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SCHUR CONVEXITY FOR A COMPOSITE FUNCTION OF COMPLETE SYMMETRIC FUNCTION

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Abstract. By properties of Schur-convex function, Schur geometrically convex function and Schur harmonically convex function, Schur-convexity, Schur geometric and harmonic convexities of a composite function for the complete symmetric function are simply proved.

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1. Introduction

Throughout the article, $\mathbb{R}$ denotes the set of real numbers, $\boldsymbol{x} = (x_1, \cdots, x_n)$ denotes n-tuple (n-dimensional real vectors), the set of vectors can be written as

$$\mathbb{R}^n = \{\boldsymbol{x} = (x_1, \cdots, x_n) : x_i \in \mathbb{R}, i = 1, \cdots, n\},$$
$$\mathbb{R}^n_+ = \{\boldsymbol{x} = (x_1, \cdots, x_n) : x_i > 0, i = 1, \cdots, n\},$$
$$\mathbb{R}^n_- = \{\boldsymbol{x} = (x_1, \cdots, x_n) : x_i < 0, i = 1, \cdots, n\}.$$

In particular, the notations $\mathbb{R}$ and $\mathbb{R}_+$ denote $\mathbb{R}^1$ and $\mathbb{R}^1_+$, respectively.

In recent years, the Schur-convexity, Schur geometric convexity and Schur harmonic convexity of various symmetric functions are hot topic of inequality research ([7]-[24]). The following complete symmetric function is an important class of symmetric functions.

For $\boldsymbol{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$, the complete symmetric function $c_n(\boldsymbol{x}, r)$ is defined as

$$c_n(\boldsymbol{x}, r) = \sum_{i_1+i_2+\cdots+i_n=r} x_1^{i_1} \cdots x_n^{i_n}, \tag{1}$$

where $c_0(\boldsymbol{x}, r) = 1,\ r \in \{1, 2, \cdots, n\},\ i_1, i_2, \cdots, i_n$ are non-negative integers.

It has been investigated by many mathematicians and there are many interesting results in the literature.

Guan [11] discussed the Schur-convexity of $c_n(\boldsymbol{x}, r)$ and proved that $c_n(\boldsymbol{x}, r)$ is increasing and Schur-convex on $\mathbb{R}^n_+$. Subsequently, Chu et al. [8] proved that $c_n(\boldsymbol{x}, r)$ is Schur geometrically convex and harmonically convex on $\mathbb{R}^n_+$.

Recently, Sun et al. [12] studied the Schur-convexity, Schur geometric and harmonic convexities of the following composite function of $c_n(\boldsymbol{x}, r)$

$$F_n(\boldsymbol{x}, r) = \sum_{i_1+i_2+\cdots+i_n=r} \left(\frac{x_1}{1-x_1}\right)^{i_1} \cdots \left(\frac{x_n}{1-x_n}\right)^{i_n}. \tag{2}$$

Using the Lemma 1, Lemma 2 and Lemma 3 in second sections, they proved the following Theorem A, Theorem B and Theorem C, respectively.

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Theorem A. *For* $\boldsymbol{x} = (x_1, \cdots, x_n) \in (0,1)^n \cup (1,+\infty)^n$ *and* $r \in \mathbb{N}$,

(*i*) $F_n(\boldsymbol{x}, r)$ *is increasing and Schur convex on* $(0,1)^n$;

(*ii*) *if* r *is even integer* (*or odd integer, respectively*), *then* $F_n(\boldsymbol{x}, r)$ *is Schur convex* (*or Schur concave, respectively*) *on* $(1,+\infty)^n$, *and is decreasing* (*or increasing, respectively*).

Theorem B. *For* $\boldsymbol{x} = (x_1, \cdots, x_n) \in (0,1)^n \cup (1,+\infty)^n$ *and* $r \in \mathbb{N}$,

(*i*) $F_n(\boldsymbol{x}, r)$ *is Schur geometrically convex on* $(0,1)^n$;

(*ii*) *if* r *is even integer* (*or odd integer, respectively*), *then* $F_n(\boldsymbol{x}, r)$ *is Schur geometrically convex* (*or concave, respectively*) *on* $(1,+\infty)^n$.

Theorem C. *For* $\boldsymbol{x} = (x_1, \cdots, x_n) \in (0,1)^n \cup (1,+\infty)^n$ *and* $r \in \mathbb{N}$,

(*i*) $F_n(\boldsymbol{x}, r)$ *is Schur harmonically convex on* $(0,1)^n$;

(*ii*) *if* r *is even integer* (*or odd integer, respectively*), *then* $F_n(\boldsymbol{x}, r)$ *is Schur harmonically convex* (*or concave, respectively*) *on* $(1,+\infty)^n$.

In this paper, by the properties of Schur convex function, Schur geometrically convex function and Schur harmonically convex function, we will be very simple to prove the above results.

2. Definitions and Lemmas

For convenience, we introduce some definitions as follows.

Definition 1. [1, 2] Let $\boldsymbol{x} = (x_1, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n$.

(*i*) $\boldsymbol{x} \geq \boldsymbol{y}$ means $x_i \geq y_i$ for all $i = 1, 2, \cdots, n$.

(*ii*) Let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be increasing if $\boldsymbol{x} \geq \boldsymbol{y}$ implies $\varphi(\boldsymbol{x}) \geq \varphi(\boldsymbol{y})$. φ is said to be decreasing if and only if $-\varphi$ is increasing.

Definition 2. [1, 2] Let $\boldsymbol{x} = (x_1, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n$.

(*i*) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \cdots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.

(*ii*) Let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function on Ω.

Definition 3. [1, 2] Let $\boldsymbol{x} = (x_1, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n$.

(*i*) $\Omega \subset \mathbb{R}^n$ is said to be a convex set if $\boldsymbol{x}, \boldsymbol{y} \in \Omega, 0 \leq \alpha \leq 1$ implies $\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} = (\alpha x_1 + (1-\alpha)y_1, \cdots, \alpha x_n + (1-\alpha)y_n) \in \Omega$.

(*ii*) Let $\Omega \subset \mathbb{R}^n$ be convex set. A function φ: $\Omega \to \mathbb{R}$ is said to be a convex function on Ω if

$$\varphi(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \leq \alpha\varphi(\boldsymbol{x}) + (1-\alpha)\varphi(\boldsymbol{y})$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, and all $\alpha \in [0,1]$. φ is said to be a concave function on Ω if and only if $-\varphi$ is convex function on Ω.

Definition 4. [1, 2]

(*i*) A set $\Omega \subset \mathbb{R}^n$ is called a symmetric set, if $\boldsymbol{x} \in \Omega$ implies $\boldsymbol{x}P \in \Omega$ for every $n \times n$ permutation matrix P.

(*ii*) A function $\varphi : \Omega \to \mathbb{R}$ is called symmetric if for every permutation matrix P, $\varphi(\boldsymbol{x}P) = \varphi(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$.

Lemma 1. (*Schur-convex function decision theorem*)[1, p. 84]*: Let* $\Omega \subset \mathbb{R}^n$ *be symmetric and have a nonempty interior convex set.* Ω^0 *is the interior of* Ω*.* $\varphi : \Omega \to \mathbb{R}$ *is continuous on* Ω *and differentiable in* Ω^0*. Then* φ *is the* $Schur-convex$ $(Schur-concave)$ *function if and only if* φ *is symmetric on* Ω *and*

$$(x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \geq 0 (\leq 0) \tag{3}$$

holds for any $\boldsymbol{x} \in \Omega^0$.

The first systematical study of the functions preserving the ordering of majorization was made by Issai Schur in 1923. In Schur's honor, such functions are said to be "Schur-convex". It can be used extensively in analytic inequalities, combinatorial optimization, quantum physics, information theory, and other related fields. See [1].

Definition 5. [3] Let $\boldsymbol{x} = (x_1, \cdots, x_n) \in \mathbb{R}_+^n$ and $\boldsymbol{y} = (y_1, \cdots, y_n) \in \mathbb{R}_+^n$.

(*i*) $\Omega \subset \mathbb{R}_+^n$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta, \cdots, x_n^\alpha y_n^\beta) \in \Omega$ for all $\boldsymbol{x}$,$\boldsymbol{y} \in \Omega$ and α,$\beta \in [0, 1]$ such that $\alpha + \beta = 1$.

(*ii*) Let $\Omega \subset \mathbb{R}_+^n$. The function φ: $\Omega \to \mathbb{R}_+$ is said to be Schur geometrically convex function on Ω if $(\log x_1, \cdots, \log x_n) \prec (\log y_1, \cdots, \log y_n)$ on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. The function φ is said to be a Schur geometrically concave function on Ω if and only if $-\varphi$ is Schur geometrically convex function.

Lemma 2. (*Schur geometrically convex function decision theorem*)[3]*: Let* $\Omega \subset \mathbb{R}_+^n$ *be a symmetric and geometrically convex set with a nonempty interior* Ω^0*. Let* $\varphi : \Omega \to \mathbb{R}_+$ *be continuous on* Ω *and differentiable in* Ω^0*. If* φ *is symmetric on* Ω *and*

$$(\log x_1 - \log x_2)\left(x_1 \frac{\partial \varphi}{\partial x_1} - x_2 \frac{\partial \varphi}{\partial x_2}\right) \geq 0 \quad (\leq 0) \tag{4}$$

holds for any $\boldsymbol{x} = (x_1, \cdots, x_n) \in \Omega^0$*, then* φ *is a Schur geometrically convex* (*Schur geometrically concave*) *function.*

The Schur geometric convexity was proposed by Zhang [3] in 2004, and was investigated by Chu et al. [4], Guan [5], Sun et al. [6], and so on. We also note that some authors use the term "Schur multiplicative convexity".

In 2009, Chu ([7], [8], [9]) introduced the notion of Schur harmonically convex function, and some interesting inequalities were obtained.

Definition 6. [7] Let $\Omega \subset \mathbb{R}_+^n$.

(*i*) A set Ω is said to be harmonically convex if $\dfrac{\boldsymbol{xy}}{\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}} \in \Omega$ for every $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\lambda \in [0, 1]$, where $\boldsymbol{xy} = \sum_{i=1}^n x_i y_i$ and $\dfrac{1}{\boldsymbol{x}} = (\dfrac{1}{x_1}, \cdots, \dfrac{1}{x_n})$.

(*ii*) A function $\varphi : \Omega \to \mathbb{R}_+$ is said to be Schur harmonically convex on Ω if $\dfrac{1}{\boldsymbol{x}} \prec \dfrac{1}{\boldsymbol{y}}$ implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. A function φ is said to be a Schur harmonically concave function on Ω if and only if $-\varphi$ is a Schur harmonically convex function.

Lemma 3. (*Schur harmonically convex function decision theorem*)[7]*: Let $\Omega \subset \mathbb{R}_+^n$ be a symmetric and harmonically convex set with inner points and let $\varphi : \Omega \to \mathbb{R}_+$ be a continuously symmetric function which is differentiable on Ω^0. Then φ is Schur harmonically convex* (*Schur harmonically concave*) *on Ω if and only if*

$$(x_1 - x_2)\left(x_1^2 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_1} - x_2^2 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_2}\right) \geq 0 \quad (\leq 0), \quad \boldsymbol{x} \in \Omega^0. \tag{5}$$

Lemma 4. *If r is even integer* (*or odd integer, respectively*)*, then $c_n(\boldsymbol{x}, r)$ is decreasing and Schur-convex* (*or increasing and Schur-concave, respectively*) *on $\mathbb{R}_-^n$.*

Proof. Notice that

$$\begin{aligned} & c_n(-\boldsymbol{x}, r) \\ & = \sum_{i_1 + \cdots + i_n = r} (-x_1)^{i_1} \cdots (-x_n)^{i_n} \\ & = (-1)^{i_1 + \cdots + i_n} \sum_{i_1 + \cdots + i_n = r} x_1^{i_1} \cdots x_n^{i_n} \\ & = (-1)^r c_n(\boldsymbol{x}, r), \end{aligned}$$

i.e.

$$c_n(-\boldsymbol{x}, r) = (-1)^r c_n(\boldsymbol{x}, r).$$

If r is even integer, then $c_n(\boldsymbol{x}, r) = c_n(-\boldsymbol{x}, r)$. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_-^n$, if $\boldsymbol{x} \prec \boldsymbol{y}$, then $-\boldsymbol{x} \prec -\boldsymbol{y}$ and $-\boldsymbol{x}, -\boldsymbol{y} \in \mathbb{R}_+^n$, but $c_n(\boldsymbol{x}, r)$ is Schur convex in $\mathbb{R}_+^n$, so that $c_n(-\boldsymbol{x}, r) \leq c_n(-\boldsymbol{y}, r)$, i.e. $c_n(\boldsymbol{x}, r) \leq c_n(\boldsymbol{y}, r)$, this shows that $c_n(\boldsymbol{x}, r)$ is Schur convex in $\mathbb{R}_-^n$. If $\boldsymbol{x} \leq \boldsymbol{y}$, then $-\boldsymbol{x} \geq -\boldsymbol{y}$, but $c_n(\boldsymbol{x}, r)$ is increasing in $\mathbb{R}_+^n$, so that $c_n(-\boldsymbol{x}, r) \geq c_n(-\boldsymbol{y}, r)$, i.e. $c_n(\boldsymbol{x}, r) \geq c_n(\boldsymbol{y}, r)$, this shows that $c_n(\boldsymbol{x}, r)$ is decreasing in $\mathbb{R}_-^n$.

If r is odd integer, then $c_n(\boldsymbol{x}, r) = -c_n(-\boldsymbol{x}, r)$. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_-^n$, if $\boldsymbol{x} \prec \boldsymbol{y}$, then $-\boldsymbol{x} \prec -\boldsymbol{y}$ and $-\boldsymbol{x}, -\boldsymbol{y} \in \mathbb{R}_+^n$, but $c_n(\boldsymbol{x}, r)$ is Schur convex in $\mathbb{R}_+^n$, so that $c_n(-\boldsymbol{x}, r) \leq c_n(-\boldsymbol{y}, r)$, i.e. $c_n(\boldsymbol{x}, r) \geq c_n(\boldsymbol{y}, r)$, this shows that $c_n(\boldsymbol{x}, r)$ is Schur concave in $\mathbb{R}_-^n$. If $\boldsymbol{x} \leq \boldsymbol{y}$, then $-\boldsymbol{x} \geq -\boldsymbol{y}$, but $c_n(\boldsymbol{x}, r)$ is increasing in $\mathbb{R}_+^n$, so that $c_n(-\boldsymbol{x}, r) \geq c_n(-\boldsymbol{y}, r)$, i.e. $c_n(\boldsymbol{x}, r) \leq c_n(\boldsymbol{y}, r)$, this shows that $c_n(\boldsymbol{x}, r)$ is increasing in $\mathbb{R}_-^n$. □

Lemma 5. ([1, p. 91], [2, p. 64-65]) *Let the set $\mathbb{A}, \mathbb{B} \subset \mathbb{R}$, $\varphi : \mathbb{B}^n \to \mathbb{R}$, $f : \mathbb{A} \to \mathbb{B}$ and $\psi(x_1, \cdots, x_n) = \varphi(f(x_1), \cdots, f(x_n)) : \mathbb{A}^n \to \mathbb{R}$.*

(*i*) *If φ is increasing and Schur-convex and f is increasing and convex, then ψ is increasing and Schur-convex.*
(*ii*) *If φ is decreasing and Schur-convex and f is increasing and concave, then ψ is decreasing and Schur-convex.*
(*iii*) *If φ is increasing and Schur-concave and f is increasing and concave, then ψ is increasing and Schur-concave.*

Lemma 6. *Let the set $\Omega \subset \mathbb{R}_+^n$. The function $\varphi : \Omega \to \mathbb{R}_+$ is differentiable.*

(*i*) *If φ is increasing and Schur-convex, then φ is Schur-geometrically convex.*
(*ii*) *If φ is decreasing and Schur-concave, then φ is Schur-geometrically concave.*

Proof. We only give the proof of Lemma 6 (*i*) in detail. Similar argument leads to the proof of Lemma 3 (*ii*).

For $\boldsymbol{x} \in I \subset \mathbb{R}_+$ and $x_1 \neq x_2$, we have

$$\begin{aligned}\Delta &= (\log x_1 - \log x_2)\left(x_1\frac{\partial\varphi}{\partial x_1} - x_2\frac{\partial\varphi}{\partial x_2}\right)\\ &= (\log x_1 - \log x_2)\left(x_1\frac{\partial\varphi}{\partial x_1} - x_1\frac{\partial\varphi}{\partial x_2} + x_1\frac{\partial\varphi}{\partial x_2} - x_2\frac{\partial\varphi}{\partial x_2}\right)\\ &= x_1\frac{\log x_1 - \log x_2}{x_1 - x_2}(x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) + \frac{\partial\varphi}{\partial x_2}(x_1 - x_2)(\log x_1 - \log x_2).\end{aligned}$$

Since φ is Schur-convex on Ω, by Lemma 1, we have

$$(x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) \geq 0.$$

Notice that φ and $y = \log x$ is increasing, we have $\frac{\partial\varphi}{\partial x_2} \geq 0$, $\frac{\log x_1 - \log x_2}{x_1 - x_2} \geq 0$ and $(x_1 - x_2)(\log x_1 - \log x_2) \geq 0$, so that $\Delta \geq 0$, by Lemma 2, it follows that φ is Schur geometrically convex on Ω. □

Lemma 7. *Let the set $\Omega \subset \mathbb{R}_+^n$. The function $\varphi : \Omega \to \mathbb{R}_+$ is differentiable.*

(i) *If φ is increasing and Schur-convex, then φ is Schur-harmonically convex.*
(ii) *If φ is decreasing and Schur-concave, then φ is Schur-harmonically concave.*

Proof. We only give the proof of Lemma 7 (ii) in detail. Similar argument leads to the proof of Lemma 4 (i).

For $\boldsymbol{x} \in I \subset \mathbb{R}_+$ and $x_1 \neq x_2$, we have

$$\begin{aligned}\Lambda &= (x_1 - x_2)\left(x_1^2\frac{\partial\varphi}{\partial x_1} - x_2^2\frac{\partial\varphi}{\partial x_2}\right)\\ &= (x_1 - x_2)\left(x_1^2\frac{\partial\varphi}{\partial x_1} - x_1^2\frac{\partial\varphi}{\partial x_2} + x_1^2\frac{\partial\varphi}{\partial x_2} - x_2^2\frac{\partial\varphi}{\partial x_2}\right)\\ &= x_1^2(x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) + \frac{\partial\varphi}{\partial x_2}(x_1 - x_2)\left(x_1^2 - x_2^2\right).\end{aligned}$$

Since φ is Schur-concave on Ω, by Lemma 1, we have

$$(x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) \leq 0.$$

Notice that φ is decreasing and $y = x^2 (x > 0)$ is increasing, we have $\frac{\partial\varphi}{\partial x_2} \leq 0$ and $(x_1 - x_2)\left(x_1^2 - x_2^2\right) \geq 0$, so that $\Lambda \leq 0$, by Lemma 3, it follows that φ is Schur harmonically concave on Ω. □

3. Simple Proof of Theorems

Proof of Theorem A: Let $g(t) = \frac{t}{1-t}$. Directly calculating yields $g'(t) = \frac{1}{(1-t)^2}$ and $g''(t) = \frac{2}{(1-t)^3}$, it is to see that g is increasing and convex on $(0, 1)$ and g is increasing and concave on $(1, +\infty)$.

Since $c_n(\boldsymbol{x}, r)$ is increasing and Schur-convex in $\mathbb{R}_+^n$, from Lemma 2 (i) it follows that $F_n(\boldsymbol{x}, r)$ is increasing and Schur-convex in $(0, 1)^n$.

If r is even integer, then from Lemma 1, we known that $c_n(\boldsymbol{x}, r)$ is decreasing and Schur-convex, moreover g is increasing and concave on $(1, +\infty)$. By Lemma 2 (ii), it follows that $F_n(\boldsymbol{x}, r)$ is decreasing and Schur-convex.

If r is odd integer, then from Lemma 1, we known that $c_n(\boldsymbol{x}, r)$ is increasing and Schur-concave, moreover g is increasing and concave on $(1, +\infty)$. By Lemma 2 (iii), it follows that $F_n(\boldsymbol{x}, r)$ is increasing and Schur-concave.

The proof of Theorem A is completed.

Proof of Theorem B: From Theorem 1(i) and Lemma 6, it follows that Theorem B (i) holds.

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$$E_n(\boldsymbol{x}, r) = \sum_{i_1+i_2+\cdots+i_n=r} \left(\frac{x_1}{x_1-1}\right)^{i_1} \cdots \left(\frac{x_n}{x_n-1}\right)^{i_n}. \tag{6}$$

Let $h(t) = \dfrac{t}{t-1}$. Directly calculating yields $h'(t) = \dfrac{1}{(t-1)^2}$ and $h''(t) = \dfrac{2}{(t-1)^3}$, it is to see that h is increasing and convex on $(1, +\infty)$. Since $c_n(\boldsymbol{x}, r)$ is increasing and Schur-convex in $\mathbb{R}^n_+$, from Lemma 2 (i) it follows that $E_n(\boldsymbol{x}, r)$ is increasing and Schur-convex in $(1, +\infty)$. And then, from Lemma 6, it follows that $E_n(\boldsymbol{x}, r)$ Schur geometrically convex on $(1, +\infty)$.

it is to see that when r is even integer, $E_n(\boldsymbol{x}, r) = F_n(\boldsymbol{x}, r)$, and when r is odd integer, $E_n(\boldsymbol{x}, r) = -F_n(\boldsymbol{x}, r)$. And then from Schur geometric convexity of $E_n(\boldsymbol{x}, r)$ on $(1, +\infty)$, it follows that Theorem B (ii) holds.

The proof of Theorem B is completed.

Proof of Theorem C: From Theorem 1(i) and Lemma 7, it follows that Theorem C (i) holds.

From the proof of Theorem A, it is known that $E_n(\boldsymbol{x}, r)$ is increasing and Schur-convex in $(1, +\infty)$, from Lemma 7, it follows that $E_n(\boldsymbol{x}, r)$ Schur harmonically convex on $(1, +\infty)$. Notice that when r is even integer, $E_n(\boldsymbol{x}, r) = F_n(\boldsymbol{x}, r)$, and when r is odd integer, $E_n(\boldsymbol{x}, r) = -F_n(\boldsymbol{x}, r)$, we immediately conclude that Theorem C (ii) holds.

The proof of Theorem C is completed.

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