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MULTI-PARAMETER GENERALIZATION OF RADO-POPOVICIU INEQUALITIES

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ABSTRACT. By using methods on the theory of majorization, new generalizations of Rado's inequality and Popoviciu's inequality which involves multi-parameter are established.
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1. INTRODUCTION

Throughout the article, $\mathbb{R}$ denotes the set of real numbers, $x = (x_1, \cdots, x_n)$ denotes n-tuple (n-dimensional real vectors), the set of vectors can be written as

$$\mathbb{R}^n = \{x = (x_1, \cdots, x_n) : x_i \in \mathbb{R}, i = 1, \cdots, n\},$$

$$\mathbb{R}^n_+ = \{x = (x_1, \cdots, x_n) : x_i \geq 0, i = 1, \cdots, n\},$$

$$\mathbb{R}^n_{++} = \{x = (x_1, \cdots, x_n) : x_i > 0, i = 1, \cdots, n\}.$$

In particular, the notations $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote $\mathbb{R}^1$, $\mathbb{R}^1_+$ and $\mathbb{R}^1_{++}$, respectively.

Let $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. The elementary symmetric functions are defined by

$$E_k(x) = E_k(x_1, \cdots, x_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod\nolimits_{j=1}^{k} x_{i_j},\ k = 1, \cdots, n,$$

$E_0(x) = 1$, and $E_k(x) = 0$ for $k < 0$ or $k > n$. The dual form of the elementary symmetric functions are

$$E^*_k(x) = E^*_k(x_1, \cdots, x_n) = \prod_{1 \leq i_1 < \cdots < i_k \leq n} \sum\nolimits_{j=1}^{k} x_{i_j},\ k = 1, \cdots, n,$$

$E^*_0(x) = 1$, and $E^*_k(x) = 0$ for $k < 0$ or $k > n$.

For $x \in \mathbb{R}^n_{++}$, let

$$A(x) = \frac{\sum_{i=1}^n x_i}{n}, G_n(x) = \sqrt[n]{\prod_{i=1}^n x_i}.$$

The following inequalities:

$$n(A_n(x) - G_n(x)) \geq (n-1)(A_{n-1}(x) - G_{n-1}(x)) \tag{1}$$

$$\left(\frac{A_n(x)}{G_n(x)}\right)^n \geq \left(\frac{A_{n-1}(x)}{G_{n-1}(x)}\right)^{n-1} \tag{2}$$

are known in the bibliography as Rado's inequality and Popoviciu's inequality respectively (see Mitrinović and Vasić [1], p. 94). Inequalities (1) and (2) furnish a good route for joining arithmetic means and geometric means of positive numbers. This pair of inequalities has attracted considerable attention by many mathematicians, and has actuated a quantity of research articles giving their simple proofs providing diverse improvements, generalizations and analogs (see [1]-[8] and references therein).

The aim of this paper is to establish a new generalization of Rado's inequality and Popoviciu's inequality which involves multi-parameter, by Schur-concavity of the elementary symmetric function and its dual formula, as well as a simple majorization relation.

Our main results are the following.

Theorem 1. *Let $x \in \mathbb{R}^n_{++}$, $n \geq 2, 0 < \alpha \leq 1, \lambda > 0$. Then for $k = 1, \cdots, n$, we have*

$$\left(A_n(x) + \frac{(\lambda-1)x_n}{n}\right)^{\alpha} \geq \left(\left(1-\frac{1}{k}\right)(A_{n-1}(x))^{\alpha} + \frac{1}{k}\lambda^{\alpha}(x_n)^{\alpha}\right)^{\frac{k}{n}} \cdot (A_{n-1}(x))^{(1-\frac{k}{n})\alpha}. \tag{3}$$

Taking $\alpha = \lambda = 1$ and $k = n$, from the inequality (3), it follows that

Corollary 1. *Let $x \in \mathbb{R}^n_{++}$, $n \geq 2$, we have*

$$nA_n(x) - (n-1)A_{n-1}(x) \geq x_n. \tag{4}$$

Remark 1. By the arithmetic-geometric mean inequality, it follows that

$$\frac{x_n + G_{n-1}(x) + \cdots + G_{n-1}(x)}{n} \geq (x_n G_{n-1}(x) \cdots G_{n-1}(x))^{\frac{1}{n}},$$

i.e.

$$x_n + (n-1)G_{n-1}(x) \geq n\left(x_n G^{n-1}_{n-1}(x)\right)^{\frac{1}{n}} = nG_n(x),$$

i.e.

$$x_n \geq nG_n(x) - (n-1)G_{n-1}(x). \tag{5}$$

(4) and (5) together give

$$nA_n(x) - (n-1)A_{n-1}(x) \geq x_n \geq nG_n(x) - (n-1)G_{n-1}(x). \tag{6}$$

The inequality (6) refine the equivalent form of Rado's inequality (1).

Taking $\alpha = \lambda = 1$ and $k = 1$, from the inequality (3), it follows that

Corollary 2. *Let $x \in \mathbb{R}^n_{++}$, $n \geq 2$, we have*

$$A_n(x) \geq (x_n)^{\frac{1}{n}} (A_{n-1}(x))^{1-\frac{1}{n}}. \tag{7}$$

Remark 2. It is clear that inequality (7) is equivalent to

$$(A_n(x))^n \geq x_n (A_{n-1}(x))^{n-1},$$

and then

$$\left(\frac{A_n(x)}{G_n(x)}\right)^n \geq \frac{x_n(A_{n-1}(x))^{n-1}}{(G_n(x))^n} = \left(\frac{A_{n-1}(x)}{G_{n-1}(x)}\right)^{n-1}.$$

This shows that the inequality (7) is equivalent to Popoviciu's inequality (2).

Theorem 2. *Let $x \in \mathbb{R}^n_{++}$, $n \geq 2, 0 < \alpha \leq 1, \lambda > 0$. Then for $k = 1, \cdots, n$, we have*

$$\frac{\left(A_n(x) + \frac{(\lambda-1)x_n}{n}\right)^{k\alpha}}{(G_n(x))^{n\alpha}} \geq \lambda^\alpha \cdot \frac{k}{n} \cdot \frac{(A_{n-1}(x))^{(k-1)\alpha}}{(G_{n-1}(x))^{(n-1)\alpha}} + \left(1 - \frac{k}{n}\right) \frac{(A_{n-1}(x))^{k\alpha}}{(G_n(x))^{n\alpha}}. \quad (8)$$

Remark 3. When $\alpha = \lambda = 1$ and $k = n$, the inequality (8) is reduces to Popoviciu's inequality (2), and when $\alpha = \lambda = 1$ and $k = 1$, the inequality (8) is reduces to (4).

2. Definitions and Lemmas

We need the following definitions and auxiliary lemmas.

Definition 1. [9, 10] Let $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$.

(i) $x \geq y$ means $x_i \geq y_i$ for all $i = 1, 2, \cdots, n$.

(ii) Let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be increasing if $x \geq y$ implies $\varphi(x) \geq \varphi(y)$. φ is said to be decreasing if and only if $-\varphi$ is increasing.

Definition 2. [9, 10] Let $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$.

(i) x is said to be majorized by y (in symbols $x \prec y$) if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \cdots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq \cdots \geq y_{[n]}$ are rearrangements of x and y in a descending order.

(ii) Let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $x \prec y$ on Ω implies $\varphi(x) \leq \varphi(y)$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function on Ω.

The Schur-convexity described the ordering of majorization, the order-preserving functions were first comprehensively studied by I. Schur in 1923. It has important applications in combinatorial analysis, geometric inequalities, matrix theory, numerical analysis, etc. See [9].

Lemma 1. [9, 10, 11] *Let $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$. Then*

$$(\bar{x}, \cdots, \bar{x}) \prec (x_1, \cdots, x_n).$$

Lemma 2. [9, p. 115] *The elementary symmetric functions $E_k(x)$ is increasing and Schur-concave on $\mathbb{R}^n_+$.*

Lemma 3. [9, p. 123] *The dual form of the elementary symmetric functions $E_k^*(x)$ is increasing and Schur-concave on $\mathbb{R}^n_+$.*

Lemma 4. [10, p. 64] *Let the set $\mathbb{A} \subset \mathbb{R}$. The function $\phi : \mathbb{R}^n_+ \to \mathbb{R}$ is increasing and Schur-concave, and the function $g : \mathbb{A} \to \mathbb{R}_+$ is concave. Then the composite function $\psi(x_1, \cdots, x_n) = \phi(g(x_1), \cdots, g(x_n)) : \mathbb{A}^n \to \mathbb{R}$ is Schur-concave.*

Lemma 5. *Let $n \geq 2, t, \lambda > 0$, $g : \mathbb{R}_{++} \to \mathbb{R}_+$ is concave. Then for $k = 1, \cdots, n$, we have*

$$g\left(\frac{t+\lambda}{n}\right) \geq \left(\left(1 - \frac{1}{k}\right) g\left(\frac{t}{n-1}\right) + \frac{1}{k} \cdot g(\lambda)\right)^{\frac{k}{n}} \cdot \left(g\left(\frac{t}{n-1}\right)\right)^{1-\frac{k}{n}} \quad (9)$$

Proof. By Lemma 1, it follows that

$$u = \Bigg(\underbrace{\frac{t+\lambda}{n}, \cdots, \frac{t+\lambda}{n}}_{n}\Bigg) \prec \Bigg(\underbrace{\frac{t}{n-1}, \cdots, \frac{t}{n-1}}_{n-1}, \lambda\Bigg) = v, \quad (10)$$

and from Lemma 3 and Lemma 4, it follows that $E_k^*(g(x_1), \cdots, g(x_n))$ is Schur-concave on $\mathbb{R}_{++}^n$, combining (10) we get

$$E_k^*\left(g\left(\frac{t+\lambda}{n}\right), \cdots, g\left(\frac{t+\lambda}{n}\right)\right) \geq E_k^*\left(g\left(\frac{t}{n-1}\right), \cdots, g\left(\frac{t}{n-1}\right), g(\lambda)\right),$$

i.e.

$$\left(kg\left(\frac{t+\lambda}{n}\right)\right)^{\binom{n}{k}} \geq \left((k-1)g\left(\frac{t}{n-1}\right) + g(\lambda)\right)^{\binom{n-1}{k-1}} \left(kg\left(\frac{t}{n-1}\right)\right)^{\binom{n-1}{k}}. \tag{11}$$

Extracting root of $\binom{n}{k}$ the both sides in the inequality (11), we get

$$k\left(g\left(\frac{t+\lambda}{n}\right)\right) \geq \left((k-1)g\left(\frac{t}{n-1}\right) + g(\lambda)\right)^{\frac{k}{n}} \cdot \left(kg\left(\frac{t}{n-1}\right)\right)^{1-\frac{k}{n}} \tag{12}$$

Dividing both sides in the inequality (12) by $k = k^{\frac{k}{n}} \cdot k^{(1-\frac{k}{n})}$, we obtain the inequality (9). □

Lemma 6. *Let $n \geq 2, t, \lambda > 0$, $g : \mathbb{R}_{++} \to \mathbb{R}_+$ is concave. Then for $k = 1, \cdots, n$, we have*

$$\left(g\left(\frac{t+\lambda}{n}\right)\right)^k \geq \frac{k}{n} \cdot g(\lambda)\left(g\left(\frac{t}{n-1}\right)\right)^{k-1} + \left(1-\frac{k}{n}\right)\left(g\left(\frac{t}{n-1}\right)\right)^k \tag{13}$$

Proof. From Lemma 2 and Lemma 4, it follows that $E_k(g(x_1), \cdots, g(x_n))$ is Schur-concave on $\mathbb{R}_{++}^n$, combining (10) we get

$$E_k\left(g\left(\frac{t+\lambda}{n}\right), \cdots, g\left(\frac{t+\lambda}{n}\right)\right) \geq E_k\left(g\left(\frac{t}{n-1}\right), \cdots, g\left(\frac{t}{n-1}\right), g(\lambda)\right),$$

i.e.

$$\binom{n}{k}\left(g\left(\frac{t+\lambda}{n}\right)\right)^k \geq \binom{n-1}{k-1} g(\lambda)\left(g\left(\frac{t}{n-1}\right)\right)^{k-1} + \binom{n-1}{k}\left(g\left(\frac{t}{n-1}\right)\right)^k. \tag{14}$$

Dividing both sides in the inequality (14) by $\binom{n}{k}$, we obtain the inequality (13). □

3. Proof of Main Results

Proof of Theorem 1: Taking $g(y) = y^\alpha, 0 < \alpha \leq 1$ and $t = (\Sigma_{i=1}^{n-1} x_i)/x_n$, from (9) it is deduced that

$$\left(\frac{A_n(x) + \frac{(\lambda-1)x_n}{n}}{x_n}\right)^\alpha \geq \left(\left(1-\frac{1}{k}\right)\left(\frac{A_{n-1}(x)}{x_n}\right)^\alpha + \frac{1}{k}\lambda^\alpha\right)^{\frac{k}{n}} \cdot \left(\frac{A_{n-1}(x)}{x_n}\right)^{(1-\frac{k}{n})\alpha}. \tag{15}$$

Multiplying both sides in the inequality (15) by $(x_n)^\alpha = (x_n)^{\frac{k}{n}\alpha}(x_n)^{(1-\frac{k}{n})\alpha}$, we obtain the inequality (3).

The proof of Theorem 1 is completed.

Proof of Theorem 2: Taking $g(y) = y^\alpha, 0 < \alpha \leq 1$ and $t = (\Sigma_{i=1}^{n-1} x_i)/x_n$, from (13) it is deduced that

$$\left(\frac{A_n(x) + \frac{(\lambda-1)x_n}{n}}{x_n}\right)^{k\alpha} \geq \lambda^\alpha \frac{k}{n}\left(\frac{A_{n-1}(x)}{x_n}\right)^{(k-1)\alpha} + \left(1-\frac{k}{n}\right)\left(\frac{A_{n-1}(x)}{x_n}\right)^{k\alpha}. \tag{16}$$

Multiplying both sides in the inequality (16) by $(x_n)^{k\alpha}\left(\prod_{i=1}^{n} x_i^{\alpha}\right)^{-1}$, we obtain the inequality (8).

The proof of Theorem 2 is completed.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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