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ON HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR n-TIMES DIFFERENTIABLE m- AND (α, m)-LOGARITHMICALLY CONVEX FUNCTIONS

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ABSTRACT. In this paper, we establish Hermite-Hadamard type inequalities for functions whose nth derivatives are m- and (α, m)-logarithmically convex functions. From our results, several results for classical trapezoidal and classical midpoint inequalities are obtained in terms second derivatives that are m- and (α, m)-logarithmically convex functions as special cases.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on I and a, $b \in \mathbb{R}$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The double inequality (1.1) was firstly discovered by Ch. Hermite [12] in 1881 in the journal Mathesis but was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result. E. F. Beckenbach [2], a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard [11] in 1893. Later on, in 1974, D. S. Mitrinović [19] found Hermite's note in Mathesis. This is why, the inequality (1.1) is now commonly referred as the Hermite-Hadamard inequality.

The inequality (1.1) has been subject of extensive research and has been refined and generalized by a number of mathematicians for over one hundred years see for instance [1], [3]-[10], [13]-[17], [20], [21]-[24], [26]-[29] and the references therein.

In recent years lot of generalizations of classical convexity have been given by a number of mathematicians, some of these are given as follows.

Definition 1. [25] *A function $f : [0, b] \to \mathbb{R}$ is said to be m-convex if*

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

holds for all x, $y \in [0, b]$, $t \in [0, 1]$ and $m \in (0, 1]$.

Definition 2. [18] *A function $f : [0, b] \to \mathbb{R}$ is said to be m-convex if*

$$f(tx + m(1-t)y) \leq t^{\alpha} f(x) + m(1-t^{\alpha})f(y)$$

holds for all x, $y \in [0, b]$, $t \in [0, 1]$ and $(\alpha, m) \in (0, 1] \times (0, 1]$.

Most recently, the above definitions are further generalized in [10] as follows.

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Definition 3. [10] *A function* $f:[0,b]\to(0,\infty)$ *is said to be* m*-logarthmically convex if*

$$f\left(tx+m\left(1-t\right)y\right)\leq\left[f\left(x\right)\right]^{t}\left[f\left(y\right)\right]^{m(1-t)}$$

holds for all $x,\ y\in[0,b]$, $t\in[0,1]$ *and* $m\in(0,1]$.

Definition 4. [10] *A function* $f:[0,b]\to(0,\infty)$ *is said to be* (α,m)*-logarithmically convex if*

$$f\left(tx+m\left(1-t\right)y\right)\leq\left[f\left(x\right)\right]^{t^{\alpha}}\left[f\left(y\right)\right]^{m(1-t^{\alpha})}$$

holds for all $x,\ y\in[0,b]$, $t\in[0,1]$ *and* $(\alpha,m)\in(0,1]\times(0,1]$.

Bai et al. obtained the following Hermite-Hadamard type inequalites for m- and (α,m)-logarithmically convex functions.

Theorem 1. [10] *Let* $I\subseteq[0,\infty)$ *be an open real interval and let* $f:I\to(0,\infty)$ *be a differentiable function on* I *such that* $f^{'}\in L([a,b])$ *for* $0\leq a<b<\infty$. *If* $\left|f^{'}(x)\right|^{q}$ *is* (α,m)*-logarithmically convex on* $\left[0,\frac{b}{m}\right]$ *for* $q\in[1,\infty)$, $(\alpha,m)\in(0,1]\times(0,1]$, *we have the inequality*

$$\left|\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx\right|\leq\frac{\left(b-a\right)}{2}\left|f^{'}\left(\frac{b}{m}\right)\right|^{m}\left(\frac{1}{2}\right)^{1-1/q}\left[E_{1}\left(\alpha,m,q\right)\right]^{1/q},\quad(1.2)$$

where

$$\mu=\frac{\left|f^{'}(a)\right|}{\left|f'\left(\frac{b}{m}\right)\right|^{m}},\ E_{1}\left(\alpha,m,q\right)=\begin{cases}\frac{1}{2}, & \mu=1\\ F_{1}\left(\mu,\alpha q\right), & 0<\mu<1\\ F_{1}\left(\mu,\frac{q}{\alpha}\right), & \mu>1\end{cases}$$

and

$$F_{1}\left(u,v\right)=\frac{1}{v^{2}\left(\ln u\right)^{2}}\left[v\left(u^{v}-1\right)\ln u-2\left(u^{v/2}-1\right)^{2}\right]$$

for $u,\ v>0,\ u\neq1$.

Corollary 1. [10] *Let* $I\subseteq[0,\infty)$ *be an open real interval and let* $f:I\to(0,\infty)$ *be a differentiable function on* I *such that* $f^{'}\in L([a,b])$ *for* $0\leq a<b<\infty$. *If* $\left|f^{'}(x)\right|^{q}$ *is* m*-logarthmically convex on* $\left[0,\frac{b}{m}\right]$ *for* $q\in[1,\infty)$, $m\in(0,1]$, *we have the inequality*

$$\left|\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx\right|\leq\frac{\left(b-a\right)}{2}\left|f^{'}\left(\frac{b}{m}\right)\right|^{m}\left(\frac{1}{2}\right)^{1-1/q}\left[E_{1}\left(1,m,q\right)\right]^{1/q},\quad(1.3)$$

where $E_{1}(\alpha,m,q)$ *is defined as in Theorem 1.*

Theorem 2. [10] *Let $I \subseteq [0, \infty)$ be an open real interval and let $f : I \to (0, \infty)$ be a differentiable function on I such that $f' \in L([a, b])$ for $0 \le a < b < \infty$. If $\left|f'(x)\right|^q$ is (α, m)-logarithmically convex on $\left[0, \frac{b}{m}\right]$ for $q \in [1, \infty)$, $(\alpha, m) \in (0, 1] \times (0, 1]$, we have the inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le \frac{(b-a)}{4}\left|f'\left(\frac{b}{m}\right)\right|^m \left(\frac{1}{2}\right)^{3-1/q}\left[E_2(\alpha, m, q)\right], \quad (1.4)$$

where

$$\mu = \frac{\left|f'(a)\right|}{\left|f'\left(\frac{b}{m}\right)\right|^m},\; E_1(\alpha, m, q) = \begin{cases} 2\left(\frac{1}{8}\right)^{1/q}, & \mu = 1 \\ \left[F_2(\mu, \alpha q)\right]^{1/q} + \left[F_3(\mu, \alpha q)\right]^{1/q}, & 0 < \mu < 1 \\ \left[F_2\left(\mu, \frac{q}{\alpha}\right)\right]^{1/q} + \left[F_3\left(\mu, \frac{q}{\alpha}\right)\right]^{1/q}, & \mu > 1 \end{cases}$$

and

$$F_2(u, v) = \frac{1}{v^2(\ln u)^2}\left[\frac{v}{2}u^{v/2}\ln u - u^{v/2} + 1\right]$$
$$F_3(u, v) = \frac{1}{v^2(\ln u)^2}\left[u^v - \frac{v}{2}u^{v/2}\ln u - u^{v/2}\right]$$

for $u,\ v > 0$, $u \neq 1$.

Corollary 2. *Let $I \subseteq [0, \infty)$ be an open real interval and let $f : I \to (0, \infty)$ be a differentiable function on I such that $f' \in L([a, b])$ for $0 \le a < b < \infty$. If $\left|f'(x)\right|^q$ is m-logarthmically convex on $\left[0, \frac{b}{m}\right]$ for $q \in [1, \infty)$, $m \in (0, 1]$, we have the inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \le \frac{(b-a)}{4}\left|f'\left(\frac{b}{m}\right)\right|^m \left(\frac{1}{2}\right)^{3-1/q}\left[E_2(1, m, q)\right], \quad (1.5)$$

where $E_2(\alpha, m, q)$ is defined as in Theorem 2.

The main purpose of the present paper to establish new Hermite-Hadamard type inequalities for functions whose nth derivatives in absolute value are m- and (α, m)-logarithmically convex. These results not only generalize the results from [10] but many other interesting results can be obtained for functions whose second derivatives in absolute value are m- and (α, m)-logarithmically convex which may be better than those from [10].

2. Main Results

First we quote and establish some useful lemmas to prove our mains results.

Lemma 1. [13] *Suppose $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on I° for $n \in \mathbb{N}$, $n \geq 1$. If $f^{(n)}$ is integrable on $[a, b]$, for $a, b \in I$ with $a > b$, the equality holds*

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a) = \frac{(b-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t) f^{(n)}(ta + (1-t)b)dt, \quad (2.1)$$

where the sum above takes 0 when $n = 1$ and $n = 2$.

Lemma 2. *Suppose $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on I° for $n \in \mathbb{N}$, $n \geq 1$. If $f^{(n)}$ is integrable on $[a, b]$, for $a, b \in I$ with $a > b$, the equality holds*

$$\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{(-1)(b-a)^n}{n!}\int_0^1 K_n(t) f^{(n)}(ta + (1-t)b)dt, \quad (2.2)$$

where

$$K_n(t) := \begin{cases} t^n, & t \in \left[0, \frac{1}{2}\right] \\ (t-1)^n, & t \in \left(\frac{1}{2}, 1\right] \end{cases}.$$

Proof. For $n = 1$, we have

$$\begin{aligned}
&(-1)(b-a)\int_0^1 K_1(t) f^{(n)}(ta + (1-t)b)dt \\
&= -(b-a)\int_0^{\frac{1}{2}} t f'(ta + (1-t)b)dt - (b-a)\int_{\frac{1}{2}}^1 (t-1) f'(ta + (1-t)b)dt \\
&= f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx,
\end{aligned}$$

which is the left hand side of (2.2) for $n = 1$.
Suppose that (2.2) holds for $n = m - 1$, $m > 2$, that is

$$\sum_{k=0}^{m-2} \frac{\left[(-1)^k + 1\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{(-1)(b-a)^{m-1}}{(m-1)!}\int_0^1 K_{m-1}(t) f^{(m-1)}(ta + (1-t)b)dt. \quad (2.3)$$

Now for $n = m$, by integration by parts and using (2.3), we have

$$\begin{aligned}
&\frac{(-1)(b-a)^m}{m!}\int_0^1 K_m(t) f^{(m)}(ta+(1-t)b)dt \\
&\qquad = \frac{\left[(-1)^{m-1}+1\right](b-a)^{m-1}}{2^m m!} f^{(m-1)}\left(\frac{a+b}{2}\right) \\
&\qquad + \frac{(-1)(b-a)^{m-1}}{(m-1)!}\int_0^1 K_{m-1}(t) f^{(m-1)}(ta+(1-t)b)dt \\
&\qquad = \frac{(b-a)^{m-1}\left[(-1)^m+1\right]}{2^m m!} f^{(m-1)}\left(\frac{a+b}{2}\right) \\
&\qquad + \sum_{k=0}^{m-2}\frac{\left[(-1)^{k+1}-1\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx, \qquad (2.4)
\end{aligned}$$

which is the required identity (2.2). This completes the proof of the Lemma. □

The following useful result will also help us establishing our results:

Lemma 3. *If $\mu > 0$ and $\mu \neq 1$, then*

$$\int_0^1 t^n \mu^t dt = \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n!\mu \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!(\ln \mu)^{k+1}}. \qquad (2.5)$$

Proof. For $n = 0$, we have

$$\int_0^1 \mu^t dt = \frac{\mu - 1}{\ln \mu},$$

which coincides with the right hand side of (2.5) for $n = 0$.

For $n = 1$, we have

$$\int_0^1 t\mu^t dt = \frac{\mu}{\ln \mu} - \frac{\mu}{(\ln \mu)^2} + \frac{1}{(\ln \mu)^2},$$

and it coincides with the right hand side of (2.5) for $n = 1$.

Suppose (2.5) is true for $n - 1$, i.e.

$$\int_0^1 t^{n-1}\mu^t dt = \frac{(-1)^n (n-1)!}{(\ln \mu)^n} + (n-1)!\mu \sum_{k=0}^{n-1} \frac{(-1)^k}{(n-1-k)!(\ln \mu)^{k+1}}. \qquad (2.6)$$

Now by integration by parts and using (2.6), we have

$$\begin{aligned}
\int_0^1 t^n \mu^t dt &= \frac{\mu}{\ln \mu} - \frac{n}{\ln \mu} \int_0^1 t^{n-1} \mu^t dt \\
&= \frac{\mu}{\ln \mu} - \frac{n}{\ln \mu} \left[\frac{(-1)^n (n-1)!}{(\ln \mu)^n} + (n-1)! \mu \sum_{k=0}^{n-1} \frac{(-1)^k}{(n-1-k)! (\ln \mu)^{k+1}} \right] \\
&= \frac{\mu}{\ln \mu} + \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n! \mu \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{(n-1-k)! (\ln \mu)^{k++2}} \\
&= \frac{n! \mu}{n! \ln \mu} + \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n! \mu \sum_{k=1}^{n} \frac{(-1)^k}{(n-k)! (\ln \mu)^{k+1}} \\
&= \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n! \mu \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)! (\ln \mu)^{k+1}}.
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 4. *If $\mu > 0$ and $\mu \neq 1$, then*

$$\int_0^{\frac{1}{2}} t^n \mu^t dt = \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n! \mu^{1/2} \sum_{k=0}^{n} \frac{(-1)^k}{2^{n-k} (n-k)! (\ln \mu)^{k+1}}. \tag{2.7}$$

Proof. It follows from Lemma 3 after making use of the substitution $t = \frac{u}{2}$. □

Lemma 5. *If $\mu > 0$ and $\mu \neq 1$, then*

$$\int_{\frac{1}{2}}^1 (1-t)^n \mu^t dt = \frac{n! \mu}{(\ln \mu)^{n+1}} - n! \mu^{1/2} \sum_{k=0}^{n} \frac{1}{2^{n-k} (n-k)! (\ln \mu)^{k+1}}. \tag{2.8}$$

Proof. It follows from Lemma 4 after making the substitution $1 - t = u$. □

Lemma 6. [26] *For $\alpha > 0$ and $\mu > 0$, we have*

$$I(\alpha, \mu) := \int_0^1 t^{\alpha-1} \mu^t dt = \mu \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (\ln \mu)^{k-1}}{(\alpha)_k} < \infty,$$

where

$$(\alpha)_k = \alpha (\alpha+1)(\alpha+2) \ldots (\alpha+k-1).$$

Moreover, it holds

$$\left| I(\alpha, \mu) - \mu \sum_{k=1}^{m} (-1)^{k-1} \frac{(\ln \mu)^{k-1}}{(\alpha)_k} \right| \leq \frac{|\ln \mu|}{\alpha \sqrt{2\pi (m-1)}} \left(\frac{|\ln \mu| e}{m-1} \right)^{m-1}.$$

We are now ready to set off our first result.

Theorem 3. *Let $I \subseteq [0, \infty)$ be an open real interval and let $f : I \to (0, \infty)$ be a function such that $f^{(n)}$ exists on I and $f^{(n)}$ is integrable on $[a, b]$ for $n \in \mathbb{N}$, $n \geq 2$, $0 \leq a < b < \infty$. If $\left| f^{(n)} \right|^q$ is (α, m)-logarithmically convex on $\left[0, \frac{b}{m}\right]$ for $q \in [1, \infty)$,*

$(\alpha,m)\in(0,1]\times(0,1]$, *we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$
$$\leq\frac{(b-a)^n}{2n!}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m\left(\frac{n-1}{n+1}\right)^{1-1/q}\left[E_1(\alpha,\mu,n,q)\right]^{1/q},\quad(2.9)$$

where

$$\mu=\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m},\ E_1(\alpha,m,n,q)=\begin{cases}\frac{n-1}{n+1}, & \mu=1\\ F_1(\mu,\alpha q,n), & 0<\mu<1\\ \mu^{q(1-\alpha)}F_1(\mu,\alpha q,n), & \mu>1\end{cases}$$

and

$$F_1(u,v,n)=\frac{(-1)^n n!\,[v\ln u+2]}{v^{n+1}(\ln u)^{n+1}}-\frac{2u^v}{\ln u}-n!u^v\sum_{k=1}^{n}\frac{(-1)^k[v\ln u+2]}{v^{k+1}(n-k)!\,(\ln u)^{k+1}}$$

for $u,\ v>0,\ u\neq1$.

Proof. Suppose $n\geq2$ and $a,b\in I$, $0\leq a<b<\infty$. By (α,m)-logarithmically convexity of $\left|f^{(n)}\right|^q$ on $\left[0,\frac{b}{m}\right]$, Lemma 1 and Hölder inequality, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$
$$\leq\frac{(b-a)^n\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{2n!}\left(\int_0^1 t^{n-1}(n-2t)\,dt\right)^{1-1/q}\left(\int_0^1 t^{n-1}(n-2t)\,\mu^{qt^\alpha}dt\right)^{1/q}$$
$$=\frac{(b-a)^n\left|f^{(n)}(\frac{b}{m})\right|^m}{2n!}\left(\frac{n-1}{n+1}\right)^{1-1/q}\left(n\int_0^1 t^{n-1}\mu^{qt^\alpha}dt-2\int_0^1 t^n\mu^{qt^\alpha}dt\right)^{1/q},\quad(2.10)$$

where $\mu=\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}$.
For $\mu=1$, we have

$$n\int_0^1 t^{n-1}\mu^{qt^\alpha}dt-2\int_0^1 t^n\mu^{qt^\alpha}dt=n\int_0^1 t^{n-1}dt-2\int_0^1 t^n dt=\frac{n-1}{n+1}.\quad(2.11)$$

For $0<\mu<1$, $\mu^{qt^{\alpha}}\leq\mu^{\alpha qt}$ and hence by using Lemma 3

$$\begin{aligned}
n\int_0^1 t^{n-1}\mu^{qt^{\alpha}}dt-2\int_0^1 t^{n}\mu^{qt^{\alpha}}dt &\leq n\int_0^1 t^{n-1}\mu^{\alpha qt}dt-2\int_0^1 t^{n}\mu^{\alpha qt}dt\\
&=\frac{(-1)^n n!}{(\alpha q)^n(\ln\mu)^n}-n!\mu^{\alpha q}\sum_{k=1}^{n}\frac{(-1)^k}{(\alpha q)^k(n-k)!(\ln\mu)^k}\\
&\quad-\frac{2(-1)^{n+1}n!}{(\alpha q)^{n+1}(\ln\mu)^{n+1}}-2n!\mu^{\alpha q}\sum_{k=0}^{n}\frac{(-1)^k}{(\alpha q)^k(n-k)!(\ln\mu)^{k+1}}\\
&=\frac{(-1)^n n!\left[\alpha q\ln\mu+2\right]}{(\alpha q)^{n+1}(\ln\mu)^{n+1}}-\frac{2\mu^{\alpha q}}{\ln\mu}-n!\mu^{\alpha q}\sum_{k=1}^{n}\frac{(-1)^k\left[\alpha q\ln\mu+2\right]}{(\alpha q)^{k+1}(n-k)!(\ln\mu)^{k+1}}. \qquad (2.12)
\end{aligned}$$

For $\mu>1$, $\mu^{qt^{\alpha}}\leq\mu^{q\alpha t+q-q\alpha}$, and hence by Lemma 3

$$\begin{aligned}
n\int_0^1 t^{n-1}\mu^{qt^{\alpha}}dt-2\int_0^1 t^{n}\mu^{qt^{\alpha}}dt &\leq n\int_0^1 t^{n-1}\mu^{q\alpha t+q(1-\alpha)}dt-2\int_0^1 t^{n}\mu^{q\alpha t+q(1-\alpha)}dt\\
&=\mu^{q(1-\alpha)}\left[\frac{(-1)^n n!\left[(q\alpha)\ln\mu+2\right]}{(q\alpha)^{n+1}(\ln\mu)^{n+1}}-\frac{2\mu^{q\alpha}}{\ln\mu}\right.\\
&\quad\left.-n!\mu^{q\alpha}\sum_{k=1}^{n}\frac{(-1)^k\left[(q\alpha)\ln\mu+2\right]}{(q\alpha)^{k+1}(n-k)!(\ln\mu)^{k+1}}\right]. \qquad (2.13)
\end{aligned}$$

Combining (2.11), (2.12) and (2.13), we obtain the required result. This completes the proof of the theorem. □

Corollary 3. *Suppose the assumptions of Theorem 3 are satisfied and if $q=1$, we have the inequality*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|\\
&\qquad\leq\frac{(b-a)^n}{2n!}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m E_1(\alpha,\mu,n,1), \qquad (2.14)
\end{aligned}$$

where $E_1(\alpha,\mu,n,q)$ is as defined in Theorem 3.

Corollary 4. *Under the assumptions of Theorem 3, if $n=2$, we have the inequality*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\
&\qquad\leq\frac{(b-a)^2}{4}\left|f''\left(\frac{b}{m}\right)\right|^m\left(\frac{1}{3}\right)^{1-1/q}\left[E_1(\alpha,\mu,2,q)\right]^{1/q}, \qquad (2.15)
\end{aligned}$$

where

$$\mu=\frac{\left|f''(a)\right|}{\left|f''\left(\frac{b}{m}\right)\right|^m},\ E_1(\alpha,m,2,q)=\begin{cases}\frac{1}{3}, & \mu=1\\ F_1(\mu,\alpha q,2), & 0<\mu<1\\ \mu^{q(1-\alpha)}F_1(\mu,\alpha q,2), & \mu>1\end{cases}$$

and

$$F_1(u,v,2)=\frac{2v(1+u^v)\ln u+4(1-u^v)}{v^3(\ln u)^3}$$

for $u,\ v>0,\ u\neq 1$.

Corollary 5. *Under the assumptions of Theorem 3, if* $n=2$ *and* $\alpha=1$, *we have the inequality when the absolute value of the second derivative of* f *is an* m-*logarthmically convex function*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{4}\left|f''\left(\frac{b}{m}\right)\right|^m\left(\frac{1}{3}\right)^{1-1/q}\left[E_1(1,\mu,2,q)\right]^{1/q}, \quad (2.16)$$

where

$$\mu=\frac{\left|f''(a)\right|}{\left|f''\left(\frac{b}{m}\right)\right|^m},\ E_1(1,m,2,q)=\begin{cases}\frac{1}{3}, & \mu=1\\ F_1(\mu,q,2), & \mu>0,\mu\neq 1\end{cases}$$

and

$$F_1(u,v,2)=\frac{2v(1+u^v)\ln u+4(1-u^v)}{v^3(\ln u)^3}$$

for $u,\ v>0,\ u\neq 1$.

Remark 1. *The inequalities (2.15) and (2.16) may be better than those given in Theorem 1 and Corollary 1.*

Theorem 4. *Let* $I\subseteq[0,\infty)$ *be an open real interval and let* $f:I\to(0,\infty)$ *be a function such that* $f^{(n)}$ *exists on* I *and* $f^{(n)}$ *is integrable on* $[a,b]$ *for* $n\in\mathbb{N}$, $n\geq 2$, $0\leq a<b<\infty$. *If* $\left|f^{(n)}\right|^q$ *is* (α,m)-*logarithmically convex on* $\left[0,\frac{b}{m}\right]$ *for* $q\in(1,\infty)$, $(\alpha,m)\in(0,1]\times(0,1]$, *we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right| \leq \frac{(b-a)^n\left[n^{(2q-1)/(q-1)}-(n-2)^{(2q-1)/(q-1)}\right]^{1-1/q}}{2^{2-1/q}n!} \times\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m\left[E_2(\alpha,m,n,q)\right]^{1/q}, \quad (2.17)$$

where

$$\mu=\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m},\ E_2(\alpha,m,n,q)=\begin{cases}\frac{1}{nq-q+1}, & \mu=1\\ F_2(\mu,\alpha q,n), & 0<\mu<1\\ \mu^{q(1-\alpha)}F_2(\mu,\alpha q,n), & \mu>1,\end{cases}$$

$$F_2(u,v,n)=u^v\sum_{k=1}^{\infty}\frac{(-1)^{k-1}v^{k-1}(\ln u)^{k-1}}{(nq-q+1)_k}<\infty$$

for $u,\ v>0,\ u\neq 1$ *and* $(nq-q+1)_k=(nq-q+1)(nq-q+2)\cdots(nq-q+k)$.

Proof. Since $\left|f^{(n)}\right|^q$ is (α,m)-logarithmically convex on $\left[0,\frac{b}{m}\right]$ for $q\in(1,\infty)$, $(\alpha,m)\in(0,1]\times(0,1]$, Lemma 1 and Hölder inequality, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$
$$\leq\frac{(b-a)^n}{2n!}\left(\int_0^1(n-2t)^{q/(q-1)}\,dt\right)^{1-1/q}\left(\int_0^1 t^{q(n-1)}\left|f^{(n)}(ta+(1-t)\,b\right|^q dt\right)^{1/q}$$
$$\leq\frac{(b-a)^n\left[n^{(2q-1)/(q-1)}-(n-2)^{(2q-1)/(q-1)}\right]^{1-1/q}}{2^{2-1/q}n!}$$
$$\times\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m\left(\int_0^1 t^{q(n-1)}\mu^{qt^\alpha}\,dt\right)^{1/q},\quad(2.18)$$

where $\mu=\frac{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{\left|f^{(n)}(a)\right|}$.
For $\mu=1$, we have

$$\int_0^1 t^{q(n-1)}\mu^{qt^\alpha}dt=\int_0^1 t^{q(n-1)}dt=\frac{1}{nq-q+1}.$$

For $0<\mu<1$, $\mu^{qt^\alpha}\leq\mu^{\alpha qt}$ and hence by using Lemma 6, we have

$$\int_0^1 t^{q(n-1)}\mu^{qt^\alpha}dt\leq\int_0^1 t^{q(n-1)}\mu^{q\alpha t}dt$$
$$\leq\mu^{\alpha q}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(\alpha q)^{k-1}(\ln\mu)^{k-1}}{(nq-q+1)_k}<\infty.$$

For $\mu>1$, $\mu^{qt^\alpha}\leq\mu^{q\alpha t+q(1-\alpha)}$, and hence by Lemma 6, we obtain

$$\int_0^1 t^{q(n-1)}\mu^{qt^\alpha}dt\leq\int_0^1 t^{q(n-1)}\mu^{q\alpha t+q(1-\alpha)}dt$$
$$\leq\mu^{q(1-\alpha)}\left(\mu^{q\alpha}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(q\alpha)^{k-1}(\ln\mu)^{k-1}}{(nq-q+1)_k}\right)<\infty.$$

Thus the inequality (2.17) follows. This completes the proof of the theorem. □

Corollary 6. *Suppose the assumptions of Theorem 4 are satisfied and $n=2$. Then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\frac{(b-a)^2}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left|f''\left(\frac{b}{m}\right)\right|^m\left[E_2(\alpha,m,2,q)\right]^{1/q},\quad(2.19)$$

where

$$\mu=\frac{\left|f''(a)\right|}{\left|f''\left(\frac{b}{m}\right)\right|^m},\ E_2(\alpha,m,2,q)=\begin{cases}\frac{1}{q+1}, & \mu=1\\ F_2(\mu,\alpha q,2), & 0<\mu<1\\ \mu^{q(1-\alpha)}F_2(\mu,\alpha q,2), & \mu>1,\end{cases}$$

$$F_2(u,v,2)=u^v\sum_{k=1}^{\infty}\frac{(-1)^{k-1}v^{k-1}(\ln u)^{k-1}}{(q+1)_k}<\infty$$

for $u,\ v>0,\ u\neq 1$ *and* $(q+1)_k=(q+1)(q+2)\cdots(q+k)$.

Corollary 7. *Suppose the assumptions of Theorem 4 are satisfied and* $n=2$, $\alpha=1$. *Then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left|f''\left(\frac{b}{m}\right)\right|^m\left[E_2(1,m,2,q)\right]^{1/q}, \quad (2.20)$$

where

$$\mu=\frac{\left|f''(a)\right|}{\left|f''\left(\frac{b}{m}\right)\right|^m},\ E_2(1,m,2,q)=\begin{cases}\frac{1}{q+1}, & \mu=1\\ F_2(\mu,q,2), & \mu>1,\mu\neq 1\end{cases}$$

$$F_2(u,v,2)=u^v\sum_{k=1}^{\infty}\frac{(-1)^{k-1}v^{k-1}(\ln u)^{k-1}}{(q+1)_k}<\infty$$

for $u,\ v>0,\ u\neq 1$ *and* $(q+1)_k=(q+1)(q+2)\cdots(q+k)$.

Now we give some results related to left-side of Hermite-Hadamard's inequality for n-times differentiable log-preinvex functions.

Theorem 5. *Let* $I\subseteq[0,\infty)$ *be an open real interval and let* $f:I\to(0,\infty)$ *be a function such that* $f^{(n)}$ *exists on* I *and* $f^{(n)}$ *is integrable on* $[a,b]$ *for* $n\in\mathbb{N}$, $n=1$, $0\leq a<b<\infty$. *If* $\left|f^{(n)}\right|^q$ *is* (α,m)*-logarithmically convex on* $\left[0,\frac{b}{m}\right]$ *for* $q\in[1,\infty)$, $(\alpha,m)\in(0,1]\times(0,1]$, *we have the inequality*

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^n\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{2^{(n+1)(q-1)/q}(n+1)^{1-1/q}n!}E_3(\alpha,m,n,q), \quad (2.21)$$

where

$$\mu=\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m},\ E_3(\alpha,m,n,q)=\begin{cases}\frac{2}{2^{(n+1)/q}(n+1)^{1/q}}, & \mu=1\\ \left[F_3(\mu,\alpha q,n)\right]^{1/q}+\left[F_4(\mu,\alpha q,n)\right]^{1/q}, & 0<\mu<1\\ \mu^{1-\alpha}\left\{\begin{array}{c}\left[F_3(\mu,\alpha q,n)\right]^{1/q}\\ +\left[F_4(\mu,\alpha q,n)\right]^{1/q}\end{array}\right\}, & \mu>1,\end{cases}$$

and

$$F_3(u,v,n)=\frac{(-1)^{n+1}n!}{v^{n+1}(\ln u)^{n+1}}+n!u^{v/2}\sum_{k=0}^{n}\frac{(-1)^k}{2^{n-k}v^{k+1}(n-k)!(\ln u)^{k+1}},$$

$$F_4(u,v,n)=\frac{n!u}{v^{n+1}(\ln u)^{n+1}}-n!u^{v/2}\sum_{k=0}^{n}\frac{1}{2^{n-k}v^{k+1}(n-k)!(\ln u)^{k+1}}$$

for $u,\ v>0,\ u\neq 1$.

Proof. Suppose $n\geq 1$. By using Lemma 2, the (α,m)-logarithmically convexity of $\left|f^{(n)}\right|$ and the Hölder inequality, we have

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\
&\leq\frac{(b-a)^n}{n!}\left[\int_0^{\frac12}t^n\left|f^{(n)}(ta+(1-t)b)\right|dt+\int_{\frac12}^{1}(1-t)^n\left|f^{(n)}(ta+(1-t)b)\right|dt\right]\\
&\qquad\leq\frac{(b-a)^n\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{n!}\left[\left(\int_0^{\frac12}t^n dt\right)^{1-1/q}\left(\int_0^{\frac12}t^n\mu^{qt^\alpha}dt\right)^{1/q}\right.\\
&\qquad\qquad\left.+\left(\int_{\frac12}^{1}(1-t)^n dt\right)^{1-1/q}\left(\int_{\frac12}^{1}(1-t)^n\mu^{qt^\alpha}dt\right)^{1/q}\right]\\
&=\frac{(b-a)^n\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{n!2^{(n+1)(q-1)/q}(n+1)^{1-1/q}}\left[\left(\int_0^{\frac12}t^n\mu^{qt^\alpha}dt\right)^{1/q}+\left(\int_{\frac12}^{1}(1-t)^n\mu^{qt^\alpha}dt\right)^{1/q}\right],
\end{aligned}\tag{2.22}$$

where $\mu=\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}$.

For $\mu=1$, we have

$$\begin{aligned}
&\left(\int_0^{\frac12}t^n\mu^{qt^\alpha}dt\right)^{1/q}+\left(\int_{\frac12}^{1}(1-t)^n\mu^{qt^\alpha}dt\right)^{1/q}\\
&=\left(\int_0^{\frac12}t^n dt\right)^{1/q}+\left(\int_{\frac12}^{1}(1-t)^n dt\right)^{1/q}=\frac{2}{2^{(n+1)/q}(n+1)^{1/q}}.
\end{aligned}$$

For $0<\mu<1$, $\mu^{qt^\alpha}\leq\mu^{\alpha qt}$ and hence by using Lemma 4 and Lemma 5, we have

$$\begin{aligned}
&\left(\int_0^{\frac12}t^n\mu^{qt^\alpha}dt\right)^{1/q}+\left(\int_{\frac12}^{1}(1-t)^n\mu^{qt^\alpha}dt\right)^{1/q}\\
&\leq\left(\frac{(-1)^{n+1}n!}{(\alpha q)^{n+1}(\ln\mu)^{n+1}}+n!\mu^{\alpha q/2}\sum_{k=0}^{n}\frac{(-1)^k}{2^{n-k}(\alpha q)^{k+1}(n-k)!(\ln\mu)^{k+1}}\right)^{1/q}\\
&+\left(\frac{n!\mu}{(\alpha q)^{n+1}(\ln\mu)^{n+1}}-n!\mu^{\alpha q/2}\sum_{k=0}^{n}\frac{1}{2^{n-k}(\alpha q)^{k+1}(n-k)!(\ln\mu)^{k+1}}\right)^{1/q}.
\end{aligned}$$

For $\mu > 1$, $\mu^{qt^{\alpha}} \leq \mu^{q\alpha t + q(1-\alpha)}$ and hence by using Lemma 4 and Lemma 5, we have

$$\left(\int_0^{\frac{1}{2}} t^n \mu^{qt^{\alpha}} dt\right)^{1/q} + \left(\int_{\frac{1}{2}}^1 (1-t)^n \mu^{qt^{\alpha}} dt\right)^{1/q}$$
$$\leq \mu^{1-\alpha} \left(\frac{(-1)^{n+1} n!}{(q\alpha)^{n+1} (\ln \mu)^{n+1}} + n! \mu^{\alpha q/2} \sum_{k=0}^{n} \frac{(-1)^k}{2^{n-k} (q\alpha)^{k+1} (n-k)! (\ln \mu)^{k+1}}\right)^{1/q}$$
$$+ \mu^{1-\alpha} \left(\frac{n! \mu}{(q\alpha)^{n+1} (\ln \mu)^{n+1}} - n! \mu^{\alpha q/2} \sum_{k=0}^{n} \frac{1}{2^{n-k} (q\alpha)^{k+1} (n-k)! (\ln \mu)^{k+1}}\right)^{1/q}.$$

Hence the inequality (2.21) follows from the above facts. This completes the proof of the theorem. □

Corollary 8. *Suppose the assumptions of Theorem 5 are fulfilled and if $q = 1$, we have*

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right] (b-a)^k}{2^{k+1} (k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)\, dx\right|$$
$$\leq \frac{(b-a)^n \left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{n!} E_3(\alpha, m, n, 1), \quad (2.23)$$

where $E_3(\alpha, m, n, q)$ is as defined in Theorem 5.

Remark 2. *Suppose the conditions of Theorem 5 are satisfied and if $n = 1$, we get the corrected inequality given in Theorem 2.*

Remark 3. *Suppose the conditions of Theorem 5 are satisfied and if $\alpha = 1$, we get the corrected inequality given in Corollary 2.*

Corollary 9. *Let $I \subseteq [0, \infty)$ be an open real interval and let $f : I \to (0, \infty)$ be a function such that $f^{(n)}$ exists on I and $f^{(n)}$ is integrable on $[a, b]$ for $n \in \mathbb{N}$, $n \geq 1$, $0 \leq a < b < \infty$. If $\left|f^{(n)}\right|^q$ is m-logarthmically convex on $\left[0, \frac{b}{m}\right]$ for $q \in [1, \infty)$, $m \in (0, 1]$, we have the inequality*

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right] (b-a)^k}{2^{k+1} (k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)\, dx\right|$$
$$\leq \frac{(b-a)^n \left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{2^{(n+1)(q-1)/q} (n+1)^{1-1/q} n!} E_3(1, m, n, q), \quad (2.24)$$

where

$$\mu = \frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}, \quad E_3(1, m, n, q) = \begin{cases} \frac{2}{2^{(n+1)/q} (n+1)^{1/q}}, & \mu = 1 \\ \left[F_3(\mu, q, n)\right]^{1/q} + \left[F_4(\mu, q, n)\right]^{1/q}, & \mu > 0, \mu \neq 1 \end{cases}$$

and

$$F_3(u,v,n)=\frac{(-1)^{n+1}n!}{v^{n+1}(\ln u)^{n+1}}+n!u^{v/2}\sum_{k=0}^{n}\frac{(-1)^k}{2^{n-k}v^{k+1}(n-k)!(\ln u)^{k+1}},$$
$$F_4(u,v,n)=\frac{n!u}{v^{n+1}(\ln u)^{n+1}}-n!u^{v/2}\sum_{k=0}^{n}\frac{1}{2^{n-k}v^{k+1}(n-k)!(\ln u)^{k+1}}$$

for $u,\ v>0,\ u\neq 1.$

Theorem 6. *Let* $I\subseteq[0,\infty)$ *be an open real interval and let* $f:I\to(0,\infty)$ *be a function such that* $f^{(n)}$ *exists on* I *and* $f^{(n)}$ *is integrable on* $[a,b]$ *for* $n\in\mathbb{N}$, $n\geq 1$, $0\leq a<b<\infty$. *If* $\left|f^{(n)}\right|^q$ *is* (α,m)*-logarithmically convex on* $\left[0,\frac{b}{m}\right]$ *for* $q\in(1,\infty)$, $(\alpha,m)\in(0,1]\times(0,1]$, *we have the inequality*

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\frac{(b-a)^n}{2^{n+1/p}(np+1)^{1/p}n!}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m\left\{\left[\frac{\alpha q\left(\frac{1}{2}\right)^{\alpha-1}\mu^{q\left(\frac{1}{2}\right)^{\alpha}}}{\ln\mu}\right]^{1/q}\right.$$
$$\left.+\left[\frac{\alpha q\mu^q-\alpha q\left(\frac{1}{2}\right)^{\alpha-1}\mu^{q\left(\frac{1}{2}\right)^{\alpha}}}{\ln\mu}\right]^{1/q}\right\},\quad(2.25)$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $\mu=\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}$.

Proof. From Lemma 2, the Hölder integral inequality and (α,m)-logarithmically convexity of $\left|f^{(n)}\right|^q$, we have

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\frac{(\eta(b,a))^n\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{n!}\left[\left(\int_0^{\frac{1}{2}}t^{np}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}\right)^{qt^{\alpha}}dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1(1-t)^{np}\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}\right)^{qt^{\alpha}}dt\right)^{\frac{1}{q}}\right]\quad(2.26)$$

from which the required inequality follows. This completes the proof of the theorem. □

Corollary 10. *Under the assumptions of Theorem 6, if $n = 1$, we have the inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{2^{1+1/p}(p+1)^{1/p}}\left|f'\left(\frac{b}{m}\right)\right|^m\left\{\left[\frac{\alpha q\left(\frac{1}{2}\right)^{\alpha-1}\mu^{q\left(\frac{1}{2}\right)^{\alpha}}}{\ln\mu}\right]^{1/q} + \left[\frac{\alpha q\mu^q-\alpha q\left(\frac{1}{2}\right)^{\alpha-1}\mu^{q\left(\frac{1}{2}\right)^{\alpha}}}{\ln\mu}\right]^{1/q}\right\}, \quad (2.27)$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $\mu=\frac{\left|f'(a)\right|}{\left|f'\left(\frac{b}{m}\right)\right|^m}$.

Corollary 11. *Let $I \subseteq [0, \infty)$ be an open real interval and let $f : I \to (0, \infty)$ be a functions such that $f^{(n)}$ exists on I and $f^{(n)}$ is integrable on $[a, b]$ for $n \in \mathbb{N}$, $n \geq 1$, $0 \leq a < b < \infty$. If $\left|f^{(n)}\right|^q$ is m-logarthmically convex on $\left[0, \frac{b}{m}\right]$ for $q \in (1, \infty)$, $m \in (0, 1]$, we have the inequality*

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^n\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{2^{n+1/p}(np+1)^{1/p}n!}\left(\frac{q}{\ln\mu}\right)^{1/q}\mu^{1/2}\left\{1+\left[\mu^{q/2}-1\right]^{1/q}\right\}, \quad (2.28)$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $\mu=\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}$.

References

[1] M. K. Bakula, M. E. Özdemir, J. Pečarić, Hadamard type inequalities for m-convex and (α, m)-convex functions, J. Inequal. Pure Appl. Math. 9 (2008), no. 4, Art. 96, 12 pages.

[2] E. F. Beckenbach, Convex functions, Bull. Amer. Math. Soc. 54(1948), 439–460.

[3] S. S. Dragomir, On some new inequalities of Hermite-Hadamard type for m-convex functions, Tamkang J. Math. 33 (2002) 45–55.

[4] S. S. Dragomir, G. Toader, Some inequalities for m-convex functions, Studia Univ. Babeş-Bolyai Math. 38 (1993) 21–28.

[5] S.S. Dragomir and R.P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and trapezoidal formula, Appl. Math. Lett., 11(5) (1998), 91-95.

[6] S. S. Dragomir and C. E. M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000.

[7] S. S. Dragomir, R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. Lett. 11 (1998) 91–95.

[8] J. Deng and J. R. Wang, Fractional Hermite-Hadamard inequalities for (α, m)-logarithmically convex functions, J Inequal Appl 2013, 2013:364.

[9] W.-D. Jiang, D.-W. Niu, Y. Hua, and F. Qi, Generalizations of Hermite-Hadamard inequality to n-time differentiable functions which are s-convex in the second sense, Analysis (Munich) 32 (2012), 1001–1012.

[10] R.- F. Bai, F. Qi and B. -Y. Xi, Hermite-Hadamard type inequalities for the m- and (α, m)-logarithmically convex functions, Filomat 27:1 (2013), 1-7.

[11] J. Hadamard, Étude sur les propriétés des fonctions entières et en particulier d'une fonction considerée par Riemann, J. Math Pures Appl., 58 (1893), 171–215.

[12] Ch. Hermite, Sur deux limites d'une intégrale définie, Mathesis 3 (1883), 82.

[13] D.-Y. Hwang, Some inequalities for n-time differentiable mappings and applications, Kyugpook Math. J. 43(2003), 335-343.

[14] U.S. Kırmacı, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp., 147 (2004), 137-146.

[15] U.S. Kırmacı and M.E. Özdemir, On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp., 153 (2004), 361-368.

[16] U.S. Kırmacı, Improvement and further generalization of inequalities for differentiable mappings and applications, Comp and Math. with Appl., 55 (2008), 485-493.

[17] M. A. Latif and S. S. Dragomir, New inequalities of Hermite-Hadamard type for functions whose derivatives in absolute value are convex with applications, Acta Univ. M. Belii, ser. Math. (2013), 24–39.

[18] V. G. Mihesan, A generalization of the convexity, Seminar on Functional Equations, Approx. Convex, Cluj-Napoca, 1993. (Romania)

[19] D. S. Mitrinović and I. B. Lacković, Hermite and convexity, Aequationes Math. 28 (1985), 229–232.

[20] C.E.M. Pearce and J. Pečarić, Inequalities for differentiable mappings with application to special means and quadrature formulae, Appl. Math. Lett., 13(2) (2000), 51–55.

[21] M. Z. Sarikaya, A. Saglam and H. Yıldırım, On some Hadamard-type inequalities for h-convex functions, J Math Inequal, Vol. 2, No. 3 (2008), 335-341.

[22] M. Z. Sarikaya and N. Aktan, On the generalization some integral inequalities and their applications Mathematical and Computer Modelling, Volume 54, Issues 9-10, November 2011, Pages 2175-2182.

[23] M. Z. Sarikaya, E. Set and M. E. Ozdemir, On some new inequalities of Hadamard type involving h-convex functions, Acta Mathematica Universitatis Comenianae, Vol. LXXIX, 2(2010), pp. 265-272.

[24] A. Saglam, M. Z. Sarikaya and H. Yildirim, Some new inequalities of Hermite-Hadamard's type, Kyungpook Mathematical Journal, 50(2010), 399-410.

[25] G. Toader, Some generalizations of the convexity, Proceedings of the Colloquium on Approximation and Optimization, Univ. Cluj-Napoca, Cluj-Napoca, 1985, 329–338.

[26] J. Wang, J. Deng, M. Fečkan, Exploring s-e-condition and applications to some Ostrowski type inequalities via Hadamard fractional integrals. Math. Slovaca (2013, in press).

[27] S.-H. Wang, B.-Y. Xi, F. Qi, Some new inequalities of Hermite-Hadamard type for n-time differentiable functions which are m-convex, Analysis (Munich) 32 (2012) 247–262.

[28] B.-Y. Xi, R.-F. Bai, F. Qi, Hermite-Hadamard type inequalities for the m- and (α, m)-geometrically convex functions, Aequationes Math. 39 (2012), in press.

[29] T.-Y. Zhang, A.-P. Ji, F. Qi, On integral inequalities of Hermite-Hadamard type for s-geometrically convex functions, Abstr. Appl. Anal. 2012 (2012).

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