_{\mathcal{F}}$ is the neural element with respect to $\oplus$, i.e.,

$$u\oplus\widetilde{0}=\widetilde{0}\oplus u=u,\ \ \forall\, u\in\mathbb{R}_{\mathcal{F}}.$$

Denote

$$D^{*}(f,g)=\sup_{x\in X\subseteq\mathbb{R}}D(f,g),$$

where $f,g:X\to\mathbb{R}_{\mathcal{F}}$.

We mention

Definition 47 *Let $f : X \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, X interval, we define the (first) fuzzy modulus of continuity of f by*

$$\omega_1^{(\mathcal{F})}(f,\delta)_X = \sup_{x,y \in X,\ |x-y| \leq \delta} D(f(x), f(y)), \quad \delta > 0.$$

When $g : X \subseteq \mathbb{R} \to \mathbb{R}$, we define

$$\omega_1(g,\delta)_X = \sup_{x,y \in X,\ |x-y| \leq \delta} |g(x) - g(y)|.$$

We define by $C_{\mathcal{F}}^U(\mathbb{R})$ the space of fuzzy uniformly continuous functions from $\mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, also $C_{\mathcal{F}}(\mathbb{R})$ is the space of fuzzy continuous functions on $\mathbb{R}$, and $C_b(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ is the fuzzy continuous and bounded functions.

We mention

Proposition 48 *([7]) Let $f \in C_{\mathcal{F}}^U(X)$. Then $\omega_1^{(\mathcal{F})}(f,\delta)_X < \infty$, for any $\delta > 0$.*

By [11], p. 129 we have that $C_{\mathcal{F}}^U([a,b]) = C_{\mathcal{F}}([a,b])$, fuzzy continuous functions on $[a,b] \subset \mathbb{R}$.

Proposition 49 *([7]) It holds*

$$\lim_{\delta \to 0} \omega_1^{(\mathcal{F})}(f,\delta)_X = \omega_1^{(\mathcal{F})}(f,0)_X = 0,$$

iff $f \in C_{\mathcal{F}}^U(X)$.

Proposition 50 *([7]) Here $[f]^r = \left[f_-^{(r)}, f_+^{(r)}\right]$, $r \in [0,1]$. Let $f \in C_{\mathcal{F}}(\mathbb{R})$. Then $f_\pm^{(r)}$ are equicontinuous with respect to $r \in [0,1]$ over $\mathbb{R}$, respectively in $\pm$.*

Note 51 *It is clear by Propositions 49, 50, that if $f \in C_{\mathcal{F}}^U(\mathbb{R})$, then $f_\pm^{(r)} \in C_U(\mathbb{R})$ (uniformly continuous on $\mathbb{R}$). Also if $f \in C_b(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ implies $f_\pm^{(r)} \in C_b(\mathbb{R})$ (continuous and bounded functions on $\mathbb{R}$).*

Proposition 52 *Let $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$. Assume that $\omega_1^{\mathcal{F}}(f,\delta)_X$, $\omega_1\left(f_-^{(r)},\delta\right)_X$, $\omega_1\left(f_+^{(r)},\delta\right)_X$ are finite for any $\delta > 0$, $r \in [0,1]$, where X any interval of $\mathbb{R}$.*

Then

$$\omega_1^{(\mathcal{F})}(f,\delta)_X = \sup_{r \in [0,1]} \max\left\{\omega_1\left(f_-^{(r)},\delta\right)_X, \omega_1\left(f_+^{(r)},\delta\right)_X\right\}.$$

Proof. Similar to Proposition 14.15, p. 246 of [11]. ∎

We need

Remark 53 *([4]). Here* $r \in [0,1]$*,* $x_i^{(r)}, y_i^{(r)} \in \mathbb{R}$*,* $i = 1, ..., m \in \mathbb{N}$*. Suppose that*

$$\sup_{r\in[0,1]} \max\left(x_i^{(r)}, y_i^{(r)}\right) \in \mathbb{R}, \text{ for } i = 1, ..., m.$$

Then one sees easily that

$$\sup_{r\in[0,1]} \max\left(\sum_{i=1}^{m} x_i^{(r)}, \sum_{i=1}^{m} y_i^{(r)}\right) \leq \sum_{i=1}^{m} \sup_{r\in[0,1]} \max\left(x_i^{(r)}, y_i^{(r)}\right). \tag{233}$$

We need

Definition 54 *Let* $x, y \in \mathbb{R}_{\mathcal{F}}$*. If there exists* $z \in \mathbb{R}_{\mathcal{F}} : x = y \oplus z$*, then we call* z *the* H*-difference on* x *and* y*, denoted* $x - y$*.*

Definition 55 *([40]) Let* $T := [x_0, x_0 + \beta] \subset \mathbb{R}$*, with* $\beta > 0$*. A function* $f : T \to \mathbb{R}_{\mathcal{F}}$ *is* H*-difference at* $x \in T$ *if there exists an* $f'(x) \in \mathbb{R}_{\mathcal{F}}$ *such that the limits (with respect to* D*)*

$$\lim_{h\to 0+} \frac{f(x+h) - f(x)}{h}, \quad \lim_{h\to 0+} \frac{f(x) - f(x-h)}{h} \tag{234}$$

exist and are equal to $f'(x)$*.*

We call f' *the* H*-derivative or fuzzy derivative of* f *at* x*.*

Above is assumed that the H-differences $f(x+h) - f(x)$, $f(x) - f(x-h)$ exists in $\mathbb{R}_{\mathcal{F}}$ in a neighborhood of x.

Higher order H-fuzzy derivatives are defined the obvious way, like in the real case.

We denote by $C_{\mathcal{F}}^N(\mathbb{R})$, $N \geq 1$, the space of all N-times continuously H-fuzzy differentiable functions from $\mathbb{R}$ into $\mathbb{R}_{\mathcal{F}}$, similarly is defined $C_{\mathcal{F}}^N([a,b])$, $[a,b] \subset \mathbb{R}$.

We mention

Theorem 56 *([34]) Let* $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ *be* H*-fuzzy differentiable. Let* $t \in \mathbb{R}$*,* $0 \leq r \leq 1$*. Clearly*

$$[f(t)]^r = \left[f(t)_-^{(r)}, f(t)_+^{(r)}\right] \subseteq \mathbb{R}.$$

Then $(f(t))_{\pm}^{(r)}$ *are differentiable and*

$$[f'(t)]^r = \left[\left(f(t)_-^{(r)}\right)', \left(f(t)_+^{(r)}\right)'\right].$$

I.e.

$$(f')_{\pm}^{(r)} = \left(f_{\pm}^{(r)}\right)', \quad \forall\, r \in [0,1].$$

Remark 57 *([6]) Let $f \in C_{\mathcal{F}}^{N}(\mathbb{R})$, $N \geq 1$. Then by Theorem 56 we obtain*

$$\left[f^{(i)}(t)\right]^{r} = \left[\left(f(t)_{-}^{(r)}\right)^{(i)}, \left(f(t)_{+}^{(r)}\right)^{(i)}\right],$$

for $i = 0, 1, 2, ..., N$, and in particular we have that

$$\left(f^{(i)}\right)_{\pm}^{(r)} = \left(f_{\pm}^{(r)}\right)^{(i)},$$

for any $r \in [0, 1]$, all $i = 0, 1, 2, ..., N$.

Note 58 *([6]) Let $f \in C_{\mathcal{F}}^{N}(\mathbb{R})$, $N \geq 1$. Then by Theorem 56 we have $f_{\pm}^{(r)} \in C^{N}(\mathbb{R})$, for any $r \in [0, 1]$.*

Items 56-58 are valid also on $[a, b]$.

By [11], p. 131, if $f \in C_{\mathcal{F}}([a, b])$, then f is a fuzzy bounded function.

For the definition of general fuzzy integral we follow [35] next.

Definition 59 *Let (Ω, Σ, μ) be a complete σ-finite measure space. We call $F : \Omega \to R_{\mathcal{F}}$ measurable iff $\forall$ closed $B \subseteq \mathbb{R}$ the function $F^{-1}(B) : \Omega \to [0, 1]$ defined by*

$$F^{-1}(B)(w) := \sup_{x \in B} F(w)(x), \text{ all } w \in \Omega$$

is measurable, see [35].

Theorem 60 *([35]) For $F : \Omega \to \mathbb{R}_{\mathcal{F}}$,*

$$F(w) = \left\{\left(F_{-}^{(r)}(w), F_{+}^{(r)}(w)\right) | 0 \leq r \leq 1\right\},$$

the following are equivalent

(1) F is measurable,

(2) $\forall\ r \in [0, 1]$, $F_{-}^{(r)}$, $F_{+}^{(r)}$ are measurable.

Following [35], given that for each $r \in [0, 1]$, $F_{-}^{(r)}$, $F_{+}^{(r)}$ are integrable we have that the parametrized representation

$$\left\{\left(\int_{A} F_{-}^{(r)} d\mu, \int_{A} F_{+}^{(r)} d\mu\right) | 0 \leq r \leq 1\right\} \tag{235}$$

is a fuzzy real number for each $A \in \Sigma$.

The last fact leads to

Definition 61 *([35]) A measurable function $F : \Omega \to \mathbb{R}_{\mathcal{F}}$,*

$$F(w) = \left\{\left(F_{-}^{(r)}(w), F_{+}^{(r)}(w)\right) | 0 \leq r \leq 1\right\}$$

is integrable if for each $r \in [0, 1]$, $F_{\pm}^{(r)}$ are integrable, or equivalently, if $F_{\pm}^{(0)}$ are integrable.

In this case, the fuzzy integral of F over $A \in \Sigma$ is defined by

$$\int_A F d\mu := \left\{ \left(\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu \right) | 0 \leq r \leq 1 \right\}.$$

By [35], F is integrable iff $w \to \|F(w)\|_{\mathcal{F}}$ is real-valued integrable.

Here denote

$$\|u\|_{\mathcal{F}} := D\left(u, \widetilde{0}\right), \ \forall \, u \in \mathbb{R}_{\mathcal{F}}.$$

We need also

Theorem 62 *([35]) Let $F, G : \Omega \to \mathbb{R}_{\mathcal{F}}$ be integrable. Then*

(1) Let $a, b \in \mathbb{R}$, then $aF + bG$ is integrable and for each $A \in \Sigma$,

$$\int_A (aF + bG) \, d\mu = a \int_A F d\mu + b \int_A G d\mu;$$

(2) $D(F, G)$ is a real- valued integrable function and for each $A \in \Sigma$,

$$D\left(\int_A F d\mu, \int_A G d\mu \right) \leq \int_A D(F, G) \, d\mu.$$

In particular,

$$\left\| \int_A F d\mu \right\|_{\mathcal{F}} \leq \int_A \|F\|_{\mathcal{F}} \, d\mu.$$

Above μ could be the Lebesgue measure, with all the basic properties valid here too.

Basically here we have

$$\left[\int_A F d\mu \right]^r = \left[\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu \right], \tag{236}$$

i.e.

$$\left(\int_A F d\mu \right)_{\pm}^{(r)} = \int_A F_{\pm}^{(r)} d\mu, \quad \forall \, r \in [0, 1]. \tag{237}$$

We need

Definition 63 *([13]) Let $f \in C_{\mathcal{F}}([a, b])$ (fuzzy continuous on $[a, b] \subset \mathbb{R}$), $\nu > 0$.*

We define the Fuzzy Fractional left Riemann-Liouville operator as

$$J_a^\nu f(x) := \frac{1}{\Gamma(\nu)} \odot \int_a^x (x - t)^{\nu - 1} \odot f(t) \, dt, \quad x \in [a, b], \tag{238}$$

$$J_a^0 f := f.$$

Also, we define the Fuzzy Fractional right Riemann-Liouville operator as

$$I_{b-}^\nu f(x) := \frac{1}{\Gamma(\nu)} \odot \int_x^b (t - x)^{\nu - 1} \odot f(t) \, dt, \quad x \in [a, b], \tag{239}$$

$$I_{b-}^0 f := f.$$

We need

Definition 64 *([13]) We define the Fuzzy Fractional left Caputo derivative, $x \in [a,b]$.*

Let $f \in C_{\mathcal{F}}^{n}([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$ ($\lceil \cdot \rceil$ denotes the ceiling). We define

$$D_{*a}^{\nu \mathcal{F}} f(x) := \frac{1}{\Gamma(n-\nu)} \odot \int_a^x (x-t)^{n-\nu-1} \odot f^{(n)}(t)\, dt \tag{240}$$

$$= \left\{ \left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f^{(n)}\right)_-^{(r)}(t)\, dt, \right.\right.$$

$$\left.\left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f^{(n)}\right)_+^{(r)}(t)\, dt \right) | 0 \le r \le 1 \right\} =$$

$$= \left\{ \left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_-^{(r)}\right)^{(n)}(t)\, dt, \right.\right.$$

$$\left.\left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_+^{(r)}\right)^{(n)}(t)\, dt \right) | 0 \le r \le 1 \right\}. \tag{241}$$

So, we get

$$\left[D_{*a}^{\nu \mathcal{F}} f(x)\right]^r = \left[\left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_-^{(r)}\right)^{(n)}(t)\, dt, \right.\right.$$

$$\left.\left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_+^{(r)}\right)^{(n)}(t)\, dt \right) \right], \quad 0 \le r \le 1. \tag{242}$$

That is

$$\left(D_{*a}^{\nu \mathcal{F}} f(x)\right)_{\pm}^{(r)} = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_{\pm}^{(r)}\right)^{(n)}(t)\, dt = \left(D_{*a}^{\nu}\left(f_{\pm}^{(r)}\right)\right)(x),$$

see [10], [28].

I.e. we get that

$$\left(D_{*a}^{\nu \mathcal{F}} f(x)\right)_{\pm}^{(r)} = \left(D_{*a}^{\nu}\left(f_{\pm}^{(r)}\right)\right)(x), \tag{243}$$

$\forall\, x \in [a,b]$, in short

$$\left(D_{*a}^{\nu \mathcal{F}} f\right)_{\pm}^{(r)} = D_{*a}^{\nu}\left(f_{\pm}^{(r)}\right), \quad \forall\, r \in [0,1]. \tag{244}$$

We need

Lemma 65 *([13]) $D_{*a}^{\nu \mathcal{F}} f(x)$ is fuzzy continuous in $x \in [a,b]$.*

We need

Definition 66 *([13]) We define the Fuzzy Fractional right Caputo derivative, $x \in [a,b]$.*

Let $f \in C^n_{\mathcal{F}}([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$. We define

$$D^{\nu \mathcal{F}}_{b-} f(x) := \frac{(-1)^n}{\Gamma(n-\nu)} \odot \int_x^b (t-x)^{n-\nu-1} \odot f^{(n)}(t)\,dt$$

$$= \left\{ \left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f^{(n)}\right)_-^{(r)}(t)\,dt, \right.\right.$$

$$\left.\left. \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f^{(n)}\right)_+^{(r)}(t)\,dt \right) | 0 \le r \le 1 \right\} \tag{245}$$

$$= \left\{ \left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_-^{(r)}\right)^{(n)}(t)\,dt, \right.\right.$$

$$\left.\left. \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_+^{(r)}\right)^{(n)}(t)\,dt \right) | 0 \le r \le 1 \right\}.$$

We get

$$\left[D^{\nu \mathcal{F}}_{b-} f(x)\right]^r = \left[\left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_-^{(r)}\right)^{(n)}(t)\,dt, \right.\right.$$

$$\left.\left. \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_+^{(r)}\right)^{(n)}(t)\,dt \right) \right], \quad 0 \le r \le 1.$$

That is

$$\left(D^{\nu \mathcal{F}}_{b-} f(x)\right)_\pm^{(r)} = \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_\pm^{(r)}\right)^{(n)}(t)\,dt = \left(D^{\nu}_{b-}\left(f_\pm^{(r)}\right)\right)(x),$$

see [9].

I.e. we get that

$$\left(D^{\nu \mathcal{F}}_{b-} f(x)\right)_\pm^{(r)} = \left(D^{\nu}_{b-}\left(f_\pm^{(r)}\right)\right)(x), \tag{246}$$

$\forall\ x \in [a,b]$, in short

$$\left(D^{\nu \mathcal{F}}_{b-} f\right)_\pm^{(r)} = D^{\nu}_{b-}\left(f_\pm^{(r)}\right), \quad \forall\ r \in [0,1]. \tag{247}$$

Clearly,

$$D^{\nu}_{b-}\left(f_-^{(r)}\right) \le D^{\nu}_{b-}\left(f_+^{(r)}\right), \quad \forall\ r \in [0,1].$$

We need

Lemma 67 *([13]) $D^{\nu \mathcal{F}}_{b-} f(x)$ is fuzzy continuous in $x \in [a,b]$.*

2.6 Fuzzy and Fuzzy-Fractional Univariate Neural Network Approximation and Interpolation

We give

Definition 68 *Let $f \in C_{\mathcal{F}}([a,b])$. We set*

$$\left(H_n^{\mathcal{F}}(f)\right)(x) := \frac{\sum_{k=0}^{n^*} f(x_k) \odot B\left(\frac{Tn(x-x_k)}{b-a}\right)}{\sum_{k=0}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}, \tag{248}$$

and we call it fuzzy interpolation univariate Neural Network operator.

Comment
We observe that

$$\left[\left(H_n^{\mathcal{F}}(f)\right)(x)\right]^r = \sum_{k=0}^{n} [f(x_k)]^r \frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} =$$

$$\sum_{k=0}^{n} \left[f_-^{(r)}(x_k), f_+^{(r)}(x_k)\right] \frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} =$$

$$\left[\sum_{k=0}^{n} f_-^{(r)}(x_k) \frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}, \sum_{k=0}^{n} f_+^{(r)}(x_k) \frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}\right] =$$

$$\left[\left(H_n\left(f_-^{(r)}\right)\right)(x), \left(H_n\left(f_+^{(r)}\right)\right)(x)\right]. \tag{249}$$

We have proved that

$$\left(H_n^{\mathcal{F}}(f)\right)_{\pm}^{(r)} = H_n\left(f_{\pm}^{(r)}\right), \tag{250}$$

$\forall\, r \in [0,1]$, respectively.

Comment
We notice also that

$$\left(\left(H_n^{\mathcal{F}}(f)\right)(x_i)\right)_{\pm}^{(r)} = \left(H_n\left(f_{\pm}^{(r)}\right)\right)(x_i) = f_{\pm}^{(r)}(x_i), \;\; i = 0,1,...,n, \;\forall\, r \in [0,1]. \tag{251}$$

Conclusion 69 *(by [33], [35])*

$$\left(H_n^{\mathcal{F}}(f)\right)(x_i) = f(x_i), \;\; i = 0,1,...,n,$$

the interpolation property is true at fuzzy setting.

We make

Remark 70 *Let $f \in C_{\mathcal{F}}([a,b])$. We notice that*

$$D\left(\left(H_n^{\mathcal{F}}(f)\right)(x), f(x)\right) =$$

$$\sup_{r\in[0,1]} \max\left\{\left|\left(H_n(f)\right)_-^{(r)}(x) - f_-^{(r)}(x)\right|, \left|\left(H_n(f)\right)_+^{(r)}(x) - f_+^{(r)}(x)\right|\right\} =$$

$$\sup_{r\in[0,1]} \max\left\{\left|\left(H_n\left(f_-^{(r)}\right)\right)(x) - f_-^{(r)}(x)\right|, \left|\left(H_n\left(f_+^{(r)}\right)\right)(x) - f_+^{(r)}(x)\right|\right\} \leq \tag{252}$$

(hence $f_\pm^{(r)} \in C([a,b])$)

$$\frac{2B^*}{B\left(\frac{T}{2}\right)} \sup_{r\in[0,1]} \max\left\{\omega_1\left(f_-^{(r)}, \frac{b-a}{n}\right), \omega_1\left(f_+^{(r)}, \frac{b-a}{n}\right)\right\} =$$

(by Theorem 10 and Proposition 52)

$$\frac{2B^*}{B\left(\frac{T}{2}\right)} \omega_1^{(\mathcal{F})}\left(f, \frac{b-a}{n}\right). \tag{253}$$

We have proved that

Theorem 71 *Let $f \in C_{\mathcal{F}}([a,b])$, $x \in [a,b]$. Then*

1)

$$D\left(\left(H_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq \frac{2B^*}{B\left(\frac{T}{2}\right)} \omega_1^{(\mathcal{F})}\left(f, \frac{b-a}{n}\right), \tag{254}$$

so that $\left(H_n^{\mathcal{F}}(f)\right)(x) \stackrel{D}{\rightarrow} f(x)$, as $n \to \infty$, pointwise,

and

2)

$$D^*\left(H_n^{\mathcal{F}}(f), f\right) \leq \frac{2B^*}{B\left(\frac{T}{2}\right)} \omega_1^{(\mathcal{F})}\left(f, \frac{b-a}{n}\right), \tag{255}$$

so that $H_n^{\mathcal{F}}(f) \stackrel{D^*}{\rightarrow} f$, as $n \to \infty$, uniformly.

Taking into account fuzzy smoothness of f we give

Theorem 72 *Let $f \in C_{\mathcal{F}}^N([a,b])$, $N \in \mathbb{N}$, $x \in [a,b]$. Then*

1)

$$D\left(\left(H_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq$$

$$\frac{2B^*}{B\left(\frac{T}{2}\right)} \left\{\sum_{j=1}^{N} \frac{(b-a)^j}{j!n^j} D\left(f^{(j)}(x), \widetilde{o}\right) + \frac{(b-a)^N}{N!n^N} \omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{b-a}{n}\right)\right\}, \tag{256}$$

2) assume more that $D\left(f^{(j)}(x), \widetilde{o}\right) = 0$, $j = 1, ..., N$, where $x \in [a,b]$ is fixed, we get

$$D\left(\left(H_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq \frac{2B^*}{B\left(\frac{T}{2}\right)} \frac{(b-a)^N}{N!n^N} \omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{b-a}{n}\right), \tag{257}$$

a fuzzy pointwise convergence at high speed $\frac{1}{n^{N+1}}$,

3)

$$D^{*}\left(H_{n}^{\mathcal{F}}(f), f\right) \leq$$

$$\frac{2B^{*}}{B\left(\frac{T}{2}\right)}\left\{\sum_{j=1}^{N}\frac{(b-a)^{j}}{j!n^{j}}D^{*}\left(f^{(j)}, \widetilde{o}\right)+\frac{(b-a)^{N}}{N!n^{N}}\omega_{1}^{(\mathcal{F})}\left(f^{(N)}, \frac{b-a}{n}\right)\right\}. \tag{258}$$

Proof. Here clearly $f_{\pm}^{(r)} \in C^{N}([a,b])$, $\forall\ r \in [0,1]$. Then

$$D\left(\left(H_{n}^{\mathcal{F}}(f)\right)(x), f(x)\right) =$$

$$\sup_{r\in[0,1]}\max\left\{\left|\left(H_{n}^{\mathcal{F}}(f)\right)_{-}^{(r)}(x)-f_{-}^{(r)}(x)\right|, \left|\left(H_{n}^{\mathcal{F}}(f)\right)_{+}^{(r)}(x)-f_{+}^{(r)}(x)\right|\right\} =$$

$$\sup_{r\in[0,1]}\max\left\{\left|\left(H_{n}\left(f_{-}^{(r)}\right)\right)(x)-f_{-}^{(r)}(x)\right|, \left|\left(H_{n}\left(f_{+}^{(r)}\right)\right)(x)-f_{+}^{(r)}(x)\right|\right\} \overset{\text{(by (34))}}{\leq} \tag{259}$$

$$\frac{2B^{*}}{B\left(\frac{T}{2}\right)}\sup_{r\in[0,1]}\max\left\{\sum_{j=1}^{N}\frac{\left|\left(f_{-}^{(r)}\right)^{(j)}(x)\right|}{j!}\frac{(b-a)^{j}}{n^{j}}+\omega_{1}\left(\left(f_{-}^{(r)}\right)^{(N)}, \frac{b-a}{n}\right)\frac{(b-a)^{N}}{N!n^{N}},\right.$$

$$\left.\sum_{j=1}^{N}\frac{\left|\left(f_{+}^{(r)}\right)^{(j)}(x)\right|}{j!}\frac{(b-a)^{j}}{n^{j}}+\omega_{1}\left(\left(f_{+}^{(r)}\right)^{(N)}, \frac{b-a}{n}\right)\frac{(b-a)^{N}}{N!n^{N}}\right\} =$$

$$\frac{2B^{*}}{B\left(\frac{T}{2}\right)}\sup_{r\in[0,1]}\max\left\{\sum_{j=1}^{N}\frac{\left|\left(f_{-}^{(j)}\right)^{(r)}(x)\right|}{j!}\frac{(b-a)^{j}}{n^{j}}+\omega_{1}\left(\left(f_{-}^{(N)}\right)^{(r)}, \frac{b-a}{n}\right)\frac{(b-a)^{N}}{N!n^{N}},\right. \tag{260}$$

$$\left.\sum_{j=1}^{N}\frac{\left|\left(f_{+}^{(j)}\right)^{(r)}(x)\right|}{j!}\frac{(b-a)^{j}}{n^{j}}+\omega_{1}\left(\left(f_{+}^{(N)}\right)^{(r)}, \frac{b-a}{n}\right)\frac{(b-a)^{N}}{N!n^{N}}\right\} \leq$$

$$\frac{2B^{*}}{B\left(\frac{T}{2}\right)}\left\{\sum_{j=1}^{N}\frac{(b-a)^{j}}{j!n^{j}}\sup_{r\in[0,1]}\max\left\{\left|\left(f_{-}^{(j)}\right)^{(r)}(x)\right|, \left|\left(f_{+}^{(j)}\right)^{(r)}(x)\right|\right\}+\right.$$

$$\left.\frac{(b-a)^{N}}{N!n^{N}}\sup_{r\in[0,1]}\max\left\{\omega_{1}\left(\left(f_{-}^{(N)}\right)^{(r)}, \frac{b-a}{n}\right), \omega_{1}\left(\left(f_{+}^{(N)}\right)^{(r)}, \frac{b-a}{n}\right)\right\}\right\} = \tag{261}$$

$$\frac{2B^{*}}{B\left(\frac{T}{2}\right)}\left\{\sum_{j=1}^{N}\frac{(b-a)^{j}}{j!n^{j}}D\left(f^{(j)}(x), \widetilde{o}\right)+\frac{(b-a)^{N}}{N!n^{N}}\omega_{1}^{(\mathcal{F})}\left(f^{(N)}, \frac{b-a}{n}\right)\right\},$$

proving theorem. ■

The related fuzzy-fractional results follow.

Theorem 73 *Let $\beta > 0$, $N = \lceil \beta \rceil$, $\beta \notin \mathbb{N}$, $f \in C_{\mathcal{F}}^{N}([a,b])$, $x \in [a,b]$. Then*

$$D\left(\left(H_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq$$

$$\frac{B^*}{B\left(\frac{T}{2}\right)}\left[2\sum_{j=1}^{N-1}\frac{D\left(f^{(j)}(x), \widetilde{o}\right)}{j!}\frac{(b-a)^j}{n^j}+\right. \tag{262}$$

$$\left.\frac{(b-a)^{\beta}}{\Gamma(\beta+1)n^{\beta}}\left[\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\beta\mathcal{F}}f\right), \frac{b-a}{n}\right)+\omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\beta\mathcal{F}}f\right), \frac{b-a}{n}\right)\right]\right].$$

Proof. We get that $f_{\pm}^{(r)} \in C^N([a,b])$, $\forall\ r \in [0,1]$, and $D_{x-}^{\beta\mathcal{F}}f$, $D_{*x}^{\beta\mathcal{F}}f$ are fuzzy continuous on $[a,b]$, $\forall\ x \in [a,b]$, so that $\left(D_{x-}^{\beta\mathcal{F}}f\right)_{\pm}^{(r)}$, $\left(D_{*x}^{\beta\mathcal{F}}f\right)_{\pm}^{(r)} \in C([a,b])$, $\forall\ x \in [a,b]$, $\forall\ r \in [0,1]$. By (74) we get

$$\left|H_n\left(f_{\pm}^{(r)}, x\right) - f_{\pm}^{(r)}(x)\right| \leq$$

$$\frac{B^*}{B\left(\frac{T}{2}\right)}\left[2\sum_{j=1}^{N-1}\frac{\left|\left(f_{\pm}^{(r)}\right)^{(j)}(x)\right|}{j!}\frac{(b-a)^j}{n^j}+\right. \tag{263}$$

$$\left.\frac{(b-a)^{\beta}}{\Gamma(\beta+1)n^{\beta}}\left[\omega_1\left(D_{x-}^{\beta}\left(f_{\pm}^{(r)}\right), \frac{b-a}{n}\right)+\omega_1\left(D_{*x}^{\beta}\left(f_{\pm}^{(r)}\right), \frac{b-a}{n}\right)\right]\right] =$$

$$\frac{B^*}{B\left(\frac{T}{2}\right)}\left[2\sum_{j=1}^{N-1}\frac{\left|\left(f^{(j)}(x)\right)_{\pm}^{(r)}\right|}{j!}\frac{(b-a)^j}{n^j}+\right.$$

$$\left.\frac{(b-a)^{\beta}}{\Gamma(\beta+1)n^{\beta}}\left[\omega_1\left(\left(D_{x-}^{\beta\mathcal{F}}f\right)_{\pm}^{(r)}, \frac{b-a}{n}\right)+\omega_1\left(\left(D_{*x}^{\beta\mathcal{F}}f\right)_{\pm}^{(r)}, \frac{b-a}{n}\right)\right]\right] \leq \tag{264}$$

$$\frac{B^*}{B\left(\frac{T}{2}\right)}\left[2\sum_{j=1}^{N-1}\frac{D\left(f^{(j)}(x), \widetilde{o}\right)}{j!}\frac{(b-a)^j}{n^j}+\right.$$

$$\left.\frac{(b-a)^{\beta}}{\Gamma(\beta+1)n^{\beta}}\left[\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\beta\mathcal{F}}f\right), \frac{b-a}{n}\right)+\omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\beta\mathcal{F}}f\right), \frac{b-a}{n}\right)\right]\right], \tag{265}$$

proving the claim. ■

Corollary 74 *(to Theorem 73) Assume more that* $D\left(f^{(j)}(x), \widetilde{o}\right) = 0$, *for* $j = 1, ..., N-1$, *for a fixed* $x \in [a,b]$. *Then*

$$D\left(\left(H_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq \frac{B^*}{B\left(\frac{T}{2}\right)} \frac{(b-a)^{\beta}}{\Gamma(\beta+1)\, n^{\beta}} \cdot$$

$$\left[\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\beta\mathcal{F}} f\right), \frac{b-a}{n}\right) + \omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\beta\mathcal{F}} f\right), \frac{b-a}{n}\right)\right], \tag{266}$$

fuzzy pointwise convergence at high speed of $\frac{1}{n^{\beta+1}}$.

Theorem 75 *Let* $\beta > 0$, $N = \lceil \beta \rceil$, $\beta \notin \mathbb{N}$, $f \in C_{\mathcal{F}}^{N}([a,b])$. *Then*

$$D^*\left(H_n^{\mathcal{F}}(f), f\right) \leq$$

$$\frac{B^*}{B\left(\frac{T}{2}\right)} \left[2 \sum_{j=1}^{N-1} \frac{D^*\left(f^{(j)}, \widetilde{o}\right)}{j!} \frac{(b-a)^j}{n^j} + \frac{(b-a)^{\beta}}{\Gamma(\beta+1)\, n^{\beta}} \cdot \right.$$

$$\left.\left[\sup_{x\in[a,b]} \omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\beta\mathcal{F}} f\right), \frac{b-a}{n}\right) + \sup_{x\in[a,b]} \omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\beta\mathcal{F}} f\right), \frac{b-a}{n}\right)\right]\right] < +\infty. \tag{267}$$

Proof. We notice the following

$$\left(D_{x-}^{\beta\mathcal{F}} f\right)_{\pm}^{(r)}(t) = \left(D_{x-}^{\beta}\left(f_{\pm}^{(r)}\right)\right)(t) =$$

$$\frac{(-1)^N}{\Gamma(N-\beta)} \int_t^x (s-t)^{N-\beta-1} \left(f_{\pm}^{(r)}\right)^{(N)}(s)\, ds, \tag{268}$$

all $a \leq t \leq x$.

Hence it holds

$$\left|\left(D_{x-}^{\beta\mathcal{F}} f\right)_{\pm}^{(r)}(t)\right| \leq \frac{1}{\Gamma(N-\beta)} \int_t^x (s-t)^{N-\beta-1} \left|\left(f_{\pm}^{(r)}\right)^{(N)}(s)\right| ds \leq$$

$$\frac{\left\|\left(f^{(N)}\right)_{\pm}^{(r)}\right\|_{\infty} (b-a)^{N-\beta}}{\Gamma(N-\beta+1)} \leq \frac{D^*\left(f^{(N)}, \widetilde{o}\right)}{\Gamma(N-\beta+1)} (b-a)^{N-\beta}, \tag{269}$$

$a \leq t \leq x$.

Thus

$$\left\|\left(D_{x-}^{\beta\mathcal{F}} f\right)_{\pm}^{(r)}\right\|_{\infty} \leq \frac{D^*\left(f^{(N)}, \widetilde{o}\right)}{\Gamma(N-\beta+1)} (b-a)^{N-\beta} \tag{270}$$

(notice $\left(D_{x-}^{\beta\mathcal{F}} f\right)_{\pm}^{(r)}(t) = 0$, for $x \leq t \leq b$), $\forall\, r \in [0,1]$.

So that

$$D^{*}\left(\left(D_{x-}^{\beta\mathcal{F}}f\right),\widetilde{o}\right)\leq\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\beta+1\right)}\left(b-a\right)^{N-\beta}. \tag{271}$$

Similarly we have

$$\left(D_{*x}^{\beta\mathcal{F}}f\right)_{\pm}^{(r)}(t)=\left(D_{*x}^{\beta}\left(f_{\pm}^{(r)}\right)\right)(t)=$$

$$\frac{1}{\Gamma\left(N-\beta\right)}\int_{x}^{t}\left(t-s\right)^{N-\beta-1}\left(f_{\pm}^{(r)}\right)^{(N)}(s)\,ds, \tag{272}$$

where $x\leq t\leq b$.

Thus

$$\left|\left(D_{*x}^{\beta\mathcal{F}}f\right)_{\pm}^{(r)}(t)\right|\leq\frac{1}{\Gamma\left(N-\beta\right)}\int_{x}^{t}\left(t-s\right)^{N-\beta-1}\left|\left(f_{\pm}^{(r)}\right)^{(N)}(s)\right|ds=$$

$$\frac{1}{\Gamma\left(N-\beta\right)}\int_{x}^{t}\left(t-s\right)^{N-\beta-1}\left|\left(f^{(N)}\right)_{\pm}^{(r)}(s)\right|ds\leq\frac{\left\|\left(f^{(N)}\right)_{\pm}^{(r)}\right\|_{\infty}}{\Gamma\left(N-\beta+1\right)}\left(b-a\right)^{N-\beta}\leq \tag{273}$$

$$\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\beta+1\right)}\left(b-a\right)^{N-\beta},\quad x\leq t\leq b. \tag{274}$$

So that

$$\left|\left(D_{*x}^{\beta\mathcal{F}}f\right)_{\pm}^{(r)}(t)\right|\leq\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\beta+1\right)}\left(b-a\right)^{N-\beta}, \tag{275}$$

$x\leq t\leq b$.

(notice $\left(D_{*x}^{\beta\mathcal{F}}f\right)_{\pm}^{(r)}(t)=0$, for $a\leq t\leq x$, $\forall\ r\in[0,1]$.)

Thus

$$\left\|\left(D_{*x}^{\beta\mathcal{F}}f\right)_{\pm}^{(r)}\right\|_{\infty}\leq\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\beta+1\right)}\left(b-a\right)^{N-\beta}, \tag{276}$$

$\forall\ r\in[0,1]$.

Therefore

$$D^{*}\left(\left(D_{*x}^{\beta\mathcal{F}}f\right),\widetilde{o}\right)\leq\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\beta+1\right)}\left(b-a\right)^{N-\beta}. \tag{277}$$

We have proved that

$$\left\{\begin{array}{l}D^{*}\left(\left(D_{x-}^{\beta\mathcal{F}}f\right),\widetilde{o}\right)\\D^{*}\left(\left(D_{*x}^{\beta\mathcal{F}}f\right),\widetilde{o}\right)\end{array}\right.\leq\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma\left(N-\beta+1\right)}\left(b-a\right)^{N-\beta}. \tag{278}$$

Next we see that

$$\omega_{1}^{(\mathcal{F})}\left(\left(D_{x-}^{\beta\mathcal{F}}f\right),\frac{b-a}{n}\right)=\sup_{\substack{z_{1},z_{2}\in[a,b]\\:|z_{1}-z_{2}|\leq\frac{b-a}{n}}}D\left(\left(D_{x-}^{\beta\mathcal{F}}f\right)(z_{1}),\left(D_{x-}^{\beta\mathcal{F}}f\right)(z_{2})\right)\leq \tag{279}$$

$$\sup_{\substack{z_1,z_2\in[a,b]\\ :|z_1-z_2|\leq\frac{b-a}{n}}} \left\{ D\left(\left(D_{x-}^{\beta\mathcal{F}}f\right)(z_1),\widetilde{o}\right) + D\left(\left(D_{x-}^{\beta\mathcal{F}}f\right)(z_2),\widetilde{o}\right)\right\} \leq$$

$$2D^*\left(\left(D_{x-}^{\beta\mathcal{F}}f\right),\widetilde{o}\right) \leq \frac{2D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\beta+1)}(b-a)^{N-\beta} =: \gamma < \infty. \tag{280}$$

Therefore it holds

$$\sup_{x\in[a,b]} \omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\beta\mathcal{F}}f\right),\frac{b-a}{n}\right) \leq \gamma < \infty. \tag{281}$$

Totally similar we get

$$\sup_{x\in[a,b]} \omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\beta\mathcal{F}}f\right),\frac{b-a}{n}\right) \leq \gamma < \infty. \tag{282}$$

Using (262), (281), (282) we have established (267). ■

2.7 Multivariate Fuzzy Analysis background

Let $f,g : \prod_{i=1}^{d}[a_i,b_i] \to \mathbb{R}_{\mathcal{F}}$. We define the distance

$$D^*(f,g) := \sup_{x\in\prod_{i=1}^{d}[a_i,b_i]} D(f(x),g(x)). \tag{283}$$

Definition 76 *Let $f \in C\left(\prod_{i=1}^{d}[a_i,b_i]\right)$, $d \in \mathbb{N}$, we define ($h>0$)*

$$\omega_1(f,h) := \sup_{all\ x_i,x_i'\in[a_i,b_i],\left|x_i-x_i'\right|\leq h,\ for\ i=1,\dots,d} \left|f(x_1,\dots,x_d) - f(x_1',\dots,x_d')\right|. \tag{284}$$

For convenience call $Q := \prod_{i=1}^{d}[a_i,b_i]$.

Definition 77 *Let $f : Q \to \mathbb{R}_{\mathcal{F}}$, we define the fuzzy modulus of continuity of f by*

$$\omega_1^{(\mathcal{F})}(f,\delta) = \sup_{x,y\in Q,|x_i-y_i|\leq\delta,\ for\ i=1,\dots,d} D(f(x),f(y)),\quad \delta>0, \tag{285}$$

where $x=(x_1,\dots,x_d)$, $y=(y_1,\dots,y_d)$.

For $f : Q \to \mathbb{R}_{\mathcal{F}}$, we use

$$[f]^r = \left[f_-^{(r)}, f_+^{(r)}\right], \tag{286}$$

where $f_{\pm}^{(r)} : Q \to \mathbb{R}$, $\forall\ r \in [0,1]$.
We need

Proposition 78 *Let $f : Q \rightarrow \mathbb{R}_{\mathcal{F}}$. Assume that $\omega_1^{\mathcal{F}}(f,\delta)$, $\omega_1\left(f_-^{(r)},\delta\right)$, $\omega_1\left(f_+^{(r)},\delta\right)$ are finite for any $\delta > 0$, $r \in [0,1]$.*

Then

$$\omega_1^{(\mathcal{F})}(f,\delta) = \sup_{r\in[0,1]} \max\left\{\omega_1\left(f_-^{(r)},\delta\right), \omega_1\left(f_+^{(r)},\delta\right)\right\}. \tag{287}$$

Proof. By [11], p. 128. ■

We define $C_{\mathcal{F}}(Q)$ the space of fuzzy continuous functions on Q.

We mention

Proposition 79 *Let $f \in C_{\mathcal{F}}(Q)$. Then $\omega_1^{(\mathcal{F})}(f,\delta) < \infty$, for any $\delta > 0$.*

Proof. By [11], p. 129. ■

Proposition 80 *It holds*

$$\lim_{\delta\rightarrow 0}\omega_1^{(\mathcal{F})}(f,\delta) = \omega_1^{(\mathcal{F})}(f,0) = 0, \tag{288}$$

iff $f \in C_{\mathcal{F}}(Q)$.

Proof. By [11], p. 129. ■

Proposition 81 *Let $f \in C_{\mathcal{F}}(Q)$. Then $f_{\pm}^{(r)}$ are equicontinuous with respect to $r \in [0,1]$ over Q, respectively in $\pm$. Also f is a fuzzy bounded function.*

Proof. By [11], pp. 131, 132. ■

We call $C_{\mathcal{F}}^N(Q)$, $N \in \mathbb{N}$, the space of all N-times fuzzy continuously differentiable functions from Q into $\mathbb{R}_{\mathcal{F}}$.

Let $f \in C_{\mathcal{F}}^N(Q)$, denote $f_{\widetilde{\alpha}} := \frac{\partial^{\widetilde{\alpha}} f}{\partial x^{\widetilde{\alpha}}}$, where $\widetilde{\alpha} := (\alpha_1, ..., \alpha_d)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., d$ and

$$0 < |\widetilde{\alpha}| := \sum_{i=1}^{d} \alpha_i \leq N, \ \ N > 1.$$

Then by Theorem 56 we get that

$$\left(f_{\pm}^{(r)}\right)_{\widetilde{\alpha}} = (f_{\widetilde{\alpha}})_{\pm}^{(r)}, \ \forall\, r \in [0,1], \tag{289}$$

and any $\widetilde{\alpha} : |\widetilde{\alpha}| \leq N$. Here $f_{\pm}^{(r)} \in C^N(Q)$.

Notation 82 *We denote*

$$\left(\sum_{i=1}^{2} D\left(\frac{\partial}{\partial x_i}, \widetilde{0}\right)\right)^2 f(x) :=$$

$$D\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2}, \widetilde{0}\right) + D\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2}, \widetilde{0}\right) + 2D\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}, \widetilde{0}\right). \tag{290}$$

In general we denote $(j = 1, ..., N)$

$$\left(\sum_{i=1}^{d} D \left(\frac{\partial}{\partial x_i}, \widetilde{0} \right) \right)^j f(x) :=$$

$$\sum_{(j_1,...,j_d) \in \mathbb{Z}_+^d : \sum_{i=1}^{d} j_i = j} \frac{j!}{j_1! j_2! ... j_d!} D \left(\frac{\partial^j f(x_1, ..., x_d)}{\partial x_1^{j_1} \partial x_2^{j_2} ... \partial x_d^{j_d}}, \widetilde{0} \right). \tag{291}$$

Let

$$f_{\widetilde{\alpha}}(x) = \widetilde{o}, \text{ for all } \widetilde{\alpha} : |\widetilde{\alpha}| = 1, ..., N,$$

for $x \in Q$ *fixed.*

The last implies $D(f_{\widetilde{\alpha}}(x), \widetilde{o}) = 0$, *and by (291) we obtain*

$$\left[\left(\sum_{i=1}^{d} D \left(\frac{\partial}{\partial x_i}, \widetilde{o} \right) \right)^j f(x) \right] = 0, \tag{292}$$

for $j = 1, ..., N$.

2.8 Multivariate Fuzzy Neural Network Approximation and Interpolation

Let $f \in C_{\mathcal{F}} \left(\prod_{i=1}^{d} [a_i, b_i] \right)$, $x \in \prod_{i=1}^{d} [a_i, b_i]$, we define

$$M_n^{\mathcal{F}}(f, x) := M_n^{\mathcal{F}}(f, x_1, ..., x_d) :=$$

$$\frac{\sum_{k_1=0}^{n*} ... \sum_{k_d=0}^{n*} f(x_{k_1 1}, ..., x_{k_d d}) \odot E\left(\frac{T_1 n (x_1 - x_{k_1 1})}{b_1 - a_1}, ..., \frac{T_d n (x_d - x_{k_d d})}{b_d - a_d} \right)}{\sum_{k_1=0}^{n} ... \sum_{k_d=0}^{n} E\left(\frac{T_1 n (x_1 - x_{k_1 1})}{b_1 - a_1}, ..., \frac{T_d n (x_d - x_{k_d d})}{b_d - a_d} \right)}, \tag{293}$$

the multivariate fuzzy neural network interpolation operator, $\forall\, n \in \mathbb{N}$.

Remark 83 *We observe that*

$$\left[M_n^{\mathcal{F}}(f, x) \right]^r =$$

$$\frac{\sum_{k_1=0}^{n} ... \sum_{k_d=0}^{n} \left[f(x_{k_1 1}, ..., x_{k_d d}) \right]^r E\left(\frac{T_1 n (x_1 - x_{k_1 1})}{b_1 - a_1}, ..., \frac{T_d n (x_d - x_{k_d d})}{b_d - a_d} \right)}{W} \tag{294}$$

$$= \sum_{k_1=0}^{n} ... \sum_{k_d=0}^{n} \left[f_-^{(r)}(x_{k_1 1}, ..., x_{k_d d}), f_+^{(r)}(x_{k_1 1}, ..., x_{k_d d}) \right].$$

$$\frac{E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1 - a_1}, ..., \frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d - a_d}\right)}{W}$$

$$= \left[\sum_{k_1=0}^{n} ... \sum_{k_d=0}^{n} f_{-}^{(r)}\left(x_{k_1 1}, ..., x_{k_d d}\right) \frac{E\left(>>\right)}{W},\right.$$

$$\left.\sum_{k_1=0}^{n} ... \sum_{k_d=0}^{n} f_{+}^{(r)}\left(x_{k_1 1}, ..., x_{k_d d}\right) \frac{E\left(>>\right)}{W}\right] \tag{295}$$

$$= \left[\left(M_n\left(f_{-}^{(r)}\right)\right)(x), \left(M_n\left(f_{+}^{(r)}\right)\right)(x)\right].$$

Hence it holds

$$\left(M_n^{\mathcal{F}}(f)\right)_{\pm}^{(r)} = M_n\left(f_{\pm}^{(r)}\right), \tag{296}$$

$\forall\ r \in [0,1]$, *respectively.*

Remark 84 *Let* $(k_1, ..., k_d) \in \{0, 1, ..., n\}^d$. *Then*

$$\left(M_n^{\mathcal{F}}\left(f, x_{k_1 1}, ..., x_{k_d d}\right)\right)_{\pm}^{(r)} = M_n\left(f_{\pm}^{(r)}\right)\left(x_{k_1 1}, ..., x_{k_d d}\right) \tag{297}$$

$$= f_{\pm}^{(r)}\left(x_{k_1 1}, ..., x_{k_d d}\right), \quad \forall\ r \in [0,1],$$

proving

$$M_n^{\mathcal{F}}\left(f, x_{k_1 1}, ..., x_{k_d d}\right) = f\left(x_{k_1 1}, ..., x_{k_d d}\right), \tag{298}$$

the interpolation property.

Remark 85 *Let* $f \in C_{\mathcal{F}}\left(\prod_{i=1}^{d} [a_i, b_i]\right)$. *Then*

$$D\left(\left(M_n^{\mathcal{F}}(f)\right)(x), f(x)\right) =$$

$$\sup_{r \in [0,1]} \max\left\{\left|\left(M_n^{\mathcal{F}}(f)\right)_{-}^{(r)}(x) - f_{-}^{(r)}(x)\right|, \left|\left(M_n^{\mathcal{F}}(f)\right)_{+}^{(r)}(x) - f_{+}^{(r)}(x)\right|\right\} =$$

$$\sup_{r \in [0,1]} \max\left\{\left|\left(M_n^{\mathcal{F}}\left(f_{-}^{(r)}\right)\right)(x) - f_{-}^{(r)}(x)\right|, \left|\left(M_n^{\mathcal{F}}\left(f_{+}^{(r)}\right)\right)(x) - f_{+}^{(r)}(x)\right|\right\} \overset{(148)}{\leq} \tag{299}$$

(we have $f_{\pm}^{r} \in C\left(\prod_{i=1}^{d} [a_i, b_i]\right)$*)*

$$\sup_{r \in [0,1]} \max\left\{\frac{2^d E^*}{E\left(\frac{T_1}{2}, ..., \frac{T_d}{2}\right)} \omega_1\left(f_{-}^{(r)}, \frac{\|b-a\|_\infty}{n}\right),\right.$$

$$\left.\frac{2^d E^*}{E\left(\frac{T_1}{2}, ..., \frac{T_d}{2}\right)} \omega_1\left(f_{+}^{(r)}, \frac{\|b-a\|_\infty}{n}\right)\right\} = \tag{300}$$

$$\frac{2^d E^*}{E\left(\frac{T_1}{2},...,\frac{T_d}{2}\right)} \sup_{r\in[0,1]} \max\left\{\omega_1\left(f_-^{(r)}, \frac{\|b-a\|_\infty}{n}\right), \omega_1\left(f_+^{(r)}, \frac{\|b-a\|_\infty}{n}\right)\right\} \quad (301)$$

$$\overset{(287)}{=} \frac{2^d E^*}{E\left(\frac{T_1}{2},...,\frac{T_d}{2}\right)} \omega_1^{(\mathcal{F})}\left(f, \frac{\|b-a\|_\infty}{n}\right).$$

We have proved

Theorem 86 *Let $f \in C_{\mathcal{F}}\left(\prod_{i=1}^{d}[a_i,b_i]\right)$. Then*

$$D\left(\left(M_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq$$

$$\frac{2^d E^*}{E\left(\frac{T_1}{2},...,\frac{T_d}{2}\right)} \omega_1^{(\mathcal{F})}\left(f, \frac{\|b-a\|_\infty}{n}\right) =: \lambda, \quad (302)$$

and

$$D^*\left(M_n^{\mathcal{F}}(f), f\right) \leq \lambda. \quad (303)$$

We make

Remark 87 *Let $f \in C_{\mathcal{F}}^N\left(\prod_{i=1}^{d}[a_i,b_i]\right)$, $N \in \mathbb{N}$, $x \in \prod_{i=1}^{d}[a_i,b_i]$ (so that $f_\pm^{(r)} \in C^N\left(\prod_{i=1}^{d}[a_i,b_i]\right)$).*

We get

$$\left|M_n\left(f_\pm^{(r)}, x\right) - f_\pm^{(r)}(x)\right| \overset{(156)}{\leq}$$

$$\frac{2^d E^*}{E\left(\frac{T_1}{2},...,\frac{T_d}{2}\right)} \left[\sum_{j=1}^{N} \frac{1}{j!}\left(\frac{\|b-a\|_\infty^j}{n^j}\right)\left(\left(\sum_{i=1}^{d}\left|\frac{\partial}{\partial x_i}\right|\right)^j f_\pm^{(r)}(x)\right)\right. \quad (304)$$

$$\left. + \frac{\|b-a\|_\infty^N d^N}{N! n^N} \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1\left(\left(f_\pm^{(r)}\right)_{\widetilde{\alpha}}, \frac{\|b-a\|_\infty}{n}\right)\right] =$$

$$\frac{2^d E^*}{E\left(\frac{T_1}{2},...,\frac{T_d}{2}\right)} \left[\sum_{j=1}^{N} \frac{1}{j!}\left(\frac{\|b-a\|_\infty^j}{n^j}\right)\left(\left(\sum_{i=1}^{d}\left|\frac{\partial}{\partial x_i}\right|\right)^j f(x)\right)_\pm^{(r)}\right. \quad (305)$$

$$\left. + \frac{\|b-a\|_\infty^N d^N}{N! n^N} \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1\left(\left(f_{\widetilde{\alpha}}\right)_\pm^{(r)}, \frac{\|b-a\|_\infty}{n}\right)\right] \leq$$

$$\frac{2^d E^*}{E\left(\frac{T_1}{2},...,\frac{T_d}{2}\right)} \left[\sum_{j=1}^{N} \frac{1}{j!}\left(\frac{\|b-a\|_\infty^j}{n^j}\right)\left[\left(\sum_{i=1}^{d} D\left(\frac{\partial}{\partial x_i}, \widetilde{o}\right)\right)^j f(x)\right]\right. \quad (306)$$

$$\left. + \frac{\|b-a\|_\infty^N d^N}{N! n^N} \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \frac{\|b-a\|_\infty}{n}\right)\right].$$

We have proved

Theorem 88 *Let* $f \in C_{\mathcal{F}}^{N}\left(\prod_{i=1}^{d}[a_i, b_i]\right)$, $N \in \mathbb{N}$, $x \in \prod_{i=1}^{d}[a_i, b_i]$. *Then*

$$D\left(M_n^{\mathcal{F}}(f)(x), f(x)\right) \leq$$

$$\frac{2^d E^*}{E\left(\frac{T_1}{2}, ..., \frac{T_d}{2}\right)}\left[\sum_{j=1}^{N}\frac{1}{j!}\left(\frac{\|b-a\|_\infty^j}{n^j}\right)\left[\left(\sum_{i=1}^{d} D\left(\frac{\partial}{\partial x_i}, \widetilde{o}\right)\right)^j f(x)\right]\right. \tag{307}$$

$$\left. + \frac{\|b-a\|_\infty^N d^N}{N! n^N}\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \frac{\|b-a\|_\infty}{n}\right)\right].$$

Corollary 89 *(to Theorem 88) Additionally assume that* $f_{\widetilde{\alpha}}(x) = \widetilde{o}$, *for all* $\widetilde{\alpha} : |\widetilde{\alpha}| = 1, ..., N$, *where* $x \in \prod_{i=1}^{d}[a_i, b_i]$ *is fixed.*

[Then $D(f_{\widetilde{\alpha}}(x), \widetilde{o}) = 0$, *and* $\left[\left(\sum_{i=1}^{d} D\left(\frac{\partial}{\partial x_i}, \widetilde{o}\right)\right)^j f(x)\right] = 0$, $j = 1, ..., N$*].*

Hence

$$D\left(M_n^{\mathcal{F}}(f)(x), f(x)\right) \leq$$

$$\frac{2^d E^*}{E\left(\frac{T_1}{2}, ..., \frac{T_d}{2}\right)}\frac{\|b-a\|_\infty^N d^N}{N! n^N}\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \frac{\|b-a\|_\infty}{n}\right). \tag{308}$$

Corollary 90 *(to Theorem 88) We get*

$$D^*\left(M_n^{\mathcal{F}}(f), f\right) \leq$$

$$\frac{2^d E^*}{E\left(\frac{T_1}{2}, ..., \frac{T_d}{2}\right)}\left[\sum_{j=1}^{N}\frac{1}{j!}\left(\frac{\|b-a\|_\infty^j}{n^j}\right)\left\|\left(\sum_{i=1}^{d} D\left(\frac{\partial}{\partial x_i}, \widetilde{o}\right)\right)^j f(x)\right\|_\infty\right. \tag{309}$$

$$\left. + \frac{\|b-a\|_\infty^N d^N}{N! n^N}\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \frac{\|b-a\|_\infty}{n}\right)\right].$$

Corollary 91 *(to Theorem 88) Case of* $N = 1$. *We derive*

$$D\left(\left(M_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq \frac{2^d E^* \|b-a\|_\infty}{n E\left(\frac{T_1}{2}, ..., \frac{T_d}{2}\right)}\cdot$$

$$\left[\sum_{i=1}^{d} D\left(\frac{\partial f}{\partial x_i}, \widetilde{o}\right) + d\max_{i\in\{1,...,d\}}\omega_1^{(\mathcal{F})}\left(\frac{\partial f}{\partial x_i}, \frac{\|b-a\|_\infty}{n}\right)\right]. \tag{310}$$

2.9 Fuzzy-Random Analysis background

Define

$$D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_+ \cup \{0\}$$

by

$$D(u,v) := \sup_{r \in [0,1]} \max \left\{ \left| u_-^{(r)} - v_-^{(r)} \right|, \left| u_+^{(r)} - v_+^{(r)} \right| \right\}, \tag{311}$$

where $[v]^r = \left[v_-^{(r)}, v_+^{(r)} \right]$; $u, v \in \mathbb{R}_{\mathcal{F}}$. We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$. Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [40], with the properties

$$\begin{aligned} D(u \oplus w, v \oplus w) &= D(u,v), \quad \forall\, u,v,w \in \mathbb{R}_{\mathcal{F}}, \\ D(k \odot u, k \odot v) &= |k|\, D(u,v), \quad \forall\, u,v \in \mathbb{R}_{\mathcal{F}}, \forall\, k \in \mathbb{R}, \\ D(u \oplus v, w \oplus e) &\leq D(u,w) + D(v,e), \quad \forall\, u,v,w,e \in \mathbb{R}_{\mathcal{F}}. \end{aligned}$$

Let $U^* := \prod_{i=1}^{d} [a_i, b_i]$, $d \in \mathbb{N}$, $f, g : U^* \rightarrow \mathbb{R}_{\mathcal{F}}$ be fuzzy real number valued functions. The distance between f, g is defined by

$$D^*(f,g) := \sup_{x \in U^*} D(f(x), g(x)).$$

On $\mathbb{R}_{\mathcal{F}}$ we define a partial order by "$\leq$": $u, v \in \mathbb{R}_{\mathcal{F}}$, $u \leq v$ iff $u_-^{(r)} \leq v_-^{(r)}$ and $u_+^{(r)} \leq v_+^{(r)}$, $\forall\, r \in [0,1]$.

We need

Lemma 92 *([24]) For any $a, b \in \mathbb{R} : a \cdot b \geq 0$ and any $u \in \mathbb{R}_{\mathcal{F}}$ we have*

$$D(a \odot u, b \odot u) \leq |a-b| \cdot D(u, \widetilde{o}), \tag{312}$$

where $\widetilde{o} \in \mathbb{R}_{\mathcal{F}}$ is defined by $\widetilde{o} := \chi_{\{0\}}$.

Lemma 93 *([24])*

(i) If we denote $\widetilde{o} := \chi_{\{0\}}$, then $\widetilde{o} \in \mathbb{R}_{\mathcal{F}}$ is the neutral element with respect to $\oplus$, i.e., $u \oplus \widetilde{o} = \widetilde{o} \oplus u = u$, $\forall\, u \in \mathbb{R}_{\mathcal{F}}$.

(ii) With respect to $\widetilde{o}$, none of $u \in \mathbb{R}_{\mathcal{F}}$, $u \neq \widetilde{o}$ has opposite in $\mathbb{R}_{\mathcal{F}}$.

(iii) Let $a, b \in \mathbb{R} : a \cdot b \geq 0$, and any $u \in \mathbb{R}_{\mathcal{F}}$, we have $(a+b) \odot u = a \odot u \oplus b \odot u$. For general $a, b \in \mathbb{R}$, the above property is false.

(iv) For any $\lambda \in \mathbb{R}$ and any $u, v \in \mathbb{R}_{\mathcal{F}}$, we have $\lambda \odot (u \oplus v) = \lambda \odot u \oplus \lambda \odot v$.

(v) For any $\lambda, \mu \in \mathbb{R}$ and $u \in \mathbb{R}_{\mathcal{F}}$, we have $\lambda \odot (\mu \odot u) = (\lambda \cdot \mu) \odot u$.

(vi) If we denote $\|u\|_{\mathcal{F}} := D(u, \widetilde{o})$, $\forall\, u \in \mathbb{R}_{\mathcal{F}}$, then $\|\cdot\|_{\mathcal{F}}$ has the properties of a usual norm on $\mathbb{R}_{\mathcal{F}}$, i.e.,

$$\begin{aligned} &\|u\|_{\mathcal{F}} = 0 \text{ iff } u = \widetilde{o}, \quad \|\lambda \odot u\|_{\mathcal{F}} = |\lambda| \cdot \|u\|_{\mathcal{F}}, \\ &\|u \oplus v\|_{\mathcal{F}} \leq \|u\|_{\mathcal{F}} + \|v\|_{\mathcal{F}}, \quad \|u\|_{\mathcal{F}} - \|v\|_{\mathcal{F}} \leq D(u,v). \end{aligned} \tag{313}$$

Notice that $(\mathbb{R}_{\mathcal{F}}, \oplus, \odot)$ is not a linear space over $\mathbb{R}$; and consequently $(\mathbb{R}_{\mathcal{F}}, \|\cdot\|_{\mathcal{F}})$ is not a normed space.

As in Remark 4.4 ([24]) one can show easily that a sequence of operators of the form

$$L_n(f)(x) := \sum_{k=0}^{n*} f(x_{k_n}) \odot w_{n,k}(x), \quad n \in \mathbb{N}, \tag{314}$$

($\sum^{*}$ denotes the fuzzy summation) where $f : U^* \to \mathbb{R}_{\mathcal{F}}$, $x_{k_n} \in U^*$, $w_{n,k}(x)$ real valued weights, are linear over U^*, i.e.,

$$L_n(\lambda \odot f \oplus \mu \odot g)(x) = \lambda \odot L_n(f)(x) \oplus \mu \odot L_n(g)(x), \tag{315}$$

$\forall \lambda, \mu \in \mathbb{R}$, any $x \in U^*$; $f, g : U^* \to \mathbb{R}_{\mathcal{F}}$. (Proof based on Lemma 93 (iv).)

We further need

Definition 94 *(see also [32], Definition 13.16, p. 654) Let $(X, \mathcal{B}, P)$ be a probability space. A fuzzy-random variable is a $\mathcal{B}$-measurable mapping $g : X \to \mathbb{R}_{\mathcal{F}}$ (i.e., for any open set $Z \subseteq \mathbb{R}_{\mathcal{F}}$, in the topology of $\mathbb{R}_{\mathcal{F}}$ generated by the metric D, we have*

$$g^{-1}(Z) = \{s \in X; g(s) \in Z\} \in \mathcal{B}). \tag{316}$$

The set of all fuzzy-random variables is denoted by $\mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$. Let $g_n, g \in \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $n \in \mathbb{N}$ and $0 < q < +\infty$. We say $g_n(s) \overset{\text{"q-mean"}}{\underset{n\to+\infty}{\to}} g(s)$ if

$$\lim_{n\to+\infty} \int_X D(g_n(s), g(s))^q P(ds) = 0. \tag{317}$$

Remark 95 *(see [32], p. 654) If $f, g \in \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, let us denote $F : X \to \mathbb{R}_+ \cup \{0\}$ by $F(s) = D(f(s), g(s))$, $s \in X$. Here, F is $\mathcal{B}$-measurable, because $F = G \circ H$, where $G(u, v) = D(u, v)$ is continuous on $\mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$, and $H : X \to \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$, $H(s) = (f(s), g(s))$, $s \in X$, is $\mathcal{B}$-measurable. This shows that the above convergence in q-mean makes sense.*

Definition 96 *(see [32], p. 654, Definition 13.17) Let $(T, \mathcal{T})$ be a topological space. A mapping $f : T \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$ will be called fuzzy-random function (or fuzzy-stochastic process) on T. We denote $f(t)(s) = f(t, s)$, $t \in T$, $s \in X$.*

Remark 97 *(see [32], p. 655) Any usual fuzzy real function $f : T \to \mathbb{R}_{\mathcal{F}}$ can be identified with the degenerate fuzzy-random function $f(t, s) = f(t)$, $\forall\, t \in T$, $s \in X$.*

Remark 98 *(see [32], p. 655) Fuzzy-random functions that coincide with probability one for each $t \in T$ will be considered equivalent.*

Remark 99 *(see [32], p. 655) Let* $f, g : T \rightarrow \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$*. Then* $f \oplus g$ *and* $k \odot f$ *are defined pointwise, i.e.,*

$$
\begin{aligned}
(f \oplus g)(t, s) &= f(t, s) \oplus g(t, s), \\
(k \odot f)(t, s) &= k \odot f(t, s), \quad t \in T, s \in X.
\end{aligned}
$$

Definition 100 *(see also Definition 13.18, pp. 655-656, [32]) For a fuzzy-random function* $f : U^* \rightarrow \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $d \in \mathbb{N}$*, we define the (first) fuzzy-random modulus of continuity*

$$
\Omega_1^{(\mathcal{F})}(f, \delta)_{L^q} =
$$

$$
\sup \left\{ \left(\int_X D^q \left(f(x, s), f(y, s) \right) P(ds) \right)^{\frac{1}{q}} : x, y \in U^*, \ \ \|x - y\|_{l_1} \leq \delta \right\}, \quad (318)
$$

$0 < \delta$, $1 \leq q < \infty$.

Definition 101 *(as in [22]) Here* $1 \leq q < +\infty$*. Let* $f : U^* \rightarrow \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $d \in \mathbb{N}$*, be a fuzzy random function. We call* f *a (q-mean) uniformly continuous fuzzy random function over* U^**, iff* $\forall \ \varepsilon > 0 \ \exists \ \delta > 0$ *:whenever* $\|x - y\|_{l_1} \leq \delta$, $x, y \in U^*$*, implies that*

$$
\int_X \left(D\left(f(x, s), f(y, s) \right) \right)^q P(ds) \leq \varepsilon. \quad (319)
$$

We denote it as $f \in C_{FR}^{U_q}(U^*)$.

Proposition 102 *(as in [22]) Let* $f \in C_{FR}^{U_q}(U^*)$*. Then* $\Omega_1^{(\mathcal{F})}(f, \delta)_{L^q} < \infty$*, any* $\delta > 0$.

Proposition 103 *(as in [22]) Let* $f, g : U^* \rightarrow \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $d \in \mathbb{N}$*, be fuzzy random functions. It holds*

(i) $\Omega_1^{(\mathcal{F})}(f, \delta)_{L^q}$ *is nonnegative and nondecreasing in* $\delta > 0$.

(ii) $\lim_{\delta \downarrow 0} \Omega_1^{(\mathcal{F})}(f, \delta)_{L^q} = \Omega_1^{(\mathcal{F})}(f, 0)_{L^q} = 0$*, iff* $f \in C_{FR}^{U_q}(U^*)$.

(iii) $\Omega_1^{(\mathcal{F})}(f, \delta_1 + \delta_2)_{L^q} \leq \Omega_1^{(\mathcal{F})}(f, \delta_1)_{L^q} + \Omega_1^{(\mathcal{F})}(f, \delta_2)_{L^q}$, $\delta_1, \delta_2 > 0$.

(iv) $\Omega_1^{(\mathcal{F})}(f, n\delta)_{L^q} \leq n\Omega_1^{(\mathcal{F})}(f, \delta)_{L^q}$, $\delta > 0$, $n \in \mathbb{N}$.

(v) $\Omega_1^{(\mathcal{F})}(f, \lambda\delta)_{L^q} \leq \lceil \lambda \rceil \Omega_1^{(\mathcal{F})}(f, \delta)_{L^q} \leq (\lambda + 1) \Omega_1^{(\mathcal{F})}(f, \delta)_{L^q}$, $\lambda > 0$, $\delta > 0$, *where* $\lceil \cdot \rceil$ *is the ceilling of the number.*

(vi) $\Omega_1^{(\mathcal{F})}(f \oplus g, \delta)_{L^q} \leq \Omega_1^{(\mathcal{F})}(f, \delta)_{L^q} + \Omega_1^{(\mathcal{F})}(g, \delta)_{L^q}$, $\delta > 0$*. Here* $f \oplus g$ *is a fuzzy random function.*

(vii) $\Omega_1^{(\mathcal{F})}(f, \cdot)_{L^q}$ *is continuous on* $\mathbb{R}_+$*, for* $f \in C_{FR}^{U_q}(U^*)$.

According to [29], p. 94 we have the following

Definition 104 *Let* $(Y, \mathcal{T})$ *be a topological space, with its* σ*-algebra of Borel sets* $\mathcal{B} := \mathcal{B}(Y, \mathcal{T})$ *generated by* $\mathcal{T}$. *If* $(X, \mathcal{S})$ *is a measurable space, a function* $f : X \to Y$ *is called measurable iff* $f^{-1}(B) \in \mathcal{S}$ *for all* $B \in \mathcal{B}$.

By Theorem 4.1.6 of [29], p. 89 f as above is measurable iff

$$f^{-1}(C) \in \mathcal{S} \text{ for all } C \in \mathcal{T}.$$

We would need

Theorem 105 *(see [29], p. 95) Let* $(X, \mathcal{S})$ *be a measurable space and* (Y, d) *be a metric space. Let* f_n *be measurable functions from* X *into* Y *such that for all* $x \in X$, $f_n(x) \to f(x)$ *in* Y. *Then* f *is measurable. I.e.,* $\lim_{n\to\infty} f_n = f$ *is measurable.*

We need also

Proposition 106 *Let* f, g *be fuzzy random variables from* $\mathcal{S}$ *into* $\mathbb{R}_{\mathcal{F}}$. *Then*
(i) Let $c \in \mathbb{R}$, *then* $c \odot f$ *is a fuzzy random variable.*
(ii) $f \oplus g$ *is a fuzzy random variable.*

2.10 Multivariate Fuzzy Random Neural Network Approximation and Interpolation

We need

Definition 107 *Let here* $(X, \mathcal{B}, P)$ *be a probability space,* $s \in X$, $n \in \mathbb{N}$, $f \in C_{\mathcal{F}R}^{U_q}\left(\prod_{i=1}^{d}[a_i, b_i]\right)$, $1 \leq q < \infty$, *and* $x \in \prod_{i=1}^{d}[a_i, b_i]$.
We define

$$M_n^{\mathcal{F}R}(f, x, s) := M_n^{\mathcal{F}R}(f, x_1, ..., x_d, s) :=$$

$$\frac{\sum_{k_1=0}^{n*} ... \sum_{k_d=0}^{n*} f(x_{k_1 1}, ..., x_{k_d d}, s) \odot E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1 - a_1}, ..., \frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d - a_d}\right)}{\sum_{k_1=0}^{n} ... \sum_{k_d=0}^{n} E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1 - a_1}, ..., \frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d - a_d}\right)}. \quad (320)$$

We make

Remark 108 *Clearly here it holds*

$$M_n^{\mathcal{F}R}(f, x_{k_1 1}, ..., x_{k_d d}, s) = \frac{f(x_{k_1 1}, ..., x_{k_d d}, s) \odot E^*}{E^*}$$

$$= f(x_{k_1 1}, ..., x_{k_d d}, s) \odot 1 = f(x_{k_1 1}, ..., x_{k_d d}, s), \quad (321)$$

proving the interpolation property of operators $M_n^{\mathcal{F}R}$.

We make

Remark 109 *Let* $f \in C_{\mathcal{F}R}^{U_q}\left(\prod_{i=1}^{d}[a_i,b_i]\right)$, $1 \leq q < \infty$, $x \in \prod_{i=1}^{d}[a_i,b_i]$, $n \in \mathbb{N}$. *We observe that*

$$D\left(M_n^{\mathcal{F}R}(f,x,s), f(x,s)\right) =$$

$$D\left(\sum_{k_1=0}^{n*}...\sum_{k_d=0}^{n*} f\left(x_{k_1 1},...,x_{k_d d},s\right) \odot \frac{E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1-a_1},...,\frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d-a_d}\right)}{W},\right. \tag{322}$$

$$\left. f(x,s) \odot \frac{W}{W}\right) =$$

$$D\left(\sum_{k_1=0}^{n*}...\sum_{k_d=0}^{n*} f\left(x_{k_1 1},...,x_{k_d d},s\right) \odot \frac{E(>>)}{W}, \sum_{k_1=0}^{n*}...\sum_{k_d=0}^{n*} f(x,s) \odot \frac{E(>>)}{W}\right) \leq \tag{323}$$

$$\frac{\sum_{k_1=0}^{n}...\sum_{k_d=0}^{n} E(>>)}{W} D\left(f\left(x_{k_1 1},...,x_{k_d d},s\right), f(x,s)\right).$$

So it holds

$$D\left(M_n^{\mathcal{F}R}(f,x,s), f(x,s)\right) \leq$$

$$\frac{\sum_{k_1=0}^{n}...\sum_{k_d=0}^{n} E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1-a_1},...,\frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d-a_d}\right)}{W} D\left(f\left(x_{k_1 1},...,x_{k_d d},s\right), f(x,s)\right). \tag{324}$$

Therefore we derive

$$\left(\int_X D^q\left(\left(M_n^{\mathcal{F}R}(f,x,s), f(x,s)\right)\right) P(ds)\right)^{\frac{1}{q}} \leq$$

$$\frac{\sum_{k_1=0}^{n}...\sum_{k_d=0}^{n} E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1-a_1},...,\frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d-a_d}\right)}{W} \cdot$$

$$\left(\int_X D^q\left(f\left(x_{k_1 1},...,x_{k_d d},s\right), f(x,s)\right) P(ds)\right)^{\frac{1}{q}} \leq \tag{325}$$

$$\frac{2^d E^*}{E\left(\frac{T_1}{2},...,\frac{T_d}{2}\right)} \Omega_1^{(\mathcal{F})}\left(f, \frac{\sum_{i=1}^{d}(b_i - a_i)}{n}\right)_{L^q}.$$

We have proved the following approximation result.

Theorem 110 *Let* $(X, \mathcal{B}, P)$ *probability space,* $f \in C^{U_q}_{\mathcal{F}R}\left(\prod_{i=1}^{d}[a_i, b_i]\right)$, $1 \leq q < \infty$. *Then*

$$\left\| \left(\int_X D^q \left(\left(M_n^{\mathcal{F}R}(f, x, s), f(x, s) \right) \right) P(ds) \right)^{\frac{1}{q}} \right\|_{\infty, x} \leq \tag{326}$$

$$\frac{2^d E^*}{E\left(\frac{T_1}{2}, ..., \frac{T_d}{2}\right)} \Omega_1^{(\mathcal{F})} \left(f, \frac{\sum_{i=1}^{d}(b_i - a_i)}{n} \right)_{L^q},$$

where $x \in \prod_{i=1}^{d}[a_i, b_i]$, $\forall\ n \in \mathbb{N}$.

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