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SEVERAL INEQUALITIES FOR POSITIVE OPERATORS ON HILBERT SPACES

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Abstract. In this paper, several inequalities for positive definite operators defined on Hilbert spaces will be presented under suitable assumptions, starting from some refinements of the Kittaneh-Manasrah inequality which improves the well-known inequality of Young.

1. Introduction

It is necessary to recall the following results which are given in the papers [4] and [5] and will be used below in the demonstration of inequalities from Proposition 1, Theorem 2 and Proposition 3. In these demonstrations the same method as in the paper [1] will be utilized.

Lemma 1. *([4]) Let a and b be such that $a, b \geq 0$ and $0 \leq \nu \leq 1$. Then the following inequality holds:*

$$\nu a^2 + (1-\nu)b^2 \leq (a^\nu b^{1-\nu})^2 + s_0(a-b)^2,$$

where $s_0 = \max\{\nu, 1-\nu\}$.

Lemma 2. *([5]) For all x, y positive real numbers and $\lambda \in (0,1,)$ we have the inequality*

$$2rE\left(x, y, \frac{1}{2}\right) \leq E(x,y,\lambda) \leq 2(1-r)E\left(x, y, \frac{1}{2}\right),$$

where

$$E(x,y,\lambda) = \lambda \exp x + (1-\lambda)\exp y - \exp(\lambda x + (1-\lambda)y) - \frac{\lambda(1-\lambda)}{2}(x-y)^2$$

and $r = \min\{\lambda, 1-\lambda\}$.

Theorem 1. *([5]) For $a, b \geq 1$, and $\lambda \in (0,1)$ we have*

$$r(\sqrt{a}-\sqrt{b})^2 + A_1(\lambda)\log^2\left(\frac{a}{b}\right) \leq \lambda a + (1-\lambda)b - a^\lambda b^{1-\lambda} \leq$$

$$\leq (1-r)(\sqrt{a}-\sqrt{b})^2 + B_1(\lambda)\log^2\left(\frac{a}{b}\right)$$

where $r = \min\{\lambda, 1-\lambda\}$, $A_1(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{r}{4}$ and $B_1(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{1-r}{4}$.

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First, it is necessary to recall that for selfadjoint operators $A, B \in B(H)$ we write $A \leq B$ (or $B \geq A$) if $< Ax, x > \leq < Bx, x >$ for every vector $x \in H$. In this paper, we will consider A as being a selfadjoint linear operator on a complex Hilbert space $(H; < .,. >)$ as in [1] and the references therein. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, an the C^*- algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (): For any $f, f \in C(Sp(A))$ and for any $\alpha, \beta \in \mathbf{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;
(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(f) = \Phi(f^*)$;
(iii) $||\Phi(f)|| = ||f|| := \sup_{t \in Sp(A)} |f(t)|$;
(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$ for $t \in Sp(A.)$

Using this notation, as in [1] for example, we define

$$f(A) := \Phi(f) \quad \text{for all} \quad f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A. It is known that if A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a *positive operator* on H. In addition, if and f and g are real valued functions on $Sp(A)$ then the following property holds:

$$(1) \qquad f(t) \geq g(t) \quad \text{for any} \quad t \in Sp(A) \quad \text{implies that} \quad f(A) \geq g(A)$$

in the operator order of $B(H)$.

2. **Main results**

The following results present several inequalities for functions of positive operators.

Proposition 1. *Let A and B be two positive definite operators on H. Then we have*

$$\nu < A^2x, x > + (1-\nu) < B^2y, y > \leq < A^{2\nu}x, x >< B^{2(1-\nu)}y, y > +$$
$$+ s_0 \left[< A^2x, x > -2 < Ax, x >< By, y > + < B^2y, y > \right],$$

for each $x, y \in H$ with $||x|| = ||y|| = 1$, where $0 \leq \nu \leq 1$ and $s_0 = \max\{\nu, 1-\nu\}$.

Proof. We consider the continue function $f(a) = (a^\nu b^{1-\nu})^2 + s_0(a-b)^2 - (\nu a^2 + (1-\nu)b^2)$, which is positive for $a \geq 0$ and we fix $b \geq 0$ and then by the property (1) for each $x \in H$ with $||x|| = 1$ we have that

$$< (\nu A^2 + (1-\nu)b^2 I)x, x > \leq < [A^{2\nu}b^{2(1-\nu)} + s_0(A^2 - 2Ab + b^2 I)]x, x >$$

which is equivalent with

$$\nu < A^2x, x > + (1-\nu)b^2 \leq$$
$$\leq b^{2(1-\nu)} < A^{2\nu}x, x > + s_0[< A^2x, x > -2b < Ax, x > + b^2 < x, x >]$$

for each $b > 0$.

If we apply again the property (1) for last inequality, then for any $y \in H$ with $||y|| = 1$ we get

$$< [\nu < A^2x, x > + (1-\nu)b^2]y, y > \leq$$
$$\leq < [b^{2(1-\nu)} < A^{2\nu}x, x > + s_0(< A^2x, x > -2b < Ax, x > + b^2 < x, x >)]y, y >$$

and this inequality is equivalent with

$$\nu < A^2x, x > +(1-\nu) < B^2y, y > \leq$$

$$\leq < A^{2\nu}x, x >< B^{2(1-\nu)}y, y > +s_0[< A^2x, x > -2 < Ax, x >< By, y > + < B^2y, y >]$$

for each $x, y \in H$ with $||x|| = ||y|| = 1$.

Taking now in previous inequality $x = y$ we obtain the desired inequality.

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As an interesting application of previous result, we have the following particular cases:

Remark 1. *(i) If we take in previous inequality $y = x$ then we have:*

$$\nu < A^2x, x > +(1-\nu) < B^2x, x > \leq < A^{2\nu}x, x >< B^{2(1-\nu)}x, x > +$$
$$+s_0\left[< A^2x, x > -2 < Ax, x >< Bx, x > + < B^2x, x >\right]$$

for each $x \in H$ with $||x|| = 1$, where $s_0 = \max\{\nu, 1-\nu\}$.

(ii) If in addition $A = B$ then in previous inequality we obtain:

$$1 - 2s_0\left[< A^2x, x > -(< Ax, x >)^2\right] \leq < A^{2\nu}x, x >< A^{2(1-\nu)}x, x >$$

for each $x \in H$ with $||x|| = 1$, where $s_0 = \max\{\nu, 1-\nu\}$.

Theorem 2. *Let A be a positive definite operator on H. Then the following inequality holds:*

$$r\left[2 < \exp(A)x, x > -2\left(< \exp\left(\frac{A}{2}\right)x, x >\right)^2 - \frac{1}{2}\left(< A^2x, x > -(< Ax, x >)^2\right)\right] \leq$$

$$\leq 1 - < \exp(\lambda A)x, x >< \exp(1-\lambda)Ax, x > -\lambda(1-\lambda)\left[< A^2x, x > -(< Ax, x >)^2\right] \leq$$

$$\leq (1-r)\left[2 < \exp(A)x, x > -2\left(< \exp\left(\frac{A}{2}\right)x, x >\right)^2 - \frac{1}{2}\left(< A^2x, x > -(< Ax, x >)^2\right)\right]$$

for each $x \in H$ with $||x|| = 1$, where $r = \min\{\lambda, 1-\lambda\}$.

Proof. We write and then use the inequality from Lemma 2 with x replaced by a and y replaced by b obtaining:

$$r\left[\exp(a) + \exp(b) - 2\exp\left(\frac{a+b}{2}\right) - \frac{1}{4}(a-b)^2\right] \leq$$

$$\leq \lambda\exp(a) + (1-\lambda)\exp(b) - \exp(\lambda a + (1-\lambda)b) - \frac{\lambda(1-\lambda)}{2}(a-b)^2 \leq$$

$$\leq (1-r)\left[\exp(a) + \exp(b) - 2\exp\left(\frac{a+b}{2}\right) - \frac{1}{4}(a-b)^2\right].$$

We fix $b > 0$ and apply the property (1) for previous inequality obtaining :

$$< r[\exp(A) + \exp(b)1_H - 2\exp(\frac{b}{2})\exp\left(\frac{A}{2}\right) - \frac{1}{4}(A^2 - 2bA + b^2 1_H)]x, x > \leq$$

$$\leq < [\lambda\exp(A) + (1-\lambda)\exp(b)1_H - \exp(\lambda A)\exp((1-\lambda)b) - \frac{\lambda(1-\lambda)}{2}(A^2 - 2bA + b^2 1_H)]x, x >$$

$$\leq < (1-r)[\exp(A) + \exp(b)1_H - 2\exp(\frac{b}{2})\exp\left(\frac{A}{2}\right) - \frac{1}{4}(A^2 - 2bA + b^2 1_H)]x, x >$$

which is equivalent with the following

$$r[< \exp(A)x, x > + \exp(b) - 2\exp(\frac{b}{2}) < \exp(\frac{A}{2})x, x > -$$

$$-\frac{1}{4}(< A^2x, x > -2b < Ax, x > +b^2)] \leq$$

$$\leq \lambda < \exp(A)x, x > +(1-\lambda)\exp(b) - \exp((1-\lambda)b) < \exp(\lambda A)x, x > -$$

$$-\frac{\lambda(1-\lambda)}{2}(< A^2x, x > -2b < Ax, x > +b^2) \leq$$

$$\leq (1-r)[< \exp(A)x, x > + \exp(b) - 2\exp(\frac{b}{2}) < \exp(\frac{A}{2})x, x > -$$

$$-\frac{1}{4}(< A^2x, x > -2b < Ax, x > +b^2)],$$

for any $x \in H$ with $||x|| = 1$.

If we apply again the property (1) for previous inequality for the variable b, then we have for any $y \in H$ with $||y|| = 1$ that

$$r[< \exp(A)x, x > + < \exp(B)y, y > -2 < \exp(\frac{B}{2})y, y >< \exp(\frac{A}{2})x, x > -$$

$$-\frac{1}{4}(< A^2x, x > -2 < Ax, x >< By, y > + < B^2y, y >)] \leq$$

$$\leq \lambda < \exp(A)x, x > +(1-\lambda) < \exp(B)y, y > - < \exp((1-\lambda)B)y, y >< \exp(\lambda A)x, x > -$$

$$-\frac{\lambda(1-\lambda)}{2}(< A^2x, x > -2 < Ax, x >< By, y > + < B^2y, y >) \leq$$

$$\leq (1-r)[< \exp(A)x, x > + < \exp(B)y, y > -2 < \exp(\frac{B}{2})y, y >< \exp(\frac{A}{2})x, x > -$$

$$-\frac{1}{4}(< A^2x, x > -2 < Ax, x >< By, y > + < B^2y, y >)],$$

If we take now $x = y$ in the above inequality we will obtained the desired inequality.
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A multiple operator version of Proposition 1 takes place also below:

Proposition 2. *Assume that $A_j, j \in \{1, ..., n\}$ are positive operators on the Hilbert space H. If $0 \leq \nu \leq 1$ then for each $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n ||x_j||^2 = 1$ we have the inequality*

$$1 \leq \sum_{j=1}^{n} < A_j^{2(1-\nu)}x_j, x_j > \sum_{j=1}^{n} < A_j^{2\nu}x_j, x_j > +$$

$$+s_0 \left[\sum_{j=1}^{n} < A_j^2 x_j, x_j > - \left(\sum_{j=1}^{n} < A_j x_j, x_j > \right)^2 \right]$$

where $s_0 = \max\{\nu, 1-\nu\}$.

Proof. As in the case of Theorem 2, see [1], we consider

$$\overline{A} := \begin{pmatrix} A_1 & . & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & . & . & . & A_n \end{pmatrix} \quad \text{and} \quad \overline{x} := \begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix}$$

having $||\overline{x}|| = 1$, and $\overline{A}$ is positive definite. Taking into account that

$$< \overline{A}^{2\nu}\overline{x}, \overline{x} >=< f_1(\overline{A})\overline{x}, \overline{x} >= \sum_{j=1}^{n} < f_1(A_j)x_j, x_j >= \sum_{j=1}^{n} < A_j^{2\nu}x_j, x_j >,$$

$$< \overline{A}^{2(1-\nu)}\overline{x}, \overline{x} >=< f_2(\overline{A})\overline{x}, \overline{x} >= \sum_{j=1}^{n} < f_2(A_j)x_j, x_j >= \sum_{j=1}^{n} < A_j^{2(1-\nu)}x_j, x_j >,$$

$$< \overline{A}\overline{x}, \overline{x} >=< f_3(\overline{A})\overline{x}, \overline{x} >= \sum_{j=1}^{n} < f_3(A_j)x_j, x_j >= \sum_{j=1}^{n} < A_jx_j, x_j >,$$

where , $f_1, f_2, f_3 : (0, \infty) \to \mathbf{R}$ are defined by $f_1(x) = x^{2\nu}$, $f_2(x) = x^{2(1-\nu)}$ and $f_3(x) = x$ respectively, and applying Remark 1 (ii) for $\overline{A}$ and $\overline{x}$ we find the desired inequality.

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Proposition 3. *Let A and B be two positive definite operators on H. If $Sp(A)$ $Sp(B) \subseteq [1, \infty)$, and $\lambda \in (0, 1)$ then we have*

$$r\left(< Ax, x > + < By, y > -2 < A^{\frac{1}{2}}x, x >< B^{\frac{1}{2}}y, y >\right) +$$

$$+A_1(\lambda)\left[< (\log^2 A)x, x > + < (\log^2 B)y, y > -2 < (\log B)y, y >< (logA)x, x >\right] \leq$$

$$\leq \lambda < Ax, x > +(1-\lambda) < By, y > - < A^{\lambda}x, x >< B^{1-\lambda}y, y >\leq$$

$$\leq (1-r)\left(< Ax, x > + < By, y > -2 < A^{\frac{1}{2}}x, x >< B^{\frac{1}{2}}y, y >\right) +$$

$$+B_1(\lambda)\left[< (\log^2 A)x, x > + < (\log^2 B)y, y > -2 < (\log B)y, y >< (logA)x, x >\right]$$

for each $x, y \in H$ with $||x|| = ||y|| = 1$, where $r = \min\{\lambda, 1-\lambda\}$, $A_1(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{r}{4}$ and $B_1(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{1-r}{4}$.

Proof. We consider the continue functions $f(a) = \lambda a + (1-\lambda)b - a^{\lambda}b^{1-\lambda} - r(a + b - 2a^{\frac{1}{2}}b^{\frac{1}{2}}) - A_1(\lambda)[\log^2 a + \log^2 b - 2\log a \log b]$, and $g(a) = (1-r)(a+b-2a^{\frac{1}{2}}b^{\frac{1}{2}}) + B_1(\lambda)[\log^2 a + \log^2 b - 2\log a \log b] - \lambda a - (1-\lambda)b + a^{\lambda}b^{1-\lambda}$ which are positive for $a \geq 1$ and we fix $b \geq 1$ and then by the property (1) for each $x \in H$ with $||x|| = 1$ we have that

$$r\left(< Ax, x > +b - 2b^{\frac{1}{2}} < A^{\frac{1}{2}}x, x >\right) +$$

$$+A_1(\lambda)\left[< (\log^2 A)x, x > + \log^2 b - 2\log b < (\log A)x, x >\right] \leq$$

$$\leq \lambda < Ax, x > +(1-\lambda)b - b^{1-\lambda} < A^{\lambda}x, x >\leq$$

$$\leq (1-r)\left(< Ax, x > +b - 2b^{\frac{1}{2}} < A^{\frac{1}{2}}x, x >\right) +$$

$$+B_1(\lambda)\left[< (\log^2 A)x, x > + \log^2 b - 2\log b < (\log A)x, x >\right]$$

for each $b > 1$.

If we apply again the property (1) for last inequality, then for any $y \in H$ with $||y|| = 1$ we get

$$r\left(< Ax, x > + < By, y > -2 < B^{\frac{1}{2}}y, y >< A^{\frac{1}{2}}x, x >\right) +$$

$$+A_1(\lambda)\left[< (\log^2 A)x, x > + < (\log^2 B)y, y > -2 < (\log B)y, y >< (\log A)x, x >\right] \leq$$

$$\leq \lambda < Ax, x > +(1-\lambda) < By, y > - < B^{1-\lambda}y, y >< A^{\lambda}x, x >\leq$$

$$\leq (1-r)\left(< Ax, x > + < By, y > -2 < B^{\frac{1}{2}}y, y >< A^{\frac{1}{2}}x, x >\right) +$$

$$+B_1(\lambda)\left[< (\log^2 A)x, x > + < (\log^2 B)y, y > -2 < (\log B)y, y >< (\log A)x, x >\right]$$

for each $x, y \in H$ with $||x|| = ||y|| = 1$.

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Next particular case of Proposition 3 may be of interest as well:

Remark 2. *Under previous conditions, if we consider $y = x$ and $A = B$ then the above inequality becomes:*

$$2r\left[< Ax, x > -\left(< A^{\frac{1}{2}}x, x >\right)^2\right] + 2A_1(\lambda)\left[< (\log^2 A)x, x > -(< (\log A)x, x >)^2\right] \leq$$

$$\leq 1- < A^{1-\lambda}x, x >< A^{\lambda}x, x >\leq$$

$$2(1-r)\left[< Ax, x > -\left(< A^{\frac{1}{2}}x, x >\right)^2\right] + 2B_1(\lambda)\left[< (\log^2 A)x, x > -(< (\log A)x, x >)^2\right].$$

Remark 3. *Assume that $A_j, j \in \{1, ..., n\}$ are positive operators on the Hilbert space H. If $0 \leq \lambda \leq 1$ then for each $x_j \in H,\ j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} ||x_j||^2 = 1$ we have the inequality*

$$r[2\sum_{j=1}^{n} < \exp(A_j)x_j, x_j > -2(\sum_{j=1}^{n} < \exp(\frac{A_j}{2})x_j, x_j >)^2 -$$

$$-\frac{1}{2}(\sum_{j=1}^{n} < A_j^2 x_j, x_j > -(\sum_{j=1}^{n} < A_j x_j, x_j >)^2)] \leq$$

$$\leq 1 - \sum_{j=1}^{n} < \exp(\lambda A_j)x_j, x_j > \sum_{j=1}^{n} < \exp((1-\lambda)A_j)x_j, x_j > -$$

$$-\lambda(1-\lambda)\left[\sum_{j=1}^{n} < A_j^2 x_j, x_j > -\left(\sum_{j=1}^{n} < A_j x_j, x_j >\right)^2\right] \leq$$

$$\leq (1-r)[2\sum_{j=1}^{n} < \exp(A_j)x_j, x_j > -2(\sum_{j=1}^{n} < \exp(\frac{A_j}{2})x_j, x_j >)^2 -$$

$$-\frac{1}{2}(\sum_{j=1}^{n} < A_j^2 x_j, x_j > -(\sum_{j=1}^{n} < A_j x_j, x_j >)^2)],$$

where $r = \min\{\lambda, 1-\lambda\}$.

Proof. The proof will be as in Proposition 2 if we consider the following functions $f_1, f_2, f_3, f_4, f_5 : (0, \infty) \to \mathbf{R}$ defined by $f_1(x) = x^2$, $f_2(x) = x$, $f_3(x) = \exp((1-\lambda)x)$, $f_4(x) = \exp(\lambda x)$ and $f_4(x) = \exp\left(\frac{x}{2}\right)$ respectively. ∎

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