Received 01/12/14

SOME WEIGHTED INTEGRAL INEQUALITIES FOR DIFFERENTIABLE h-PREINVEX FUNCTIONS

WAJEEHA IRSHAD [1], M. A. LATIF [2], AND MUHAMMAD IQBAL BHATTI [3]

Abstract. In this paper, we present weighted integral inequalities of Hermite-Hadamard type for differentiable h-preinvex functions. We have established the weighted generalization of recent results for preinvex functions as well as we extend several results connected with the Hermite-Hadamard type integral inequalities by weighted identity of functions defined on open invex subset of set of reals and by using h-preinvexity.

1. Introduction

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)y$$

holds for every $x, y \in I$ and $t \in [0,1]$.
Convexity plays an important role in mathematical economics, management science, engineering, and optimization theory.
The following double integral inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

holds for convex functions and is known as the Hermite-Hadamard inequality (see [28]). Both the inequalities in (1.1) hold in reversed direction if f is concave. Inequalities (1.1) are famous in mathematical literature due to their rich geometrical significance and applications.

For several results which generalize, improve and extend inequalities (1.1), we refer the interested reader to [2, 7, 8, 9], [11]-[15], [26, 27], [32]-[37].

In [7], Dragomir and Agarwal obtained the following inequalities for differentiable functions which estimate the difference between the middle and the rightmost terms in (1.1).

Theorem 1. [7] *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°. If $\left|f^{'}\right|$ is convex function on $[a,b]$, with a, $b \in I$ and $a < b$, and $f^{'} \in L\left([a,b]\right)$ then following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}\left[\left|f^{'}(a)\right| + \left|f^{'}(b)\right|\right]. \tag{1.2}$$

Date: Today.

2000 *Mathematics Subject Classification.* 26D15, 26D20, 26D07.

Key words and phrases. Hermite-Hadamard's inequality, invex set, preinvex function, h-preinvex function.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Theorem 2. [7] *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°. If $\left|f^{'}\right|^{\frac{p}{p-1}}$ is a convex function on $[a, b]$, , with $a,\, b \in I$ and $a < b$,and $f^{'} \in L([a, b])$ the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|\leq\frac{b-a}{2\,(p+1)^{\frac{1}{p}}}\left[\left|f^{'}\,(a)\right|^{\frac{p}{p-1}}+\left|f^{'}\,(b)\right|^{\frac{p}{p-1}}\right], \tag{1.3}$$

where $p > 1$ and $\frac{1}{p}+\frac{1}{q}=1$.

In [26], Pearce and Pečarić gave an refined and simplified form of the constant in Theorem 2 and these results are strengthen with Theorem 1. The following is the main result from [26].

Theorem 3. [26] *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°. If $\left|f^{'}\right|^{q}$ is a convex function on $[a, b]$, for some $q \geq 1$, with $a,\, b \in I$ and $a < b$, and $f^{'} \in L([a, b])$ the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|\leq\frac{b-a}{4}\left[\frac{\left|f^{'}\,(a)\right|^{q}+\left|f^{'}\,(b)\right|^{q}}{2}\right]^{\frac{1}{q}}. \tag{1.4}$$

If $\left|f^{'}\right|^{q}$ is concave on $[a, b]$, for some $q \geq 1$. Then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|\leq\frac{b-a}{4}\left|f^{'}\left(\frac{a+b}{2}\right)\right|. \tag{1.5}$$

2. Inequalities For h-Preinvex Functions

We have witnessed a rapid growth of the research on generalized convexity in the past decade. A significant generalization of convex functions termed preinvex functions was introduced in Weir and Mond [29]and Weir and Jeyakumar [30] Extensive work has been reported in the literature on generalized convex functions Žsee and the references therein. Hanson [6] has introduced a new class of generalized convex functions, subsequently called by Craven [3] "invex functions", with the aim to extend the validity of the sufficiency of the Kuhn-Tucker conditions. Since the papers of Hanson and Craven, many authors have studied invex functions, their generalizations and related functions: see, e.g., [2, 4, 5, 8, 14] and, for what concerns Romanian mathematicians, [9, 10, 11, 12, 15].
Here we are presenting some basic definitions of preinvex and h-preinvex functions before we proceed to prove our main results.

Definition 1. [5] *Let K be a subset in $\mathbb{R}^n$ and let $f : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}^n$ be continuous functions. Let $x \in K$, then the set K is said to be invex at x with respect to $\eta(\cdot, \cdot)$, if*

$$x + t\eta(y, x) \in K, \forall x, y \in K, t \in [0, 1].$$

K is said to be an invex set with respect to η if K is invex at each $x \in K$. The invex set K is also called a η-connected set.

Definition 2. [38] *The function f on the invex set K is said to be preinvex with respect to η, if*

$$f(u+t\eta(v,u))\leq(1-t)\,f(u)+t f(v),\forall u,v\in K,t\in[0,1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

It is to be noted that every convex function is preinvex with respect to the map $\eta(x,y)=x-y$ but the converse is not true see for instance [37].

Theorem 4. [20]*Let $f:[a,a+\eta(b,a)]\to(0,\infty)$ be a preinvex function on the interval of the real numbers K° (the interior of K) and a, $b\in K^{\circ}$ with $\eta(b,a)>0$. Then the following inequalities holds:*

$$f\left(\frac{2a+\eta(b,a)}{2}\right)\leq\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(x)\,dx\leq\frac{f(a)+f(b)}{2}. \tag{2.1}$$

Theorem 5. [4] *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$. Suppose that $f:K\to\mathbb{R}$ is a differentiable function. If $\left|f'\right|$ is preinvex on K, for every $a,b\in K$ with $\eta(b,a)\neq0$, the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(x)dx\right|\leq\frac{|\eta(b,a)|}{8}\left(\left|f'(a)\right|+\left|f'(b)\right|\right). \tag{2.2}$$

Theorem 6. [4] *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$. Suppose that $f:K\to\mathbb{R}$ is a differentiable function. Assume $p\in\mathbb{R}$ with $p>1$. If $\left|f'\right|^{\frac{p}{p-1}}$ is preinvex on K, for every $a,b\in K$ with $\eta(b,a)\neq0$, the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(x)dx\right|\leq\frac{|\eta(b,a)|}{2(1+p)^{\frac{1}{p}}}\left[\frac{\left|f'(a)\right|^{\frac{p}{p-1}}+\left|f'(b)\right|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}. \tag{2.3}$$

Definition 3. [16] *The function $f:K\to[0,\infty)$ on the invex set $K\subseteq[0,\infty)^{n}$ is said to be s-preinvex with respect to η, if*

$$f(u+t\eta(v,u))\leq(1-t)^{s}f(u)+t^{s}f(v),\forall u,v\in K,t\in[0,1].$$

for some fixed $s\in(0,1]$.
The function f is said to be s- preincave if and only if $-f$ is preinvex.

Definition 4. [19] *Let $h:[0,1]\to\mathbb{R}$ be a nonnegative function, $h\neq0$. The function f on the invex set K is said to be h-preinvex with respect to η, if*

$$f(u+\eta(v,u))\leq h(1-t)f(u)+h(t)f(v)$$

for each $u,v\in K$ and $t\in[0,1]$ where $f(.)>0$. If the above inequality is reversed, then f is said to be h-preconcave. Note, that every convex function is a h-preinvex function with respect to $\eta(v,u)=v-u$ and $h(t)=t$ for any $t\in[0,1]$.

Definition 5. [16] *A function $h : J \to \mathbb{R}$ is said to be a super-additive function if*

$$h(x+y) \geq h(x) + h(y)$$

for all $x, y \in J$, when $x + y \in J$

Theorem 7. [19] *Let $f : K = [a, a+\eta(b,a)] \to \mathbb{R}$ be a differential function on X°, such that $f' \in L^1([a, a+\eta(b,a)])$, where $a < a+\eta(b,a)$. If $|f'|$ is h-preinvex on $[a, a+\eta(b,a)]$, then we have*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{\eta(b,a)}{2}\left[|f'(a)|+|f'(b)|\right]\int_{\frac{1}{2}}^{1}(2t-1)(h(t)+h(1-t))\,dt. \tag{2.4}$$

Theorem 8. [16] *Let $f : K = [a, a+\eta(b,a)] \to \mathbb{R}$ be a differential function on X°, such that $f' \in L^1([a, a+\eta(b,a)])$, where $a < a+\eta(b,a)$. If $|f'|^q$ is h-preinvex on $[a, a+\eta(b,a)]$, and $q \geq 1$, then we have*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{\eta(b,a)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}\left(\int_0^1(2t-1)(h(t)+h(1-t))\,dt\right)^{\frac{1}{q}}. \tag{2.5}$$

For several recent results on inequalities for preinvex and h-preinvex functions, we refer the interested readers to [16, 17, 18, 23, 24] and [38].

3. Main Results

The following Lemma is essential in establishing our main results in this section:

Lemma 1. [16] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $\eta(b,a) > 0$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a+\eta(b,a)])$. If $h : [a, a+\eta(b,a)] \to [0, \infty)$ be a differentiable mapping, then the following equality holds:*

$$\begin{aligned}&\frac{1}{2}\left[(h(a+\eta(b,a)) - 2h(a))f(a) + h(a+\eta(b,a))f(a+\eta(b,a))\right] - \int_a^{a+\eta(b,a)} f(x)h'(x)\,dx = \frac{\eta(b,a)}{4}\\ &\times\left\{\int_0^1\left[2h\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right) - h(a+\eta(b,a))\right]f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)dt\right.\\ &\left.+\int_0^1\left[2h\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right) - h(a+\eta(b,a))\right]f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)dt\right\}.\end{aligned} \tag{3.1}$$

Remark 1. *If we take $\eta(b,a) = b - a$, then Lemma 1 reduces to Lemma 2.1 from* [9].

Now using Lemma 1, we shall propose some new upper bounds for the difference between the rightmost and middle terms of weighted version of the Hadamard's inequality (2.1) using preinvex and prequasiinvex mappings. Our results provide a weighted generalization of those results given in [3, 4] and [17].

In what follows we use the notations $L^{'}(a,b,t)=a+\left(\frac{1-t}{2}\right)\eta(b,a)$ and $U^{'}(a,b,t)=a+\left(\frac{1+t}{2}\right)\eta(b,a)$.

Theorem 9. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$ and $a,\ b\in K$ with $\eta(b,a)>0$. Suppose $f:K\to\mathbb{R}$ is a differentiable mapping on K and $w:[a,a+\eta(b,a)]\to[0,\infty)$ be continuous and symmetric to $a+\frac{1}{2}\eta(b,a)$. If $\left|f^{'}\right|$ is h-preinvex on K, we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)}w(x)\,dx-\int_a^{a+\eta(b,a)}f(x)w(x)\,dx\right| \tag{3.2}$$

$$\leq\frac{\eta(b,a)}{4}\left[\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right]\int_0^1\left(\int_{L^{'}(a,b,t)}^{U^{'}(a,b,t)}w(x)\,dx\right)\left[h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right]dt.$$

Proof. Let $h(t)=\int_a^t w(t)\,dt$ for all $t\in[a,a+\eta(b,a)]$ in Lemma 1, we obtain

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)}w(t)\,dt-\int_a^{a+\eta(b,a)}f(x)w(x)\,dx\right| \tag{3.3}$$

$$\leq\frac{\eta(b,a)}{4}\left\{\int_0^1\left|2h\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)-h(a+\eta(b,a))\right|\right.$$

$$\times\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|dt$$

$$\left.+\int_0^1\left|2h\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)-h(a+\eta(b,a))\right|\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|dt\right\}.$$

Since $w(x)$ is symmetric to $a+\frac{1}{2}\eta(b,a)$, so

$$w\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)=w\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)$$

and hence, we have

$$\left|2h\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)-h(a+\eta(b,a))\right|=\int_{L^{'}(a,b,t)}^{U^{'}(a,b,t)}w(x)\,dx \tag{3.4}$$

and

$$\left|2h\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)-h(a+\eta(b,a))\right|=\int_{L^{'}(a,b,t)}^{U^{'}(a,b,t)}w(x)\,dx \tag{3.5}$$

for all $t\in[0,1]$. Using (3.4) and (3.5) in (3.3), we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)}w(t)\,dt-\int_a^{a+\eta(b,a)}f(x)w(x)\,dx\right| \tag{3.6}$$

$$\leq\frac{\eta(b,a)}{4}\int_0^1\left(\int_{L^{'}(a,b,t)}^{U^{'}(a,b,t)}w(x)\,dx\right)\left[\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|\right.$$

$$\left.+\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|\right]dt.$$

Since $\left|f^{'}\right|$ is h-preinvex on K, hence for every $a, b \in K$ with $\eta(b,a) > 0$, we have

$$\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|+\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right| \tag{3.7}$$
$$\leq h\left(\frac{1+t}{2}\right)\left|f^{'}(a)\right|+h\left(\frac{1-t}{2}\right)\left|f^{'}(b)\right|+h\left(\frac{1-t}{2}\right)\left|f^{'}(a)\right|+h\left(\frac{1+t}{2}\right)\left|f^{'}(b)\right|$$
$$=\left(\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right)\left(h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right).$$

Using (3.7) in (3.6), we get the required inequality. This completes the proof of the theorem. □

Corollary 1. *In Theorem 9, if we take* $w(x)=\frac{1}{\eta(b,a)}$ *for all* $x \in [a, a+\eta(b,a)]$, *then (3.2) becomes the inequality*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right| \tag{3.8}$$
$$\leq\frac{\eta(b,a)}{4}\left[\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right]\int_0^1 t\left[h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right]dt.$$

Corollary 2. *If* $\eta(b,a)=b-a$ *in Theorem 9, then (3.2) reduces to the inequality*

$$\left|\frac{f(a)+f(b)}{2}\int_0^1 w(x)dx-\int_a^b f(x)\,w(x)\,dx\right| \tag{3.9}$$
$$\leq\frac{(b-a)}{4}\left[\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right]\int_0^1\left(\int_{L(a,b,t)}^{U(a,b,t)}w(x)dx\right)\left[h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right]dt.$$

where $U(a,b,t)=\frac{1-t}{2}a+\frac{1+t}{2}b$ *and* $L(a,b,t)=\frac{1+t}{2}a+\frac{1-t}{2}b$ *for all* $t\in[0,1]$

Corollary 3. *If* $\eta(b,a)=b-a$, $w(x)=\frac{1}{b-a}$ *in Theorem 9, then (3.2) reduces to the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{3.10}$$
$$\leq\frac{(b-a)}{4}\left[\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right]\int_0^1 t\left[h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right]dt.$$

Corollary 4. *Suppose* $h(t)=t^s$, $s\in[0,1]$ *in corollary 1, we have the following inequality for s-convex function*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right| \tag{3.11}$$
$$\leq\frac{\eta(b,a)\left(s2^{s+1}+1\right)}{2^{s+2}(s+1)(s+2)}\left[\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right].$$

Corollary 5. *If $\eta(b,a)=b-a$ in Corollary 4, we have the following inequality for s-convex function*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)\left(s2^{s+1}+1\right)}{2^{s+2}(s+1)(s+2)}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \tag{3.12}$$

.

Corollary 6. *Suppose the assumptions of Theorem 9 are satisfied. If h is super additive, we have the following inequality*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx-\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx\right| \leq \frac{\eta(b,a)\,h(1)}{4}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\int_0^1\left(\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dx\right)dt. \tag{3.13}$$

Theorem 10. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$ and $a,\ b\in K$ with $\eta(b,a)>0$. Suppose $f:K\to\mathbb{R}$ is a differentiable mapping on K and $w:[a,a+\eta(b,a)]\to[0,\infty)$ be continuous and symmetric to $a+\frac{1}{2}\eta(b,a)$. If $\left|f'\right|^q$ is h-preinvex on K for $q>1$, we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx-\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx\right| \leq \frac{\eta(b,a)}{2}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}} \times\left[\int_0^1\left(h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right)dt\right]^{\frac{1}{q}}\left(\int_0^1\left[\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dx\right]^p dt\right)^{\frac{1}{p}}, \tag{3.14}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Continuing from inequality (3.6) in the proof of Theorem 9 and using the well known Hölder's integral inequality, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(t)\,dt-\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx\right| \leq \frac{\eta(b,a)}{4}\left(\int_0^1\left[\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dx\right]^p dt\right)^{\frac{1}{p}}\left[\left(\int_0^1\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}} +\left(\int_0^1\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right]. \tag{3.15}$$

By the power-mean inequality $t^r+s^r<2^{1-r}(t+s)^r$ for $t>0$, $s>0$ and $r<1$, and by the the h-preinvexity of $\left|f'\right|^q$ on K for $q>1$, we have for every $a,b\in K$

with $\eta(b,a) > 0$ the following inequality

(3.16)
$$\left(\int_0^1 \left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}} + \left(\int_0^1 \left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq 2^{1-\frac{1}{q}}\left[\int_0^1 \left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt + \int_0^1 \left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right]^{\frac{1}{q}}$$
$$\leq 2^{1-\frac{1}{q}}\left[\int_0^1 \left\{h\left(\frac{1+t}{2}\right)\left|f'(a)\right|^q + h\left(\frac{1-t}{2}\right)\left|f'(b)\right|^q\right.\right.$$
$$\left.\left.+h\left(\frac{1-t}{2}\right)\left|f'(a)\right|^q + h\left(\frac{1+t}{2}\right)\left|f'(b)\right|^q\right\} dt\right]^{\frac{1}{q}}$$
$$= 2^{1-\frac{1}{q}}\left[\int_0^1 \left(h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right)dt\right]^{\frac{1}{q}}\left[\left|f'(a)\right|^q + \left|f'(b)\right|^q\right]^{\frac{1}{q}}.$$

Using the last inequality (3.16) in (3.15), we get the desired inequality. This completes the proof of the theorem as well. □

Corollary 7. *In Theorem 10 if we take $w(x) = \frac{1}{\eta(b,a)}$ for all $x \in [a, a+\eta(b,a)]$ with $\eta(b,a) > 0$, then (3.14) reduces to the inequality*

(3.17)
$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$
$$\leq \frac{\eta(b,a)}{2(1+p)^{\frac{1}{p}}}\left[\frac{\left|f'(a)\right|^{\frac{p}{p-1}} + \left|f'(b)\right|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}\left(\int_0^1 \left[h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right]dt\right)^{\frac{p-1}{p}}.$$

.

Corollary 8. *If we take $\eta(b,a) = b-a$ in Theorem 10, then (3.14) reduces to the following inequality:*

(3.18)
$$\left|\frac{f(a)+f(b)}{2}\int_a^b w(x)\,dx - \int_a^b f(x)\,w(x)\,dx\right|$$
$$\leq \frac{b-a}{2}\left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}$$
$$\times\left(\int_0^1 \left[h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right]dt\right)^{\frac{1}{q}}\left(\int_0^1 \left[\int_{L(a,b,t)}^{U(a,b,t)} w(x)\,dx\right]^p\right)^{\frac{1}{p}} dt,$$

where $\frac{1}{p}+\frac{1}{q} = 1$, $L(a,b,t) = \left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b$, $U(a,b,t) = \left(\frac{1-t}{2}\right)a + \left(\frac{1+t}{2}\right)b$, $t \in [a,b]$.

Corollary 9. *Assume that all the conditions of Theorem 9 are satisfied and in addition if h is super-additive, we have the following inequality*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx-\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx\right| \tag{3.19}$$
$$\leq \frac{\eta(b,a)\,(h(1))^{\frac{1}{q}}}{2}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}\left(\int_0^1\left[\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dx\right]^p dt\right)^{\frac{1}{p}}.$$

Corollary 10. *Suppose $h(t)=t^s$, $s\in[0,1]$ in corollary 6, we have the following inequality for s-convex function.*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \tag{3.20}$$
$$\leq \frac{\eta(b,a)}{2^{1-\frac{s}{q}}\,(1+p)^{\frac{1}{p}}\,(1+s)^{\frac{1}{q}}}\left[\frac{\left|f'(a)\right|^{\frac{p}{p-1}}+\left|f'(b)\right|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}.$$

.

A similar result may be stated as follows:

Theorem 11. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$. Suppose $f:K\to\mathbb{R}$ is a differentiable mapping on K and $w:[a,a+\eta(b,a)]\to[0,\infty)$ be continuous and symmetric to $a+\frac{1}{2}\eta(b,a)$. If $\left|f'\right|^q$ is h-preinvex on K for $q\geq 1$, then for every $a,b\in K$ with $\eta(b,a)>0$, we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx-\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx\right| \tag{3.21}$$
$$\leq \frac{\eta(b,a)}{2}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}\left(\int_0^1\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dxdt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left(\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dx\right)\left[h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right]dt\right)^{\frac{1}{q}}.$$

Proof. Continuing from inequality (3.6) in the proof of Theorem 9 and using the well known Hölder's integral inequality, we have

$$\begin{aligned}(3.22)\quad &\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)}w(t)\,dt-\int_a^{a+\eta(b,a)}f(x)w(x)\,dx\right|\\ &\le\frac{\eta(b,a)}{4}\left[\int_0^1\left(\int_{L'(a,b,t)}^{U'(a,b,t)}w(x)\,dx\right)dt\right]^{1-\frac{1}{q}}\\ &\times\left[\left(\int_0^1\left(\int_{L'(a,b,t)}^{U'(a,b,t)}w(x)\,dx\right)\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_0^1\left(\int_{L'(a,b,t)}^{U'(a,b,t)}w(x)\,dx\right)\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right].\end{aligned}$$

By the power-mean inequality $t^r+s^r<2^{1-r}(t+s)^r$ for $t>0$, $s>0$ and $r<1$, and by the the h-preinvexity of $\left|f'\right|^q$ on K for $q>1$, we have for every $a,b\in K$ with $\eta(b,a)>0$ the following inequality

$$\begin{aligned}(3.23)\quad &\left(\int_0^1\left(\int_{L'(a,b,t)}^{U'(a,b,t)}w(x)\,dx\right)\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\\ &+\left(\int_0^1\left(\int_{L'(a,b,t)}^{U'(a,b,t)}w(x)\,dx\right)\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\\ &\le 2^{1-\frac{1}{q}}\left[\int_0^1\left(\int_{L'(a,b,t)}^{U'(a,b,t)}w(x)\,dx\right)dt\right]^{\frac{1}{q}}\left[h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right]^{\frac{1}{q}}\\ &\qquad\times\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]^{\frac{1}{q}}.\end{aligned}$$

Utilizing inequality (3.23) in (3.22), we get the inequality (3.29). This completes the proof of the theorem. □

Corollary 11. *Suppose all the assumptions of Theorem 11 are satisfied and if* $w(x)=\frac{1}{\eta(b,a)}$ *for all* $x\in[a,a+\eta(b,a)]$ *with* $\eta(b,a)>0$*, then we have the following inequality:*

$$\begin{aligned}(3.24)\quad &\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|\\ &\le\frac{\eta(b,a)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}\left(\int_0^1 t\left[h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right]dt\right)^{\frac{1}{q}}.\end{aligned}$$

Corollary 12. *If we take $\eta(b,a)=b-a$ and $w(x)=\frac{1}{(b-a)}$ in Theorem 11, then the inequality reduces to the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}\int_a^b f(x)\,dx\right| \tag{3.25}$$
$$\leq\frac{(b-a)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}\left[\int_0^1 t\left(h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right)dt\right]^{\frac{1}{q}}.$$

Corollary 13. *In corollary 11, put $h(t)=t$, then we have*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \tag{3.26}$$
$$\leq\frac{\eta(b,a)}{4}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}.$$

Corollary 14. *Under the assumptions of Theorem 11, if h is super-additive, then we have the following inequality*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx-\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx\right| \tag{3.27}$$
$$\leq\frac{\eta(b,a)\,(h(1))^{\frac{1}{q}}}{2}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}\left(\int_0^1\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dxdt\right).$$

Corollary 15. *If h is super-additive in Corollary 11, then we have the following inequality*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \tag{3.28}$$
$$\leq\frac{\eta(b,a)\,h(1)^{\frac{1}{q}}}{4}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}.$$

Corollary 16. *If h is super-additive in Corollary 12, then we have the following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}\int_a^b f(x)\,dx\right| \tag{3.29}$$
$$\leq\frac{(b-a)h(1)^{\frac{1}{q}}}{4}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}.$$

Corollary 17. *Suppose $h(t) = t^s$, $s \in [0,1]$ in corollary 9, we have the following inequality for s-convex function.*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \tag{3.30}$$

$$\leq \frac{\eta(b,a)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{(s2^{s+1}+1)}{2^s(s+1)(s+2)}\right)^{\frac{1}{q}}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}.$$

References

[1] T. Antczak, Mean value in invexity analysis, Nonl. Anal., 60 (2005), 1473-1484.

[2] M. Alomari , M. Darus, U.S. Kirmaci, Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means, Computers and Mathematics with Applications 59 (2010) 225 232.

[3] A. Barani, A.G. Ghazanfari, S.S. Dragomir, Hermite-Hadamard inequality through prequasi-invex functions, RGMIA Research Report Collection, 14(2011), Article 48, 7 pp.

[4] A. Barani, A.G. Ghazanfari, S.S. Dragomir, Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex, RGMIA Research Report Collection, 14(2011), Article 64, 11 pp.

[5] A. Ben-Israel and B. Mond, What is invexity?, J. Austral. Math. Soc., Ser. B, 28(1986), No. 1, 1-9.

[6] P.S. Bullen, Handbook of Means and Their Inequalities, Kluwer Academic Publishers, Dordrecht, 2003.

[7] S. S. Dragomir, and R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and trapezoidal formula, Appl. Math. Lett., 11(5)(1998), 91–95.

[8] S. S. Dragomir, Two mappings in connection to Hadamard's inequalities, J. Math. Anal. Appl., 167(1992), 42–56.

[9] D. -Y. Hwang, Some inequalities for differentiable convex mapping with application to weighted trapezoidal formula and higher moments of random variables, Appl. Math. Comp., 217(23)(2011), 9598-9605.

[10] M. A. Hanson, On sufficiency of the Kuhn-Tucker conditions, J. Math. Anal. Appl. 80 (1981) 545-550.

[11] D. A. Ion, Some estimates on the Hermite-Hadamard inequality through quasi-convex functions, Annals of University of Craiova, Math. Comp. Sci. Ser. 34 (2007) 82 87.

[12] U. S. Kırmacı, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp., 147(1)(2004), 137-146.

[13] U. S. Kırmacı and M. E. Özdemir, On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp., 153(2)(2004), 361-368.

[14] K. C. Lee and K. L. Tseng, On a weighted generalization of Hadamard's inequality for G-convex functions, Tamsui-Oxford J. Math. Sci., 16(1)(2000), 91–104.

[15] A. Lupas, A generalization of Hadamard's inequality for convex functions, Univ. Beograd. Publ. Elek. Fak. Ser. Mat. Fiz., 544-576(1976), 115–121.

[16] M. A. latif,S.S.Dragomir, Some weighted integral inequalities for differentiable preinvex and prequasiinvex functions with applications.

[17] M. A. latif, Some inequalities for differentiable prequasiinvex functions with applications. (to appear)

[18] S. R. Mohan and S. K. Neogy, On invex sets and preinvex functions, J. Math. Anal. Appl. 189 (1995), 901–908.

[19] Marian Matloka, Inequalities for h-preinvex functions.

[20] M. A. Noor, Hermite-Hadamard integral inequalities for log-preinvex functions, Preprint.

[21] M. A. Noor, Variational-like inequalities, Optimization, 30 (1994), 323–330.

[22] M. A. Noor, Invex equilibrium problems, J. Math. Anal. Appl., 302 (2005), 463–475.

[23] M. A. Noor, Some new classes of nonconvex functions, Nonl. Funct. Anal. Appl.,11(2006),165-171
[24] M. A. Noor, On Hadamard integral inequalities involving two log-preinvex functions, J. Inequal. Pure Appl. Math., 8(2007), No. 3, 1-14.
[25] R. Pini, Invexity and generalized convexity, Optimization 22 (1991) 513-525.
[26] C. E. M. Pearce and J. Pečarić, Inequalities for differentiable mappings with application to special means and quadrature formulae, Appl. Math. Lett., 13(2)(2000), 51–55.
[27] F. Qi, Z. -L.Wei and Q. Yang, Generalizations and refinements of Hermite-Hadamard's inequality, Rocky Mountain J. Math., 35(2005), 235–251.
[28] J. Pečarić, F. Proschan and Y. L. Tong, Convex functions, partial ordering and statistical applications, Academic Press, New York, 1991.
[29] T. Weir and B. Mond, Pre-invex functions in multiple objective optimization, J. Math. Anal. Appl. 136Ž1988., 2938.
[30] T. Weir and V. Jeyakumar, A class of nonconvex functions and mathematical programming, Bull. Austral. Math. Soc. 38Ž1988., 177189.
[31] M. Z. Sarikaya, H. Bozkurt and N. Alp, On Hermite-Hadamard type integral inequalities for preinvex and log-preinvex functions, arXiv:1203.4759v1.
[32] M. Z. Sarıkaya and N. Aktan, On the generalization some integral inequalities and their applications Mathematical and Computer Modelling, 54(9-10)(2011), 2175-2182.
[33] M. Z. Sarikaya, M. Avci and H. Kavurmaci, On some inequalities of Hermite-Hadamard type for convex functions, ICMS International Conference on Mathematical Science, AIP Conference Proceedings 1309, 852(2010).
[34] M. Z. Sarikaya, O new Hermite-Hadamard Fejér type integral inequalities, Stud. Univ. Babeş-Bolyai Math. 57(2012), No. 3, 377-386.
[35] A. Saglam, M. Z. Sarikaya and H. Yıldırım and, Some new inequalities of Hermite-Hadamard's type, Kyungpook Mathematical Journal, 50(2010), 399-410.
[36] C. -L. Wang and X. -H. Wang, On an extension of Hadamard inequality for convex functions, Chin. Ann. Math., 3(1982), 567–570.
[37] S. -H. Wu , On the weighted generalization of the Hermite-Hadamard inequality and its applications, The Rocky Mountain J. of Math., 39(2009), no. 5, 1741–1749.
[38] T. Weir, and B. Mond, Preinvex functions in multiple bjective optimization, Journal of Mathematical Analysis and Applications, 136 (1998) 29-38.
[39] X. M. Yang and D. Li, On properties of preinvex functions, J. Math. Anal. Appl. 256 (2001), 229-241.
[40] X.M. Yang, X.Q. Yang and K.L. Teo, Characterizations and applications of prequasiinvex functions, properties of preinvex functions, J. Optim. Theo. Appl. 110 (2001) 645-668.
[41] X. M. Yang, X. Q. Yang, K.L. Teo, Generalized invexity and generalized invariant monotonocity, Journal of Optimization Theory and Applications 117 (2003) 607-625.

[1]Department of Mathematics, University of Engineering and Technology, Lahore, Pakistan
E-mail address: wchattah@hotmail.com

[2]School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa
E-mail address: m_amer_latif@hotmail.com

[3]Department of Mathematics, University of Engineering and Technology, Lahore, Pakistan
E-mail address: uetzone@hotmail.com