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SOME PROPERTIES RELATED TO GENERALIZED TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS WITH TWO PARAMETERS

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Abstract. In this paper, we present some integral identities and inequalities of $(p, q)-$ complete elliptic integrals, and prove some inequalities related to generalized trigonometric and hyperbolic functions with two parameters.

1. Introduction

The generalized trigonometric and hyperbolic functions depending on a parameter $p > 1$ were studied by P. Lindqvist in a highly cited paper(See [14]). Motivated by this work, many authors have studied the equalities and inequalities related to generalized trigonometric and hyperbolic functions in [6, 8, 13]. Recently, in [18], S. Takeuchi has investigated the $(p, q)-$trigonometric functions depending on two parameters and in which the case of $p = q$ coincides with the $p-$function of Lindqvist, and for $p = q = 2$ they coincide with familiar elementary functions.

For $1 < p, q < \infty$ and $0 \le x \le 1$, the arc sine may be generalized as

$$\arcsin_{p,q} x = \int_0^x \frac{1}{(1-t^q)^{1/p}} dt \tag{1.1}$$

and

$$\frac{\pi_{p,q}}{2} = \arcsin_{p,q} 1 = \int_0^1 \frac{1}{(1-t^q)^{1/p}} dt. \tag{1.2}$$

The inverse of $\arcsin_{p,q}$ on $\left[0, \frac{\pi_{p,q}}{2}\right]$ is called the generalized $(p, q)-$sine function, denoted by $\sin_{p,q}$, and may be extended to $(-\infty, \infty)$. In the same way, we can define the generalized $(p, q)-$cosine function, the generalized $(p, q)-$tangent function and their inverses. Their definitions and formulas can be found in [10, 12]. Similarly, we can define the inverse of the generalized $(p, q)-$hyperbolic sine function as follows.

$$\operatorname{arcsinh}_{p,q} x = \int_0^x \frac{1}{(1+t^q)^{1/p}} dt \tag{1.3}$$

and also other corresponding $(p, q)-$hyperbolic functions. In [7], B. A. Bhayo and M. Vuorinen establish some inequalities and present a few conjectures for the $(p, q)-$functions. Very recently, a conjecture posed in [7] was verified in [12].

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Legendre's complete elliptic integrals of the first and second kind are defined for real numbers $0 < r < 1$ by

$$\kappa(r) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - r^2 \sin^2 t}} dt = \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-r^2t^2)}} dt \tag{1.4}$$

and

$$\varepsilon(r) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2 t} dt = \int_0^1 \sqrt{\frac{1-r^2t^2}{1-t^2}} dt \tag{1.5}$$

respectively. The complete elliptic integrals have many applications in several mathematical branches as well as in engineering and physics. Motivated by problems in potential theory and in the theory of quasi-conformal mappings, many mathematicians obtain monotonicity and convexity theorems of certain combinations of $\kappa(r)$ and $\varepsilon(r)$. See [1]-[4], [9], [11], [16], [19].

In the second section of the paper, we define $(p, q)-$ complete elliptic integrals, and prove some integral identities and inequalities. In the final section, we obtain some inequalities related to generalized trigonometric and hyperbolic functions with two parameters.

2. Some properties related to (p, q)-complete elliptic integrals

Definition 2.1. *For all $p, q \in (1, \infty)$ and $r \in (0, 1)$, the following the first and second kind of (p, q)-complete elliptic integrals are defined by*

$$\begin{cases} \kappa_{p,q}(r) = \int_0^{\pi_{p,q}/2} \frac{1}{(1-r^q \sin_{p,q}^q \theta)^{1/p}} d\theta \\ \kappa_{p,q}(0) = \frac{\pi_{p,q}}{2}, \kappa_{p,q}(1) = \infty, \end{cases} \tag{2.1}$$

and

$$\begin{cases} \varepsilon_{p,q}(r) = \int_0^{\pi_{p,q}/2} \frac{1}{(1-r^q \sin_{p,q}^q \theta)^{1/p}} d\theta \\ \varepsilon_{p,q}(0) = \frac{\pi_{p,q}}{2}, \varepsilon_{p,q}(1) = 1. \end{cases} \tag{2.2}$$

respectively.

Remark 2.1. For $p = q = 2$, they coincide with the first and second kind of complete elliptic integrals.

Lemma 2.1. [10, Proposition 3.3] *For all $p, q \in (1, +\infty)$ and all $\theta \in (0, \frac{\pi_{p,q}}{2}]$, then*

$$\frac{2}{\pi_{p,q}} \le \frac{\sin_{p,q} \theta}{\theta} \le 1. \tag{2.3}$$

Theorem 2.1. *For all $p, q \in (1, \infty), r \in (0, 1)$ and $\theta \in (0, \frac{\pi_{p,q}}{2})$, we have*

$$\int_0^1 \kappa_{p,q}(r) dr = \int_0^{\pi_{p,q}/2} \frac{\theta}{\sin_{p,q} \theta} d\theta. \tag{2.4}$$

Proof. The substitution $t = xr$ turns the identity

$$\operatorname{arcsin}_{p,q} x = \int_0^x \frac{1}{(1-t^q)^{1/p}} \, \mathrm{d}t \tag{2.5}$$

into

$$\operatorname{arcsin}_{p,q} x = x \int_0^1 \frac{1}{(1-r^q x^q)^{1/p}} \, \mathrm{d}r \tag{2.6}$$

Setting $\theta = \operatorname{arcsin}_{p,q} x$, we have

$$\frac{\theta}{\sin_{p,q}\theta} = \int_0^1 \frac{1}{(1 - r^q \sin_{p,q}^q \theta)^{1/p}}\,\mathrm{d}r. \tag{2.7}$$

From (2.7), it follows that

$$\begin{aligned} &\textstyle\int_0^{\pi_{p,q}/2} \frac{\theta}{\sin_{p,q}\theta} d\theta \\ &= \textstyle\int_0^1 \left(\int_0^{\pi_{p,q}/2} \frac{1}{(1-r^q \sin_{p,q}^q \theta)^{1/p}}\,\mathrm{d}r\right) d\theta \\ &= \textstyle\int_0^{\pi_{p,q}/2} \left(\int_0^1 \frac{1}{(1-r^q \sin_{p,q}^q \theta)^{1/p}}\,\mathrm{d}\theta\right) dr \\ &= \textstyle\int_0^1 \kappa_{p,q}(r) dr \end{aligned} \tag{2.8}$$

by using Fubini theorem. □

Corollary 2.1. *For all* $p, q \in (1, \infty), r \in (0, 1)$*, we have*

$$\frac{\pi_{p,q}}{2} \le \int_0^1 \kappa_{p,q}(r) dr \le \frac{\pi_{p,q}^2}{4}. \tag{2.9}$$

Proof. Using Lemma 2.1 and Theorem 2.1, we easily obtain the inequality (2.9). □

Theorem 2.2. *For all* $p, q \in (1, \infty), r \in (0, 1)$ *and* $\theta \in (0, \frac{\pi_{p,q}}{2})$*, we have*

$$\int_0^1 \varepsilon_{p,q}(r) dr = \frac{p}{p+q} + \frac{q}{p+q}\int_0^1 \kappa_{p',q}(r) dr, \tag{2.10}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. By definite integration by part, we have

$$\int_0^x \left(1 - t^q\right)^{1/p} dt = x\left(1 - x^q\right)^{1/p} + \frac{q}{p}\int_0^x \left(1 - t^q\right)^{-1/p'} dt - \frac{q}{p}\int_0^x \left(1 - t^q\right)^{1/p} dt. \tag{2.11}$$

So, we have

$$\int_0^x \left(1 - t^q\right)^{1/p} dt = \frac{p}{p+q} x\left(1 - x^q\right)^{1/p} + \frac{q}{p+q}\int_0^x \left(1 - t^q\right)^{-1/p'} dt. \tag{2.12}$$

The substitution $t = xr$ turns (2.12) into

$$x\int_0^1 \left(1 - r^q x^q\right)^{1/p} dt = \frac{p}{p+q} x\left(1 - x^q\right)^{1/p} + \frac{qx}{p+q}\int_0^1 \left(1 - r^q x^q\right)^{-1/p'} dt. \tag{2.13}$$

Setting $\theta = \operatorname{arcsin}_{p,q} x$, we have

$$\int_0^1 (1 - r^q \sin_{p,q}^q \theta)^{1/p} dr = \frac{p}{p+q}\cos_{p,q}\theta + \frac{q}{p+q}\int_0^1 \frac{1}{(1 - r^q \sin_{p,q}^q \theta)^{1/p'}} dr. \tag{2.14}$$

Similar to the proof of Theorem 2.1, we easily obtain (2.10) by using Fubini theorem. □

Theorem 2.3. *For all* $p, q \in (1, \infty), r \in (0, 1)$*, we have*

$$\varepsilon'_{p,q}(r) = \frac{q}{pr}\left(\varepsilon_{p,q}(r) - \kappa_{p',q}(r)\right) \tag{2.15}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. For all $p,q\in(1,\infty), r\in(0,1)$, we have

$$\begin{aligned}&\varepsilon'_{p,q}(r)=-\tfrac{q}{p}\int_0^{\pi_{p,q}/2}(1-r^q\sin_{p,q}^q\theta)^{(1-p)/p}r^{q-1}\sin_{p,q}^q\theta d\theta\\&=\tfrac{q}{pr}\int_0^{\pi_{p,q}/2}(1-r^q\sin_{p,q}^q\theta)^{(1-p)/p}\left(1-r^q\sin_{p,q}^q\theta-1\right)d\theta\\&=\tfrac{q}{pr}\left(\varepsilon_{p,q}(r)-\kappa_{p',q}(r)\right).\end{aligned}$$

□

Lemma 2.2. [15, Tchebychef's inequality] *Let $f(x),g(x)$ be integrable functions in $[a,b]$, both increasing or both decreasing. Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx\le\frac{1}{b-a}\int_a^b f(x)dx\cdot\frac{1}{b-a}\int_a^b g(x)dx. \tag{2.16}$$

If one of the functions $f(x)$ or $g(x)$ is nonincreasing and the other nondecreasing, then the inequality in (2.16) is reversed.

Lemma 2.3. *For all $p,q\in(1,\infty)$ and $\theta\in(0,\frac{\pi_{p,q}}{2})$, we have*

$$\int_0^{\pi_{p,q}/2}\sin_{p,q}^{q-1}\theta d\theta=\frac{p}{(p-1)q}. \tag{2.17}$$

Proof. Putting $t=\sin_{p,q}\theta$ and $t^q=u$, we have

$$\begin{aligned}&\int_0^{\pi_{p,q}/2}\sin_{p,q}^{q-1}\theta d\theta\\&=\int_0^1 t^{q-1}\left(1-t^q\right)^{-1/p}dt\\&=\tfrac{1}{q}\mathrm{B}\left(1,1-\tfrac{1}{p}\right)\\&=\tfrac{1}{q}\tfrac{\Gamma(1-1/p)}{\Gamma(2-1/p)}\\&=\tfrac{p}{(p-1)q}.\end{aligned}$$

□

Lemma 2.4. *For all $p,q\in(1,\infty)$ and $\theta\in(0,\frac{\pi_{p,q}}{2})$, we have*

$$\int_0^{\pi_{p,q}/2}\frac{\sin_{p,q}^{q-1}\theta}{(1-r^q\sin_{p,q}^q\theta)^{1/p}}d\theta=A(p,q,r), \tag{2.18}$$

where $A(p,q,r)=\frac{(1-r^q)^{1-2/p}}{r^{q(p-1)/p}}\int_0^r\frac{u^{q-q/p-1}}{(1-u^q)^{2-2/p}}du$.

Proof. Putting $t=\cos_{p,q}\theta$ and $t^p=\frac{1-r^q}{r^q}\frac{u^q}{1-u^q}$, we have

$$\begin{aligned}&\int_0^{\pi_{p,q}/2}\frac{\sin_{p,q}^{q-1}\theta}{(1-r^q\sin_{p,q}^q\theta)^{1/p}}d\theta\\&=\tfrac{p}{q}\int_0^1\frac{t^{p-2}}{(1-r^q+r^qt^p)^{1/p}}dt\\&=\frac{(1-r^q)^{1-2/p}}{r^{q(p-1)/p}}\int_0^r\frac{u^{q-q/p-1}}{(1-u^q)^{2-2/p}}du.\end{aligned}$$

□

Theorem 2.4. *For all $p,q\in(1,\infty), r\in(0,1)$ and $\theta\in(0,\frac{\pi_{p,q}}{2})$, we have*

$$\frac{\pi_{p,q}\operatorname{arcsin}_{p,q}r}{2r}\le\kappa_{p,q}(r)\le\frac{(p-1)q\pi_{p,q}A(p,q,r)}{2p}. \tag{2.19}$$

Proof. It is easily known that the functions $f(\theta)=(1-r^q\sin_{p,q}^q\theta)^{-1/p}$ and $g(\theta)=\cos_{p,q}\theta$ are increasing and decreasing in $(0,\frac{\pi_{p,q}}{2})$. Using Tchebychef's inequality (2.16) in Lemma 2.2 and substitution of variable $t=\sin_{p,q}\theta,\ rt=u$, then we obtain

$$\begin{aligned}\kappa_{p,q}(r)&\geq \tfrac{\pi_{p,q}}{2}\int_0^{\pi_{p,q}/2}\tfrac{\cos_{p,q}\theta}{(1-r^q\sin_{p,q}^q\theta)^{1/p}}d\theta\\ &=\tfrac{\pi_{p,q}}{2}\int_0^1\tfrac{dt}{(1-r^qt^q)^{1/p}}\\ &=\tfrac{\pi_{p,q}}{2}\int_0^r\tfrac{1}{(1-u^q)^{1/p}}\tfrac{1}{r}du\\ &=\tfrac{\pi_{p,q}}{2}\tfrac{\operatorname{arcsin}_{p,q}r}{r}.\end{aligned}$$

So, the proof of the first inequality is completed. Similarly, Putting

$$f(\theta)=(1-r^q\sin_{p,q}^q\theta)^{-1/p}$$

and $g(\theta)=\sin_{p,q}^{q-1}\theta$ in Lemma 2.2 and applying Lemma 2.3 and 2.4, we easily obtain the second inequality. Thus, we accomplished the inequalities (2.19). □

Putting $f(\theta)=(1-r^q\sin_{p,q}^q\theta)^{1/p}$ and $g(\theta)=\cos_{p,q}\theta$ or $g(\theta)=\sin_{p,q}^{q-1}\theta$ in Lemma 2.2, we easily obtain the following theorem.

Theorem 2.5. *For all $p,q\in(1,\infty), r\in(0,1)$ and $\theta\in(0,\frac{\pi_{p,q}}{2})$, we have*

$$\frac{\pi_{p,q}}{2}\frac{\lambda(p,q,r)}{r}\leq\varepsilon_{p,q}(r)\leq\frac{\pi_{p,q}}{2}\frac{\mu(p,q,r)}{r},\tag{2.20}$$

where $\lambda(p,q,r)=\frac{1-r^q}{r^{q(p-1)/p}}\int_0^r\frac{u^{(pq-q-p)/p}}{(1-u^q)^2}du$ and $\mu(p,q,r)=\int_0^r(1-u^q)^{1/p}du$.

3. Some Inequalities $(p,q)-$trigonometric and hyperbolic functions

Lemma 3.1. *Let the nonempty number set $D\subseteq(0,\infty)$, the mapping $f:D\longrightarrow J\subseteq(0,\infty)$ is a bijective function. Assume that function $g(x)$ is positive increasing and $\frac{f(x)}{g(x)}$ $(x\in D,k>0)$ is strictly increasing.*

(1) If $f(x)\geq y$ for all $x\in D$, then $g(x)y\leq f(x)g(f^{-1}(y))$ where $f^{-1}:J\longrightarrow D$ denotes the inverse function of f;

(2) If $f(x)\leq y$ for all $x\in D$, then $g(x)y\geq f(x)g(f^{-1}(y))$.

Proof. The proof of Lemma is similar to Theorem 2.1 of [17]. Here we omit the detail. □

Lemma 3.2. [7, Theorem 1.1] *For all $p,q\in(1,\infty),x\in(0,1)$, we have*

$$\begin{aligned}&(1)\quad x\left(1+\tfrac{x^q}{p(1+q)}\right)<\operatorname{arcsin}_{p,q}(x)<\min\left\{\tfrac{\pi_{p,q}}{2},(1-x^q)^{-1/(p(1+q))}\right\},\\ &(2)\quad \left(\tfrac{x^p}{1+x^q}\right)^{1/p}L(p,q,x)<\operatorname{arcsinh}_{p,q}(x)<\left(\tfrac{x^p}{1+x^q}\right)^{1/p}U(p,q,x),\end{aligned}$$

where

$$\begin{aligned}L(p,q,x)&=\max\left\{\left(1-\tfrac{qx^q}{p(1+q)(1+x^q)}\right)^{-1},(1+x^q)^{1/p}\left(\tfrac{pq+p+qx^q}{p(q+1)}\right)^{-1/q}\right\},\\ U(p,q,x)&=\left(1-\tfrac{x^q}{1+x^q}\right)^{-q/(p(q+1))}.\end{aligned}$$

Theorem 3.1. *For all $p,q\in(1,\infty)$, and $x\in(0,1)$, we have*

$$\frac{e^x}{\operatorname{arcsin}_{p,q}(x)}\leq\frac{e^{\sin_{p,q}\left(x\left(1+\frac{x^q}{p(1+q)}\right)\right)}}{x\left(1+\frac{x^q}{p(1+q)}\right)}.\tag{3.1}$$

Proof. Setting $g(x) = e^x$ and $f(x) = \operatorname{arcsin}_{p,q}(x), x \in (0,1)$ in Lemma 3.1, we have

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{1}{e^x}\left((1-x^q)^{-1/p} - \operatorname{arcsin}_{p,q}(x)\right) \geq 0.$$

In fact, since the function $(1-x^q)^{-1/p}$ is strictly increasing, we easily obtain

$$\operatorname{arcsin}_{p,q}(x) = \int_0^x (1-t^q)^{-1/p} dt \leq x\,(1-x^q)^{-1/p} < (1-x^q)^{-1/p}.$$

So, $\left(\frac{f(x)}{g(x)}\right)' \geq 0$ implies that the function $\frac{f(x)}{g(x)}$ is increasing for $x \in (0,1)$. Taking $y = x\left(1 + \frac{x^q}{p(1+q)}\right)$ and applying Lemma 3.2, we have $y \leq f(x)$. By using Lemma 3.1, we easily obtain inequality (3.1). □

Theorem 3.2. *For all $p, q \in (1,\infty)$, and $x \in (0,\xi)$, we have*

$$\frac{e^x}{\operatorname{arcsinh}_{p,q}(x)} \leq \frac{e^{\sinh_{p,q}\left(\left(\frac{x^p}{1+x^q}\right)^{1/p} U(p,q,x)\right)}}{\left(\frac{x^p}{1+x^q}\right)^{1/p} U(p,q,x)}, \tag{3.2}$$

where ξ is an unique positive root of equation $1 - x\,(1+x^q)^{1/p} = 0$.

Proof. Define $h(x) = 1 - x\,(1+x^q)^{1/p}$. A direct computation yields

$$h'(x) = -\left((1+x^q)^{1/p} + \frac{q}{p}x^q\,(1+x^q)^{(1-p)/p}\right) < 0.$$

Thus, the function $h(x)$ is decreasing on $(0,1)$. Setting $g(x) = e^x$ and

$$f(x) = \operatorname{arcsinh}_{p,q}(x), x \in (0,\xi)$$

in Lemma 3.1, we have

$$\begin{aligned}
&\left(\tfrac{f(x)}{g(x)}\right)' = \tfrac{1}{e^x}\left((1+x^q)^{1/p} - \operatorname{arcsinh}_{p,q}(x)\right)\\
&> \tfrac{1}{e^x}\left((1+x^q)^{1/p} - x\right)\\
&= \tfrac{1-x(1+x^q)^{1/p}}{e^x(1+x^q)^{1/p}} \geq 0.
\end{aligned}$$

Using Lemma 3.1 and Lemma 3.2, we easily obtain the inequality (3.2). □

Theorem 3.3. *For $p > 1, q > 2$ and $x \in (0,1)$, we have*

$$q\int_0^1 \frac{\cos_{p,q} x}{\sqrt[p]{1-x^q}}dx > p\int_0^1 \frac{x^{p-2}\sin_{p,q} x}{\sqrt[q]{1-x^p}}dx. \tag{3.3}$$

Proof. Putting $t = \operatorname{arcsin}_{p,q} x$, the left integral of (3.3) becomes

$$q\int_0^1 \frac{\cos_{p,q} x}{\sqrt[p]{1-x^q}}dx = q\int_0^{\pi_{p,q}/2} \cos_{p,q}(\sin_{p,q} t)dt. \tag{3.4}$$

Similarly, taking $t = \operatorname{arccos}_{p,q} x$, the right hand side of (3.3) is reduced into

$$p\int_0^1 \frac{x^{p-2}\sin_{p,q} x}{\sqrt[q]{1-x^p}}dx = q\int_0^{\pi_{p,q}/2} \sin_{p,q}^{q-2} t \sin_{p,q}(\cos_{p,q} t)dt. \tag{3.5}$$

Making use of the monotonicity of $\sin_{p,q}$ and $\cos_{p,q}$, we have

$$\sin_{p,q}^{q-2} t\sin_{p,q}(\cos_{p,q} t) < \sin_{p,q}(\cos_{p,q} t) < \cos_{p,q} t < \cos_{p,q}(\sin_{p,q} t).$$

Thus, the inequality (3.3) is proved. □

Theorem 3.4. *Let $p > 1,\ q > 1$ satisfy $1/p + 1/p' = 1$. For any $x \in (0,1)$, we have*

$$\frac{x}{2q} B_{x^{2q}} \left(1 - \frac{1}{p}, \frac{1}{2q}\right) \leq \operatorname{arcsin}_{p,q} x \operatorname{arcsinh}_{p,q} x < \frac{x^2}{(1-t^q)^{1/p}}, \tag{3.6}$$

where $B_{x^{2q}}\left(1 - \frac{1}{p}, \frac{1}{2q}\right)$ is incomplete beta function.

Proof. For the first inequality, it is easy to see that the function $\frac{1}{(1-t^q)^{1/p}}$ is strictly increasing and $\frac{1}{(1+t^q)^{1/p}}$ is strictly decreasing for $t \in (0,1)$. Using integral expression of $\operatorname{arcsin}_{p,q} x, \operatorname{arcsinh}_{p,q} x$ and Tchebychef's inequality, we have

$$\begin{aligned}
&\operatorname{arcsin}_{p,q} x \ \operatorname{arcsinh}_{p,q} x = \int_0^x \frac{1}{(1-t^q)^{1/p}} dt \int_0^x \frac{1}{(1+t^q)^{1/p}} dt \\
&\geq x \int_0^x \frac{1}{(1-t^{2q})^{1/p}} dt = \frac{x}{2q} \int_0^{x^{2q}} (1-u)^{-1/p} u^{(1/2p)-1} du \\
&= \frac{x}{2q} B_{x^{2q}} \left(1 - \frac{1}{p}, \frac{1}{2q}\right).
\end{aligned}$$

For the second inequality, we have

$$\begin{aligned}
&\operatorname{arcsin}_{p,q} x \ \operatorname{arcsinh}_{p,q} x = \int_0^x \frac{1}{(1-t^q)^{1/p}} dt \int_0^x \frac{1}{(1+t^q)^{1/p}} dt \\
&\leqslant \left(\int_0^x \frac{1}{1-t^q} dt\right)^{1/p} \left(\int_0^x 1^{p'} dt\right)^{1/p'} \left(\int_0^x \frac{1}{1+t^q} dt\right)^{1/p} \left(\int_0^x 1^{p'} dt\right)^{1/p'} \\
&= x^{2/p'} \left(\int_0^x \frac{1}{1-t^q} dt \int_0^x \frac{1}{1+t^q} dt\right)^{1/p} \\
&< x^{2/p'} \left(\frac{x^2}{1-x^q}\right)^{1/p} = \frac{x^2}{(1-t^q)^{1/p}}
\end{aligned}$$

by using Hölder's inequality. □

Lemma 3.3. [20, Example 1.51] *If $f(x)$ is a convex function for $x \in (0,\infty)$ and satisfies $f(0) = 0$, then $\frac{f(x)}{x}$ is monotone increasing.*

Lemma 3.4. [20, Example 1.52] *If $f_1(x)$ and $f_2(x)$ are convex functions in $x \in (0,\infty)$ and $f_1(x) \geq 0,\ f_2(x) \geq 0, f_1(0) = f_2(0) = 0$, then $\frac{f_1(x) f_2(x)}{x}$ is also convex in $x \in (0,\infty)$.*

It is easily known that the functions $\operatorname{arcsin}_{p,q} x$ and $\sinh_{p,q} x$ are convex in $x \in (0,1)$(The readers may see Lemma 2.10 in [13]). Applying Lemma 3.3 and 3.4, we easily obtain the following theorems.

Theorem 3.5. *Let $p > 1,\ q > 1$, the following conclusions hold:*
(i) The function $\frac{\operatorname{arcsin}_{p,q} x}{x}$ is monotone increasing in $x \in (0,1)$.
(ii) The function $\frac{\sinh_{p,q} x}{x}$ is monotone increasing in $x \in (0,\infty)$.
In particular, the following inequalities are valid for $s < r$

$$\frac{\operatorname{arcsin}_{p,q} s}{s} < \frac{\operatorname{arcsin}_{p,q} r}{r}, \quad r \in (0,1),$$
$$\frac{\sinh_{p,q} s}{s} < \frac{\sinh_{p,q} r}{r}, \quad r \in (0,\infty).$$

Theorem 3.6. *Let $p > 1,\ q > 1$, the function $\frac{\operatorname{arcsin}_{p,q} x \sinh_{p,q} x}{x}$ is convex in $x \in (0,1)$. In particular, the following inequalities is valid for $r, s \in (0,1)$*

$$\begin{aligned} & rs \operatorname{arcsin}_{p,q}\left(\tfrac{r+s}{2}\right) \sinh_{p,q}\left(\tfrac{r+s}{2}\right) \\ & \le \tfrac{s \operatorname{arcsin}_{p,q} r \sinh_{p,q} r}{r+s} + \tfrac{r \operatorname{arcsin}_{p,q} s \sinh_{p,q} s}{r+s}. \end{aligned}$$

Remark 3.1. Let the function $y = f(x)$ be convex on the interval $(0, \infty)$. Its Legendre transform has the new variable r and denoted by $g(r)$. It is easily known that the function $F(r, x) = rx - f(x)$ attains the maximum at the point $x = x(r)$. Setting $g(r) = F(r, x(r))$, we have the following Young inequality(See [5]):

$$rx \le f(x) + g(r). \tag{3.7}$$

Because the function $\sinh_{p,q} x$ is convex on $(0, \infty)$ and $p, q > 1$, its Legendre transform is

$$g(r) = r \operatorname{arccosh}_{p,q}(r) - (1 - r^p)^{1/q}$$

by simple computation. Considering the inequality (3.7), we have

$$rx + (1 - r^p)^{1/q} \le r \operatorname{arccosh}_{p,q}(r) + \sinh_{p,q}(x). \tag{3.8}$$

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