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ON SCHUR m-POWER CONVEXITY OF A CLASS OF MULTIPLICATIVELY CONVEX FUNCTIONS

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Abstract. By properties of Schur-geometrically convex function and Schur m-power convex function, we very simple prove that the symmetric function

$$F_{n,k}(\boldsymbol{x}, r) = \prod_{1 \le i_1 < \ldots < i_k \le n} f\left((\sum\nolimits_{j=1}^{k} x_{i_j}^r)^{1/r} \right),\ k = 1, \ldots, n.$$

is Schur m-power convex, where f is a multiplicatively convex function, $\boldsymbol{x} \in \mathbb{R}_+^n, r > 0$ and $m < 0$.

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1. Introduction

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers, $\boldsymbol{x} = (x_1, , \cdots, x_n)$ denotes n-tuple (n-dimensional real vectors), the set of vectors can be written as

$$\mathbb{R}^n = \{\boldsymbol{x} = (x_1, \cdots, x_n) : x_i \in \mathbb{R}, i = 1, \cdots, n\},$$

$$\mathbb{R}_+^n = \{\boldsymbol{x} = (x_1, \cdots, x_n) : x_i > 0, i = 1, \cdots, n\}.$$

In particular, the notations $\mathbb{R}$ and $\mathbb{R}_+$ denote $\mathbb{R}^1$ and $\mathbb{R}_+^1$, respectively.

In 2013, Shi and Zhang [24] studied the symmetric function

$$F_k^*(\boldsymbol{x}) = \prod_{1 \le i_1 < \ldots < i_k \le n} \sum\nolimits_{j=1}^{k} f(x_{i_j}),\ k = 1, \ldots, n, \tag{1}$$

and obtained the following result:

Theorem A. *Let $I \subset \mathbb{R}$ is a symmetric convex set with non-empty interior and let $f : I \to \mathbb{R}$ be continuous on I and differentiable in the interior of I. If f is a log-convex function, then for any $k = 1, 2, ..., n$, $F_k^*(\boldsymbol{x})$ is Schur convex, Schur geometrically and harmonically convex on I^n.*

Recently, Wang and Yang [25] defined the following symmetric function:

$$F_{n,k}(\boldsymbol{x}, r) = \prod_{1 \le i_1 < \ldots < i_k \le n} f\left((\sum\nolimits_{j=1}^{k} x_{i_j}^r)^{1/r} \right),\ k = 1, \ldots, n. \tag{2}$$

They using Schur m-power convex function decision theorem, i.e. Lemma 4 in the second section, proved the following result:

Theorem B. *Let $\Omega \subset \mathbb{R}_+^n$ be a symmetric convex set with non-empty interior and $f : \Omega \to \mathbb{R}_+^n$ is continuous on Ω and differentiable in the interior of Ω. If f is increasing and multiplicatively convex, then for $m \le 0$ and $r > 0$, $F_{n,k}(\boldsymbol{x}, r)$ defined in (2) are Schur m-power convex on Ω, where $k = 1, 2, ..., n$.*

In this paper, by properties of Schur geometrically convex function and Schur m-power convex function, we give a very simple proof of Theorem B.

2. Definitions and lemmas

In this section we will recall usefull definitions and lemmas.

Definition 1. [1, 4] Let $\boldsymbol{x} = (x_1, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n$.

- (i) We say $\boldsymbol{y}$ majorizes $\boldsymbol{x}$ ($\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$), denoted by $\boldsymbol{x} \prec \boldsymbol{y}$, if $\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}$ for $k = 1, 2, \cdots, n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, where $x_{[1]} \ge \cdots \ge x_{[n]}$ and $y_{[1]} \ge \cdots \ge y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.
- (ii) Let $\Omega \subset \mathbb{R}^n$, a function φ: $\Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$. A function φ is said to be a Schur concave function on Ω if and only if $-\varphi$ is Schur convex function on Ω.

Definition 2. [1, 4] Let $\boldsymbol{x} = (x_1, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n, 0 \le \alpha \le 1$. A set $\Omega \subset \mathbb{R}^n$ is said to be a convex set if $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ implies $\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} = (\alpha x_1 + (1-\alpha)y_1, \cdots, \alpha x_n + (1-\alpha)y_n) \in \Omega$.

Definition 3. [1, 4]

- (i) A set $\Omega \subset \mathbb{R}^n$ is called a symmetric set, if $\boldsymbol{x} \in \Omega$ implies $\boldsymbol{x}P \in \Omega$ for every $n \times n$ permutation matrix P.
- (ii) A function $\varphi : \Omega \to \mathbb{R}$ is called symmetric if for every permutation matrix P, $\varphi(\boldsymbol{x}P) = \varphi(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$.

Definition 4. Let $\Omega \subset \mathbb{R}^n_+$, $\boldsymbol{x} = (x_1, \cdots, x_n) \in \Omega$ and $\boldsymbol{y} = (y_1, \cdots, y_n) \in \Omega$.

- (i) [2, p. 64] A set Ω is called a geometrically convex set if $(x_1^{\alpha} y_1^{\beta}, \cdots, x_n^{\alpha} y_n^{\beta}) \in \Omega$ for all $\boldsymbol{x}$, $\boldsymbol{y} \in \Omega$ and α, $\beta \in [0, 1]$ such that $\alpha + \beta = 1$.
- (ii) [2, p. 107] A function φ: $\Omega \to \mathbb{R}_+$ is said to be a Schur geometrically convex function on Ω if $(\log x_1, \cdots, \log x_n) \prec (\log y_1, \cdots, \log y_n)$ on Ω implies $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$. A function φ is said to be a Schur geometrically concave function on Ω if and only if $-\varphi$ is a Schur geometrically convex function.

Definition 5. [14, 3] Let $\Omega \subset \mathbb{R}^n_+$.

- (i) A set Ω is said to be a harmonically convex set if $\frac{\boldsymbol{x}\boldsymbol{y}}{\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}} \in \Omega$ for every $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\lambda \in [0, 1]$, where $\boldsymbol{x}\boldsymbol{y} = \sum_{i=1}^{n} x_i y_i$ and $\frac{1}{\boldsymbol{x}} = \left(\frac{1}{x_1}, \cdots, \frac{1}{x_n}\right)$.
- (ii) A function $\varphi : \Omega \to \mathbb{R}_+$ is said to be a Schur harmonically convex function on Ω if $\frac{1}{\boldsymbol{x}} \prec \frac{1}{\boldsymbol{y}}$ implies $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$. A function φ is said to be a Schur harmonically concave function on Ω if and only if $-\varphi$ is a Schur harmonically convex function.

Schur convex, Schur geometrically convex and Schur harmonically convex were introduced by Schur [1], Zhang [2] and Chu [3], respectively, and played a key role in analytic inequalities [1-28]. Moreover, the theory of convex functions and Schur convex functions is one of the most important research fields in modern analysis and geometry. Recently, Yang present the Schur f-convexity in [26] as follows.

Definition 6. [26] Let $\Omega \subset \mathbb{R}^n$ be a set with nonempty interior and f be a strictly monotone function defined on Ω. Let $f(\boldsymbol{x}) = (f(x_1), ..., f(x_n))$ and $f(\boldsymbol{y}) = (f(y_1), ..., f(y_n))$. Then function $\varphi : \Omega \to \mathbb{R}$ is said to be Schur f-convex on Ω if

$f(\boldsymbol{x}) \prec f(\boldsymbol{y})$ on Ω implies $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$. φ is said to be Schur f-concave if $-\varphi$ is Schur f-convex.

Take $f(x) = x, \log x, x^{-1}$ in Definition 6, it yields the Schur convexity, Schur geometrical convexity and Schur harmonic convexity. It is clear that the Schur f-convexity is a generalization of the Schur convexity mentioned above. In general, we have:

Definition 7. [26] Let $f : \mathbb{R}_+^n \to \mathbb{R}^n$ be defined by $f(x) = (x^m - 1)/m$ if $m \neq 0$ and $f(x) = \log x$ if $m = 0$. Then function: $\psi : \Omega \subset \mathbb{R}_+^n \to \mathbb{R}^n$ is said to be Schur m-power convex on Ω, if $f(\boldsymbol{x}) \prec f(\boldsymbol{y})$ on Ω implies $\psi(\boldsymbol{x}) \le \psi(\boldsymbol{y})$. ψ is said to be Schur m-power concave if $-\psi$ is Schur m-power convex.

Definition 8. [22] Let $I \subset \mathbb{R}_+$, $\varphi : I \to \mathbb{R}_+$ be continuous. A function φ is said to be a GA convex (concave) function on I if

$$\varphi\left(\sqrt{xy}\right) \le (\ge) \frac{\varphi(x) + \varphi(y)}{2} ,$$

for all x, $y \in I$.

Definition 9. [22] Let $I \subset \mathbb{R}_+$, $\varphi : I \to \mathbb{R}_+$ be continuous. A function φ is said to be a multiplicatively convex (concave) function on I if

$$\varphi\left(\sqrt{xy}\right) \le (\ge) \sqrt{\varphi(x)\varphi(y)} ,$$

for all x, $y \in I$.

Lemma 1. [4, p. 57] *Let $\Omega \subset \mathbb{R}^n$ be a symmetric convex set with a nonempty interior Ω^0. $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable on Ω^0. Then φ is a Schur-convex (Schur-concave) function if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \ge 0 (\le 0) \tag{3}$$

holds for any $\boldsymbol{x} = (x_1, \cdots, x_n) \in \Omega^0$.

Lemma 2. [2, p. 108] *Let $\Omega \subset \mathbb{R}_+^n$ be a symmetric geometrically convex set with a nonempty interior Ω^0. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable on Ω^0. Then φ is a Schur geometrically convex (Schur geometrically concave) function if and only if φ is symmetric on Ω and*

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial \varphi}{\partial x_1} - x_2\frac{\partial \varphi}{\partial x_2}\right) \ge 0 \quad (\le 0) \tag{4}$$

holds for any $\boldsymbol{x} = (x_1, \cdots, x_n) \in \Omega^0$.

Remark 1. Since $\frac{\log x_1 - \log x_2}{x_1 - x_2} > 0$ for $x_1 \neq x_2$, (4) is equivalent to

$$(x_1 - x_2)\left(x_1\frac{\partial \varphi}{\partial x_1} - x_2\frac{\partial \varphi}{\partial x_2}\right) \ge 0 \quad (\le 0) \tag{5}$$

Lemma 3. [5, 7] *Let $\Omega \subset \mathbb{R}_+^n$ be a symmetric harmonically convex set with a nonempty interior Ω^0. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable on Ω^0. Then φ is a Schur harmonically convex (Schur harmonically concave) function if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(x_1^2\frac{\partial \varphi}{\partial x_1} - x_2^2\frac{\partial \varphi}{\partial x_2}\right) \ge 0 \quad (\le 0) \tag{6}$$

holds for any $\boldsymbol{x} = (x_1, \cdots, x_n) \in \Omega^0$.

Lemma 4. [5, 26] *Let $\Omega \subset \mathbb{R}_+^n$ be a symmetric convex set with a nonempty interior Ω^0 and $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω^0. Then φ is Schur m-power convex (Schur m-power concave) on Ω if and only if φ is symmetric on Ω and*

$$\begin{cases} \frac{x_1^m - x_2^m}{m}\left(x_1^{1-m}\frac{\partial\varphi}{\partial x_1} - x_2^{1-m}\frac{\partial\varphi}{\partial x_2}\right) \geq 0 (\leq 0), \ if \ m \neq 0, \\ (\log x_1 - \log x_2)\left(x_1\frac{\partial\varphi}{\partial x_1} - x_2\frac{\partial\varphi}{\partial x_2}\right) \geq 0 (\leq 0), \ if \ m = 0 \end{cases} \tag{7}$$

hold for any $\boldsymbol{x} = (x_1, \cdots, x_n) \in \Omega^0$ with $x_1 \neq x_2$.

Remark 2. Ming Li pointed out that when $x_1 \neq x_2$, by Lagrange mean value theorem, it follows that $\frac{x_1^m - x_2^m}{m} = \xi^{m-1}(x_1 - x_2)$, where ξ lies between x_1 and x_2 and $\xi > 0$, hence combining Remark 1, (7) is equivalent to

$$(x_1 - x_2)\left(x_1^{1-m}\frac{\partial\varphi}{\partial x_1} - x_2^{1-m}\frac{\partial\varphi}{\partial x_2}\right) \geq 0 (\leq 0) \tag{8}$$

Lemma 5. [22] *Let $I \subset \mathbb{R}_+$ be an open subinterval and let $\varphi : I \to \mathbb{R}_+$ be differentiable. φ is GA-convex (concave) if and only if $x\varphi'(x)$ is increasing (decreasing).*

Lemma 6. [22] *Let $I \subset \mathbb{R}_+$ be an open subinterval and let $\varphi : I \to \mathbb{R}_+$ be differentiable. φ is multiplicatively convex (concave) if and only if $\frac{x\varphi'(x)}{\varphi(x)}$ is increasing (decreasing).*

Let $\pi = (\pi(1), \cdots, \pi(n))$ be a permutation of $(1, \cdots, n)$, all permutations is totally $n!$. The following conclusion is proved in [24].

Lemma 7. [24] *Let $A \subset \mathbb{R}^k$ be a symmetric geometrically convex set and let φ be a Schur geometrically convex (concave) function defined on A with the property that for each fixed $x_2, \cdots, x_k$, $\varphi(z, x_2, \cdots, x_k)$ is GA convex (concave) in z on $\{z : (z, x_2, \cdots, x_k) \in A\}$. Then for any $n > k$,*

$$\psi(x_1, \cdots, x_n) = \sum_{\pi} \varphi\left(x_{\pi(1)}, \cdots, x_{\pi(k)}\right)$$

is Schur geometrically convex (concave) on

$$B = \left\{(x_1, \cdots, x_n) : \left(x_{\pi(1)}, \cdots, x_{\pi(k)}\right) \in A, \ for \ all \ permutations \ \pi\right\}.$$

Furthermore, the symmetric function

$$\overline{\psi}(\boldsymbol{x}) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \varphi(x_{i_1}, \cdots, x_{i_k})$$

is also Schur geometrically convex (concave) on B.

Lemma 8. [21]

Let Interval $I \subset \mathbb{R}_+^n$ and $\varphi : I^n \to \mathbb{R}_+$ be a symmetric differentiable function.For $m < 0$, if φ is increasing and Schur geometrically convex, then φ is Schur m-power convex.

Proof.

$$\begin{aligned}\Lambda &:= (x_1-x_2)\left(x_1^{1-m}\frac{\partial\varphi}{\partial x_1}-x_2^{1-m}\frac{\partial\varphi}{\partial x_2}\right)\\&=(x_1-x_2)\left[x_1^m\left(x_1\frac{\partial\varphi}{\partial x_1}-x_2\frac{\partial\varphi}{\partial x_2}\right)+x_2\left(x_1^{-m}-x_2^{-m}\right)\frac{\partial\varphi}{\partial x_2}\right]\\&=x_1^m(x_1-x_2)\left(x_1\frac{\partial\varphi}{\partial x_1}-x_2\frac{\partial\varphi}{\partial x_2}\right)+x_2(x_1-x_2)\left(x_1^{-m}-x_2^{-m}\right)\frac{\partial\varphi}{\partial x_2}.\end{aligned}$$

From φ is increasing, it follows that $\frac{\partial\varphi}{\partial x_2}\geq 0$. Noting $-m>0$ yield $(x_1-x_2)\left(x_1^{-m}-x_2^{-m}\right)\geq 0$. And since φ is Schur geometrically convex, by Lemma 2 and Remark 1, we have $(x_1-x_2)\left(x_1\frac{\partial\varphi}{\partial x_1}-x_2\frac{\partial\varphi}{\partial x_2}\right)\geq 0$, and then $\Lambda\geq 0$, from Lemma 4 and Remark 2, it follows that φ is Schur m-power convex. □

3. Proofs of main results

Proof of Theorem B:

Firstly, we by using Lemma 7 to prove Schur geometric convexity of the functions $F_{n,k}(\boldsymbol{x},r)$.

Let

$$\varphi(\boldsymbol{z})=\log f\left(\left(\sum_{i=1}^k z_i^r\right)^{\frac{1}{r}}\right).$$

Then

$$\frac{\partial\varphi(\boldsymbol{z})}{\partial z_j}=\frac{z_j^{r-1}\cdot f'\left(\left(\sum_{i=1}^k z_i^r\right)^{\frac{1}{r}}\right)\cdot\left(\sum_{i=1}^k z_i^r\right)^{\frac{1}{r}-1}}{f\left(\left(\sum_{i=1}^k z_i^r\right)^{\frac{1}{r}}\right)},\ j=1,2 \tag{9}$$

and then

$$\begin{aligned}\Delta :&= (z_1-z_2)\left(z_1\frac{\partial\varphi(\boldsymbol{z})}{\partial z_1}-z_2\frac{\partial\varphi(\boldsymbol{z})}{\partial z_2}\right)\\&=(z_1-z_2)\left(z_1^r-z_2^r\right)\frac{f'\left(\left(\sum_{i=1}^k z_i^r\right)^{\frac{1}{r}}\right)\cdot\left(\sum_{i=1}^k z_i^r\right)^{\frac{1}{r}-1}}{f\left(\left(\sum_{i=1}^k z_i^r\right)^{\frac{1}{r}}\right)}.\end{aligned}$$

From f is increasing, it follows that $f'\geq 0$, and notice that $f>0, r>0$, we have $\Delta\geq 0$ on $\Omega\cap\mathbb{R}^k$. According to Lemma 2 and Remark 1, φ is Schur geometrically convex on $\Omega\cap\mathbb{R}^k$.

Let $g(t)=\log f(u)$, where $u=(t^r+a)^{\frac{1}{r}}$ and $a>0$ is a constant. Then

$$h(t):=tg'(t)=\frac{t^r}{t^r+a}\cdot\frac{uf'(u)}{f(u)}.$$

It is easy to see that $\frac{t^r}{t^r+a}$ is increasing, and since f is multiplicatively convex, by Lemma 6, it follows that $\frac{uf'(u)}{f(u)}$ is increasing, and then $h(t)$ is increasing. According to Lemma 5, $h(t)$ is GA convex in its single variable on $\Omega\cap\mathbb{R}^k$. So $\log F_{n,k}(\boldsymbol{x},r)$ is Schur geometrically convex on Ω from Lemma 7. Notice that the function $\log t$ is increasing, by the definition of Schur geometrically convex function, it is clear that $F_{n,k}(\boldsymbol{x},r)$ is also Schur geometrically convex on Ω, i.e. $F_{n,k}(\boldsymbol{x},r)$ is a 0-order power convex function on Ω. Since f is increasing, it is easy to see that $F_{n,k}(\boldsymbol{x},r)$ is also increasing. For $m < 0$, by Lemma 8, it follows that $F_{n,k}(\boldsymbol{x}, r)$ is Schur m-power convex on Ω.

The proof of Theorem B is completed.

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