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INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE FOR K-BOUNDED NORM CONVEX MAPPINGS

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Abstract. In this paper we obtain some inequalities of Hermite-Hadamard type for K-bounded norm convex mappings between two normed spaces. Applications for twice differentiable functions in Banach spaces and functions defined by power series in Banach algebras are provided as well. Some discrete inequalities of Jensen type are also obtained.

1. Introduction

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A| := (A^*A)^{1/2}$.

One of the central problems in perturbation theory is to find bounds for

$$\|f(A) - f(B)\|$$

in terms of $\|A - B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [5], [34] and the references therein.

It is known that [4] in the infinite-dimensional case the map $f(A) := |A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e. there is no constant $L > 0$ such that

$$\||A| - |B|\| \le L \|A - B\|$$

for any $A, B \in \mathcal{B}(H)$.

However, as shown by Farforovskaya in [32], [33] and Kato in [39], the following inequality holds

$$\||A| - |B|\| \le \frac{2}{\pi} \|A - B\| \left(2 + \log\left(\frac{\|A\| + \|B\|}{\|A - B\|}\right)\right) \tag{1.1}$$

for any $A, B \in \mathcal{B}(H)$ with $A \neq B$.

If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_{HS} := (\operatorname{tr} C^*C)^{1/2}$ of an operator C, then the following inequality is true [2]

$$\||A| - |B|\|_{HS} \le \sqrt{2} \|A - B\|_{HS} \tag{1.2}$$

for any $A, B \in \mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be selfadjoint, then the best coefficient is 1.

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It has been shown in [4] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\||A|-|B|\| \leq a_1 \|A-B\| + a_2 \|A-B\|^2 + O\left(\|A-B\|^3\right) \tag{1.3}$$

where

$$a_1 = \left\|A^{-1}\right\| \|A\| \text{ and } a_2 = \left\|A^{-1}\right\| + \left\|A^{-1}\right\|^3 \|A\|^2 .$$

In [3] the author also obtained the following *Lipschitz type inequality*

$$\|f(A) - f(B)\| \leq f'(a) \|A-B\| \tag{1.4}$$

where f is an *operator monotone function* on $(0,\infty)$ and $A, B \geq aI_H > 0$.

Let $(X; \|\cdot\|_X)$ and $(Y; \|\cdot\|_Y)$ be two Banach spaces over the complex number field $\mathbb{C}$. Let C be a convex set in X. For any mapping $F : C \subset X \to Y$ we can consider the associated functions $\Phi_{F,x,y,\lambda}$, $\Psi_{F,x,y,\lambda} : [0,1] \to Y$, where $x, y \in C$, $\lambda \in [0,1]$, defined by [30]

$$\begin{aligned} \Phi_{F,x,y,\lambda}(t) \quad : \quad &= (1-\lambda) F[(1-t)((1-\lambda)x + \lambda y) + ty] \\ &+ \lambda F[(1-t)x + t((1-\lambda)x + \lambda y)] \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} \Psi_{F,x,y,\lambda}(t) := &(1-\lambda) F[(1-t)((1-\lambda)x + \lambda y) + ty] \\ &+ \lambda F[tx + (1-t)((1-\lambda)x + \lambda y)] . \end{aligned} \tag{1.6}$$

We say that the mapping $F : B \subset X \to Y$ is *Lipschitzian* with the constant $L > 0$ on the subset B of X if

$$\|F(x) - F(y)\|_Y \leq L \|x-y\|_X \text{ for any } x, y \in B. \tag{1.7}$$

The following result holds:

Theorem 1. *Let $F : C \subset X \to Y$ be a Lipschitzian mapping with the constant $L > 0$ on the convex subset C of X. If $x, y \in C$, then we have*

$$\begin{aligned} &\left\|\Lambda_{F,x,y,\lambda}(t) - \int_0^1 F[sy + (1-s)x]\, ds\right\|_Y \\ &\leq 2L\left[\frac{1}{4} + \left(t - \frac{1}{2}\right)^2\right]\left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right] \|x-y\|_X \end{aligned} \tag{1.8}$$

for any $t \in [0,1]$ and $\lambda \in [0,1]$, where $\Lambda_{F,x,y,\lambda} = \Phi_{F,x,y,\lambda}$ or $\Lambda_{F,x,y,\lambda} = \Psi_{F,x,y,\lambda}$.

If we take in (1.8) $\Lambda_{F,x,y,\lambda} = \Phi_{F,x,y,\lambda}$, $\lambda = \frac{1}{2}$, then we get

$$\begin{aligned} &\left\|\frac{1}{2}\left(F\left[(1-t)\frac{x+y}{2} + ty\right] + F\left[(1-t)x + t\frac{x+y}{2}\right]\right)\right. \\ &\qquad \left. - \int_0^1 F[sy + (1-s)x]\, ds\right\| \\ &\qquad\qquad \leq \frac{1}{2} L\left[\frac{1}{4} + \left(t - \frac{1}{2}\right)^2\right] \|x-y\|_X \end{aligned} \tag{1.9}$$

for any $x, y \in C$ and $t \in [0,1]$.

If we take in (1.8) $\Lambda_{F,x,y,\lambda} = \Psi_{F,x,y,\lambda}$, $\lambda = \frac{1}{2}$, then we get

$$\left\| \frac{1}{2} \left(F\left[(1-t)\frac{x+y}{2} + ty \right] + F\left[tx + (1-t)\frac{x+y}{2} \right] \right) - \int_0^1 F[sy + (1-s)x]\, ds \right\|_Y \leq \frac{1}{2} L \left[\frac{1}{4} + \left(t - \frac{1}{2} \right)^2 \right] \|x - y\|_X \tag{1.10}$$

for any $t \in [0,1]$ and $x, y \in C$.

We also have the simpler inequalities

$$\left\| \frac{1}{2} \left[F\left(\frac{3x+y}{4} \right) + F\left(\frac{x+3y}{4} \right) \right] - \int_0^1 F[sy + (1-s)x]\, ds \right\|_Y \leq \frac{1}{8} L \|x - y\|_X , \tag{1.11}$$

$$\left\| F\left(\frac{x+y}{2} \right) - \int_0^1 F[sy + (1-s)x]\, ds \right\|_Y \leq \frac{1}{4} L \|x - y\|_X \tag{1.12}$$

and

$$\left\| \frac{1}{2} [F(x) + F(y)] - \int_0^1 F[sy + (1-s)x]\, ds \right\|_Y \leq \frac{1}{4} L \|x - y\|_X \tag{1.13}$$

for any $x, y \in C$. The constants $\frac{1}{8}$ and $\frac{1}{4}$ are best possible.

The inequalities (1.12) and (1.13) are the corresponding versions of Hermite-Hadamard inequalities for Lipschitzian functions. The scalar cases were obtained in [12] and [43]. For Hermite-Hadamard's type inequalities, see for instance [10], [12], [13], [35], [37], [38], [40], [42], [43], [46], [47], [48], [49], [50] and the references therein.

From (1.8) we also have the Ostrowski's inequality

$$\left\| F[ty + (1-t)x] - \int_0^1 F[sy + (1-s)x]\, ds \right\|_Y \leq L \left[\frac{1}{4} + \left(t - \frac{1}{2} \right)^2 \right] \|x - y\|_X \tag{1.14}$$

for any $t \in [0,1]$ and $x, y \in C$. For Ostrowski's type inequalities for the Lebesgue integral, see [1], [8]-[9] and [15]-[28]. Inequalities for the Riemann-Stieltjes integral may be found in [17], [19] while the generalization for isotonic functionals was provided in [20]. For the case of functions of self-adjoint operators on complex Hilbert spaces, see the recent monograph [23].

Motivated by the above results, we introduce here a class of functions that extends the concept of Lipschitzian function to power two of norm difference and called them K-bounded norm convex functions. Comprehensive examples of such functions are given. Integral inequalities of Hermite-Hadamard type are obtained and applications for discrete inequalities of Jensen type are provided as well.

2. K-Bounded Norm Convex Mappings

Let $(X;\|\cdot\|_X)$ and $(Y;\|\cdot\|_Y)$ be two normed linear spaces over the complex number field $\mathbb{C}$. Let C be a convex set in X. We consider the following class of functions:

Definition 1. *A mapping $F:C\subset X\to Y$ is called K-bounded norm convex, for some given $K>0$ if it satisfies the condition*

$$\|(1-\lambda)F(x)+\lambda F(y)-F((1-\lambda)x+\lambda y)\|_Y \leq \frac{1}{2}K\lambda(1-\lambda)\|x-y\|_X^2 \tag{2.1}$$

for any $x,y\in C$ and $\lambda\in[0,1]$. For simplicity, we denote this by $F\in\mathcal{BN}_K(C)$.

We have from (2.1) for $\lambda=\frac{1}{2}$ the Jensen's inequality

$$\left\|\frac{F(x)+F(y)}{2}-F\left(\frac{x+y}{2}\right)\right\|_Y \leq \frac{1}{8}K\|x-y\|_X^2 \tag{2.2}$$

for any $x,y\in C$.

We observe that $\mathcal{BN}_K(C)$ is a convex subset in the linear space of all functions defined on C and with values in Y.

We observe also that, by the triangle inequality, we have

$$\begin{aligned} &\|F((1-\lambda)x+\lambda y)\|_Y-\|(1-\lambda)F(x)+\lambda F(y)\|_Y \\ &\leq \|(1-\lambda)F(x)+\lambda F(y)-F((1-\lambda)x+\lambda y)\|_Y \end{aligned} \tag{2.3}$$

and by (2.1) we get

$$\begin{aligned} &\|F((1-\lambda)x+\lambda y)\|_Y-\|(1-\lambda)F(x)+\lambda F(y)\|_Y \\ &\leq \frac{1}{2}K\lambda(1-\lambda)\|x-y\|_X^2, \end{aligned}$$

which, again, by the triangle inequality gives

$$\begin{aligned} &\|F((1-\lambda)x+\lambda y)\|_Y \\ &\leq \frac{1}{2}K\lambda(1-\lambda)\|x-y\|_X^2+(1-\lambda)\|F(x)\|_Y+\lambda\|F(y)\|_Y \end{aligned} \tag{2.4}$$

for any $x,y\in C$ and $\lambda\in[0,1]$.

Now, if the function $t\mapsto\|F((1-\lambda)x+\lambda y)\|_Y$, for some $x,y\in C$, is Lebesgue integrable on $[0,1]$, then by taking the integral in (2.4) we get

$$\begin{aligned} &\int_0^1\|F((1-\lambda)x+\lambda y)\|_Y d\lambda \\ &\leq \frac{1}{2}K\|x-y\|_X^2\int_0^1\lambda(1-\lambda)d\lambda \\ &+\|F(x)\|_Y\int_0^1(1-\lambda)d\lambda+\|F(y)\|_Y\int_0^1\lambda d\lambda \end{aligned} \tag{2.5}$$

and since

$$\int_0^1\lambda(1-\lambda)d\lambda=\frac{1}{6},\quad \int_0^1(1-\lambda)d\lambda=\int_0^1\lambda d\lambda=\frac{1}{2},$$

then we get from (2.5) that

$$\int_0^1\|F((1-\lambda)x+\lambda y)\|_Y d\lambda \leq \frac{1}{12}K\|x-y\|_X^2+\frac{1}{2}\left[\|F(x)\|_Y+\|F(y)\|_Y\right]. \tag{2.6}$$

If we assume continuity for the function F on C in the norm topology of $(X;\|\cdot\|_X)$, then the inequality (2.6) holds for any $x,y\in C$. Moreover, if we assume that $(Y;\|\cdot\|_Y)$ is a Banach space and F is continuos on C, then we have the generalized triangle inequality

$$\left\|\int_0^1 F\left((1-\lambda)x+\lambda y\right)d\lambda\right\|_Y \le \int_0^1 \left\|F\left((1-\lambda)x+\lambda y\right)\right\|_Y d\lambda,$$

and by (2.6) we get

$$\left\|\int_0^1 F\left((1-\lambda)x+\lambda y\right)d\lambda\right\|_Y \le \frac{1}{12}K\|x-y\|_X^2+\frac{1}{2}\left[\|F(x)\|_Y+\|F(y)\|_Y\right] \tag{2.7}$$

for any $x,y\in C$.

We can improve this result as follows.

Theorem 2. *Let $(X;\|\cdot\|_X)$ and $(Y;\|\cdot\|_Y)$ be two normed linear spaces over the complex number field $\mathbb{C}$ with Y complete. Assume that the mapping $F:C\subset X\to Y$ is continuous on the convex set C in the norm topology. If $F\in\mathcal{BN}_K(C)$ for some $K>0$, then we have*

$$\left\|\frac{F(x)+F(y)}{2}-\int_0^1 F\left((1-\lambda)x+\lambda y\right)d\lambda\right\|_Y \le \frac{1}{12}K\|x-y\|_X^2 \tag{2.8}$$

and

$$\left\|\int_0^1 F\left((1-\lambda)x+\lambda y\right)d\lambda - F\left(\frac{x+y}{2}\right)\right\|_Y \le \frac{1}{24}K\|x-y\|_X^2 \tag{2.9}$$

for any $x,y\in C$.

The constants $\frac{1}{12}$ and $\frac{1}{24}$ are best possible.

Proof. From (2.1) we have successively

$$\begin{aligned}&\left\|\int_0^1 \left[(1-\lambda)F(x)+\lambda F(y)-F\left((1-\lambda)x+\lambda y\right)\right]d\lambda\right\|_Y\\ &\le \int_0^1 \left\|(1-\lambda)F(x)+\lambda F(y)-F\left((1-\lambda)x+\lambda y\right)\right\|_Y d\lambda\\ &\le \frac{1}{2}K\|x-y\|_X^2\int_0^1 \lambda(1-\lambda)\,d\lambda,\end{aligned}$$

which produces the desired result (2.8).

Utilising (2.2) we have

$$\begin{aligned}&\left\|\frac{F\left((1-\lambda)x+\lambda y\right)+F\left(\lambda x+(1-\lambda)y\right)}{2}-F\left(\frac{x+y}{2}\right)\right\|_Y\\ &\le \frac{1}{8}K\left\|(1-\lambda)x+\lambda y-\lambda x-(1-\lambda)y\right\|_X^2\\ &=\frac{1}{8}K(1-2\lambda)^2\|x-y\|_X^2=\frac{1}{2}K\left(\lambda-\frac{1}{2}\right)^2\|x-y\|_X^2\end{aligned} \tag{2.10}$$

for any $x,y\in C$ and $\lambda\in[0,1]$.

Integrating in (2.10) we get

$$\left\| \int_0^1 \left[\frac{F((1-\lambda)x+\lambda y)+F(\lambda x+(1-\lambda)y)}{2} - F\left(\frac{x+y}{2}\right)\right] d\lambda \right\|_Y \tag{2.11}$$
$$\leq \int_0^1 \left\| \frac{F((1-\lambda)x+\lambda y)+F(\lambda x+(1-\lambda)y)}{2} - F\left(\frac{x+y}{2}\right)\right\|_Y d\lambda$$
$$\leq \frac{1}{2} K \|x-y\|_X^2 \int_0^1 \left(\lambda - \frac{1}{2}\right)^2 d\lambda = \frac{1}{24} K \|x-y\|_X^2$$

and since

$$\int_0^1 F((1-\lambda)x+\lambda y)\, d\lambda = \int_0^1 F(\lambda x+(1-\lambda)y)\, d\lambda,$$

then from (2.11) we get (2.9).

Now, consider the function $F_0 : H \to \mathbb{R}$, $F_0(x) = \|x\|^2$ where $(H, \langle .,. \rangle)$ is a complex inner product space. If $x, y \in H$ and $\lambda \in [0,1]$, then

$$(1-\lambda)F_0(x)+\lambda F_0(y) - F_0((1-\lambda)x+\lambda y)$$
$$= (1-\lambda)\|x\|^2 + \lambda\|y\|^2 - \|(1-\lambda)x+\lambda y\|^2$$
$$= (1-\lambda)\|x\|^2 + \lambda\|y\|^2 - (1-\lambda)^2\|x\|^2 - 2(1-\lambda)\lambda \operatorname{Re}\langle x,y\rangle - \lambda^2\|y\|^2$$
$$= (1-\lambda)\lambda\left[\|x\|^2 - 2\operatorname{Re}\langle x,y\rangle + \|y\|^2\right] = (1-\lambda)\lambda\|x-y\|^2$$

showing that F_0 is continuous and K-bounded norm convex with $K=2$ on H.

We have

$$\int_0^1 F_0((1-\lambda)x+\lambda y)\, d\lambda$$
$$= \int_0^1 \|(1-\lambda)x+\lambda y\|^2 d\lambda$$
$$= \int_0^1 \left[(1-\lambda)^2\|x\|^2 + 2(1-\lambda)\lambda\operatorname{Re}\langle x,y\rangle + \lambda^2\|y\|^2\right] d\lambda$$
$$= \frac{1}{3}\left[\|x\|^2 + \operatorname{Re}\langle x,y\rangle + \|y\|^2\right]$$

for any $x, y \in H$.

Therefore

$$\frac{F_0(x)+F_0(y)}{2} - \int_0^1 F_0((1-\lambda)x+\lambda y)\, d\lambda$$
$$= \frac{1}{2}\left[\|x\|^2+\|y\|^2\right] - \frac{1}{3}\left[\|x\|^2 + \operatorname{Re}\langle x,y\rangle + \|y\|^2\right]$$
$$= \frac{1}{6}\|x-y\|^2$$

showing that we have the same quantity $\frac{1}{6}\|x-y\|^2$ in both sides of (2.8).

We also have

$$\int_0^1 F_0\left((1-\lambda)x+\lambda y\right)d\lambda - F_0\left(\frac{x+y}{2}\right)$$
$$= \frac{1}{3}\left[\|x\|^2 + \operatorname{Re}\langle x,y\rangle + \|y\|^2\right] - \frac{1}{4}\left[\|x\|^2 + 2\operatorname{Re}\langle x,y\rangle + \|y\|^2\right]$$
$$= \frac{1}{12}\|x-y\|^2$$

showing that we have the same quantity $\frac{1}{12}\|x-y\|^2$ in both sides of (2.9). □

3. Some Examples in Banach Algebras

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B}\to[0,\infty)$ such that $(\mathcal{B},\|\cdot\|)$ is a normed space, and, further:

$$\|ab\| \le \|a\|\,\|b\|$$

for any $a,b\in\mathcal{B}$. The normed algebra $(\mathcal{B},\|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.*

We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\|=1$.

Let $\mathcal{B}$ be a unital algebra. An element $a\in\mathcal{B}$ is *invertible* if there exists an element $b\in\mathcal{B}$ with $ab=ba=1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by Inv$\mathcal{B}$. If $a,b\in$Inv$\mathcal{B}$ then $ab\in$Inv$\mathcal{B}$ and $(ab)^{-1}=b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a\in\mathcal{B}$ and $\lim_{n\to\infty}\|a^n\|^{1/n}<1$, then $1-a\in$Inv$\mathcal{B}$;
(ii) $\{a\in\mathcal{B}:\ \|1-b\|<1\}\subset$ Inv$\mathcal{B}$;
(iii) Inv$\mathcal{B}$ is an *open subset* of $\mathcal{B}$;
(iv) The map Inv$\mathcal{B}\ni a\longmapsto a^{-1}\in$Inv$\mathcal{B}$ is continuous.

For simplicity, we denote $z1$, where $z\in\mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a\in\mathcal{B}$ is defined by

$$\rho(a) := \{z\in\mathbb{C}:\ z-a\in \text{Inv}\mathcal{B}\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a:\rho(a)\to$Inv$\mathcal{B}$,

$$R_a(z) := (z-a)^{-1}.$$

For each $z,w\in\rho(a)$ we have the identity

$$R_a(w)-R_a(z) = (z-w)\,R_a(z)\,R_a(w).$$

We also have that

$$\sigma(a)\subset\{z\in\mathbb{C}:\ |z|\le\|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a)=\sup\{|z|:z\in\sigma(a)\}.$$

If a,b are *commuting* elements in $\mathcal{B}$, i.e. $ab=ba$, then

$$\nu(ab)\le\nu(a)\,\nu(b)\ \text{ and }\nu(a+b)\le\nu(a)+\nu(b).$$

Let $\mathcal{B}$ a unital Banach algebra and $a\in\mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;

(ii) For any *bounded linear functionals* $\lambda : \mathcal{B} \rightarrow \mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) We have

$$\nu(a) = \lim_{n \to \infty} \|a^n\|^{1/n}.$$

Let f be an analytic functions on the open disk $D(0,R)$ given by the *power series*

$$f(z) := \sum_{j=0}^{\infty} \alpha_j z^j \quad (|z| < R).$$

If $\nu(a) < R$, then the series $\sum_{j=0}^{\infty} \alpha_j a^j$ converges in the Banach algebra $\mathcal{B}$ because $\sum_{j=0}^{\infty} |\alpha_j| \|a^j\| < \infty$, and we can define $f(a)$ to be its sum. Clearly $f(a)$ is well defined and there are many examples of important functions on a Banach algebra $\mathcal{B}$ that can be constructed in this way. For instance, the *exponential map* on $\mathcal{B}$ denoted exp and defined as

$$\exp a := \sum_{j=0}^{\infty} \frac{1}{j!} a^j \text{ for each } a \in \mathcal{B}.$$

If $\mathcal{B}$ is not commutative, then many of the familiar properties of the exponential function from the scalar case do not hold. The following key formula is valid, however with the additional hypothesis of commutativity for a and b from $\mathcal{B}$

$$\exp(a+b) = \exp(a)\exp(b).$$

In a general Banach algebra $\mathcal{B}$ it is difficult to determine the elements in the range of the exponential map $\exp(\mathcal{B})$, i.e. the element which have a "*logarithm*". However, it is easy to see that if a is an element in B such that $\|1-a\| < 1$, then a is in $\exp(\mathcal{B})$. That follows from the fact that if we set

$$b = -\sum_{n=1}^{\infty} \frac{1}{n}(1-a)^n,$$

then the series converges absolutely and, as in the scalar case, substituting this series into the series expansion for $\exp(b)$ yields $\exp(b) = a$.

Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [31] and [45].

Now, by the help of power series $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficients of the original series, namely, $f_a(\lambda) := \sum_{n=0}^{\infty} |\alpha_n| \lambda^n$. It is obvious that this new power series will have the same radius of convergence as the original series. We also notice that if all coefficients $\alpha_n \geq 0$, then $f_a = f$.

As some natural examples that are useful for applications, we can point out that, if

$$f(\lambda)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\lambda^n=\ln\frac{1}{1+\lambda},\ \ \lambda\in D(0,1);\tag{3.1}$$
$$g(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\lambda^{2n}=\cos\lambda,\ \ \lambda\in\mathbb{C};$$
$$h(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\lambda^{2n+1}=\sin\lambda,\ \ \lambda\in\mathbb{C};$$
$$l(\lambda)=\sum_{n=0}^{\infty}(-1)^n\lambda^n=\frac{1}{1+\lambda},\ \ \lambda\in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_a(\lambda)=\sum_{n=1}^{\infty}\frac{1}{n}\lambda^n=\ln\frac{1}{1-\lambda},\ \ \lambda\in D(0,1);\tag{3.2}$$
$$g_a(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}\lambda^{2n}=\cosh\lambda,\ \ \lambda\in\mathbb{C};$$
$$h_a(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\lambda^{2n+1}=\sinh\lambda,\ \ \lambda\in\mathbb{C};$$
$$l_a(\lambda)=\sum_{n=0}^{\infty}\lambda^n=\frac{1}{1-\lambda},\ \ \lambda\in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(\lambda)=\sum_{n=0}^{\infty}\frac{1}{n!}\lambda^n\qquad\lambda\in\mathbb{C},\tag{3.3}$$
$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1},\qquad\lambda\in D(0,1);$$
$$\sin^{-1}(\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1},\qquad\lambda\in D(0,1);$$
$$\tanh^{-1}(\lambda)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1},\qquad\lambda\in D(0,1)$$
$${}_2F_1(\alpha,\beta,\gamma,\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^n,\alpha,\beta,\gamma>0,$$
$$\lambda\in D(0,1);$$

where Γ is *Gamma function.*

The following result provides a class of functions that are K-bounded norm convex on closed balls from Banach algebras.

Theorem 3. *Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R>0$. For any $x, y \in \mathcal{B}$ with $\|x\|, \|y\| \le M < R$, $M > 0$ we have that*

$$\|(1-\lambda) f(x) + \lambda f(y) - f((1-\lambda)x + \lambda y)\| \le \frac{1}{2} \lambda (1-\lambda) f_a''(M) \|x-y\|^2 \tag{3.4}$$

for any $\lambda \in [0,1]$.

In other words the function $f : \overline{B}(0,M) \subset \mathcal{B} \to \mathcal{B}$, where $\overline{B}(0,M)$ is the closed ball $\{x \in \mathcal{B},\ \|x\| \le M\}$ defined by $f(x) = \sum_{n=0}^{\infty} \alpha_n x^n$, $x \in \overline{B}(0,M)$ is K-bounded norm convex with $K = f_a''(M)$.

Proof. We use the identity (see for instance [6, p. 254])

$$a^n - b^n = \sum_{j=0}^{n-1} a^{n-1-j} (a-b) b^j \tag{3.5}$$

that holds for any $a, b \in \mathcal{B}$ and $n \ge 1$.

Let $x, y \in \mathcal{B}$. We have by (3.5) that

$$[(1-\lambda)x + \lambda y]^n - x^n = \lambda \sum_{j=0}^{n-1} [(1-\lambda)x + \lambda y]^{n-1-j} (y-x) x^j \tag{3.6}$$

and

$$[(1-\lambda)x + \lambda y]^n - y^n = -(1-\lambda) \sum_{j=0}^{n-1} [(1-\lambda)x + \lambda y]^{n-1-j} (y-x) y^j \tag{3.7}$$

for $n \ge 1$ and $\lambda \in [0,1]$.

Multiply (3.6) by $1-\lambda$ and (3.7) by λ and add the obtained equalities to get

$$\begin{aligned} &[(1-\lambda)x + \lambda y]^n - (1-\lambda)x^n - \lambda y^n \\ &= \lambda(1-\lambda) \sum_{j=0}^{n-1} [(1-\lambda)x + \lambda y]^{n-1-j} (y-x) \left(x^j - y^j\right) \\ &= \lambda(1-\lambda) \sum_{j=1}^{n-1} [(1-\lambda)x + \lambda y]^{n-1-j} (y-x) \left(x^j - y^j\right) \end{aligned} \tag{3.8}$$

for $n \ge 2$ and $\lambda \in [0,1]$.

If $j \ge 1$ we also have

$$x^j - y^j = \sum_{\ell=0}^{j-1} x^{j-1-\ell} (x-y) y^\ell$$

and by (3.8) we have

$$\begin{aligned} &(1-\lambda)x^n + \lambda y^n - [(1-\lambda)x + \lambda y]^n \\ &= \lambda(1-\lambda) \sum_{j=1}^{n-1} \sum_{\ell=0}^{j-1} [(1-\lambda)x + \lambda y]^{n-1-j} (y-x) x^{j-1-\ell} (x-y) y^\ell \end{aligned} \tag{3.9}$$

for $n \ge 2$ and $\lambda \in [0,1]$, which is an equality of interest in itself.

Let $m \geq 2$ and $x, y \in \mathcal{B}$. Then, by utilizing (3.9), we have

$$(1-\lambda)\sum_{n=0}^{m} a_n x^n + \lambda \sum_{n=0}^{m} a_n y^n - \sum_{n=0}^{m} a_n \left[(1-\lambda)x + \lambda y\right]^n \tag{3.10}$$
$$= \sum_{n=0}^{m} a_n \left[(1-\lambda)x^n + \lambda y^n - \left[(1-\lambda)x + \lambda y\right]^n\right]$$
$$= \sum_{n=2}^{m} a_n \left[(1-\lambda)x^n + \lambda y^n - \left[(1-\lambda)x + \lambda y\right]^n\right]$$
$$= \lambda(1-\lambda)$$
$$\times \sum_{n=2}^{m} a_n \left(\sum_{j=1}^{n-1}\sum_{\ell=0}^{j-1} \left[(1-\lambda)x + \lambda y\right]^{n-1-j}(y-x)x^{j-1-\ell}(x-y)y^{\ell}\right)$$

for all $m \geq 2$, $x, y \in \mathcal{B}$ and $\lambda \in [0,1]$.

Taking the norm in (3.10) and using repeatedly the generalized triangle inequality we have

$$\left\|(1-\lambda)\sum_{n=0}^{m} a_n x^n + \lambda \sum_{n=0}^{m} a_n y^n - \sum_{n=0}^{m} a_n \left[(1-\lambda)x + \lambda y\right]^n\right\| \tag{3.11}$$
$$\leq \lambda(1-\lambda)$$
$$\times \sum_{n=2}^{m} |a_n| \left(\sum_{j=1}^{n-1}\sum_{\ell=0}^{j-1} \left\|\left[(1-\lambda)x + \lambda y\right]^{n-1-j}(y-x)x^{j-1-\ell}(x-y)y^{\ell}\right\|\right).$$

If $\|x\|, \|y\| \leq M < R$, then $\|(1-\lambda)x + \lambda y\| \leq M$ for $\lambda \in [0,1]$ and using the Banach algebra properties we have

$$\left\|\left[(1-\lambda)x + \lambda y\right]^{n-1-j}(y-x)x^{j-1-\ell}(x-y)y^{\ell}\right\| \tag{3.12}$$
$$\leq \|(1-\lambda)x + \lambda y\|^{n-1-j}\|y-x\|\|x\|^{j-1-\ell}\|x-y\|\|y\|^{\ell}$$
$$= \|y-x\|^2\|(1-\lambda)x + \lambda y\|^{n-1-j}\|x\|^{j-1-\ell}\|y\|^{\ell}$$
$$\leq \|y-x\|^2 M^{n-1-j}M^{j-1-\ell}M^{\ell} = \|y-x\|^2 M^{n-2}$$

for $n \geq 2$.

Therefore, by (3.11) and (3.12) we have

$$\left\|(1-\lambda)\sum_{n=0}^{m} a_n x^n + \lambda \sum_{n=0}^{m} a_n y^n - \sum_{n=0}^{m} a_n \left[(1-\lambda)x + \lambda y\right]^n\right\| \tag{3.13}$$
$$\leq \lambda(1-\lambda)\sum_{n=2}^{m} |a_n| \left(\sum_{j=1}^{n-1}\sum_{\ell=0}^{j-1} \|y-x\|^2 M^{n-2}\right)$$
$$= \lambda(1-\lambda)\|y-x\|^2 \sum_{n=2}^{m} |a_n| M^{n-2}\sum_{j=1}^{n-1} j$$
$$= \frac{1}{2}\lambda(1-\lambda)\|y-x\|^2 \sum_{n=2}^{m} n(n-1)|a_n| M^{n-2}$$

for any $\|x\|, \|y\| \leq M < R$, $m \geq 2$ and $\lambda \in [0,1]$.

Since the series whose partial sums involved in (3.13) are convergent and

$$\sum_{n=0}^{\infty} a_n x^n = f(x), \quad \sum_{n=0}^{\infty} a_n y^n = f(y),$$

$$\sum_{n=0}^{m} a_n [(1-\lambda)x + \lambda y]^n = f((1-\lambda)x + \lambda y),$$

and

$$\sum_{n=2}^{\infty} n(n-1)|a_n| M^{n-2} = f_a''(M),$$

then by letting $m \to \infty$ in (3.13) we deduce the desired result (3.4). □

Corollary 1. *With the assumptions from Theorem 3 we have the inequalities*

$$\left\| \frac{f(x)+f(y)}{2} - \int_0^1 f((1-\lambda)x + \lambda y)\, d\lambda \right\| \leq \frac{1}{12} f_a''(M) \|x-y\|^2 \tag{3.14}$$

and

$$\left\| \int_0^1 f((1-\lambda)x + \lambda y)\, d\lambda - f\left(\frac{x+y}{2}\right) \right\| \leq \frac{1}{24} f_a''(M) \|x-y\|^2 \tag{3.15}$$

for any $x, y \in \mathcal{B}$ with $\|x\|, \|y\| \leq M < R$, $M > 0$.

The constants $\frac{1}{12}$ and $\frac{1}{24}$ are best possible.

It is known that if x and y are commuting, i.e. $xy = yx$, then the exponential function satisfies the property

$$\exp(x)\exp(y) = \exp(y)\exp(x) = \exp(x+y).$$

Also, if z is invertible and $a, b \in \mathbb{R}$ with $a < b$ then

$$\int_a^b \exp(tz)\, dt = z^{-1}[\exp(bz) - \exp(az)].$$

Therefore, if x and y are commuting and $y - x$ is invertible, then

$$\begin{aligned} \int_0^1 \exp((1-s)x + sy)\, ds &= \int_0^1 \exp(s(y-x))\exp(x)\, ds \\ &= \left(\int_0^1 \exp(s(y-x))\, ds\right)\exp(x) \\ &= (y-x)^{-1}[\exp(y-x) - 1]\exp(x) \\ &= (y-x)^{-1}[\exp(y) - \exp(x)], \end{aligned}$$

and by (3.14) and (3.15) we get

$$\left\| \frac{\exp(x)+\exp(y)}{2} - (y-x)^{-1}[\exp(y) - \exp(x)] \right\| \leq \frac{1}{12}\exp(M)\|x-y\|^2 \tag{3.16}$$

and

$$\left\| (y-x)^{-1}[\exp(y) - \exp(x)] - \exp\left(\frac{x+y}{2}\right) \right\| \leq \frac{1}{24}\exp(M)\|x-y\|^2 \tag{3.17}$$

provided $\|x\|, \|y\| \leq M$, $M > 0$.

4. The Case of Twice Differentiable Mappings in Normed Spaces

We first recall some results concerning Taylor's formula for differentiable mappings between two normed spaces, see for instance [11] for the basic definitions and results.

Lemma 1 (Taylor's formula, Lagrange's remainder [11, p. 110 - p. 111])**.** *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed linear spaces, Ω an open subset of X and $f : \Omega \to Y$ a $(k+1)$-differentiable mapping on Ω with $k \geq 0$. Suppose that $x, y \in \Omega$ are such that the segment $[x, y] := \{(1-\lambda)x + \lambda y,\ \lambda \in [0,1]\}$ is contained in Ω. Then*

$$f(y) = f(x) + f^{(1)}(x)(y-x) + \frac{1}{2!}f^{(2)}(x)(y-x, y-x) + \dots + \frac{1}{k!}f^{(k)}(x)(y-x, \dots, y-x) + R_k(x,y), \tag{4.1}$$

where

$$\|R_k(x,y)\|_Y \leq \frac{1}{(k+1)!}\|y-x\|_X^{k+1} \sup_{\lambda\in[0,1]} \left\| f^{(k+1)}((1-\lambda)x + \lambda y)\right\|_{\mathcal{L}(X^{k+1};Y)}. \tag{4.2}$$

We observe that if Ω is open and convex, then the equality (4.1) holds for any $x, y \in \Omega$. In this case we also have the bound

$$\|R_k(x,y)\|_Y \leq \frac{1}{(k+1)!}\|y-x\|_X^{k+1} \sup_{z\in\Omega} \left\| f^{(k+1)}(z)\right\|_{\mathcal{L}(X^{k+1};Y)}, \tag{4.3}$$

for any $x, y \in \Omega$.

We can prove the following result:

Theorem 4. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed linear spaces, C an open convex subset of X and $f : C \to Y$ a twice-differentiable mapping on C. Then for any $x, y \in C$ and $\lambda \in [0,1]$ we have*

$$\|(1-\lambda)f(x) + \lambda f(y) - f((1-\lambda)x + \lambda y)\|_Y \leq \frac{1}{2}K\lambda(1-\lambda)\|y-x\|_X^2, \tag{4.4}$$

where

$$K := \sup_{z\in C}\|f''(z)\|_{\mathcal{L}(X^2;Y)} \tag{4.5}$$

is assumed to be finite.

Proof. Using the above Lemma 1 we can state that

$$\|f(u) - f(v) - f'(v)(u-v)\|_F \leq \frac{1}{2}K\|u-v\|_X^2 \tag{4.6}$$

for any $u, v \in C$, where K is given by (4.5).

Let $x, y \in C$ and $\lambda \in [0,1]$. By (4.6) we have

$$\|f(x) - f((1-\lambda)x + \lambda y) - \lambda f'((1-\lambda)x + \lambda y)(x-y)\|_Y \leq \frac{1}{2}K\lambda^2\|y-x\|_X^2 \tag{4.7}$$

and

$$\left\|f(y)-f((1-\lambda)x+\lambda y)-(1-\lambda)f'((1-\lambda)x+\lambda y)(y-x)\right\|_Y \le \frac{1}{2}K(1-\lambda)^2\|y-x\|_X^2. \tag{4.8}$$

Multiply (4.7) by $1-\lambda$ and (4.8) by λ and add the obtained inequalities to get

$$\begin{aligned}&(1-\lambda)\left\|f(x)-f((1-\lambda)x+\lambda y)-\lambda f'((1-\lambda)x+\lambda y)(x-y)\right\|_Y\\&+\lambda\left\|f(y)-f((1-\lambda)x+\lambda y)+(1-\lambda)f'((1-\lambda)x+\lambda y)(x-y)\right\|_Y\\&\le\frac{1}{2}K\lambda^2(1-\lambda)\|y-x\|_X^2+\frac{1}{2}K(1-\lambda)^2\lambda\|y-x\|_X^2\\&=\frac{1}{2}K\lambda(1-\lambda)\|y-x\|_X^2.\end{aligned}\tag{4.9}$$

By the triangle inequality we also have

$$\begin{aligned}&\|(1-\lambda)f(x)+\lambda f(y)-f((1-\lambda)x+\lambda y)\|_Y\\&\le(1-\lambda)\left\|f(x)-f((1-\lambda)x+\lambda y)-\lambda f'((1-\lambda)x+\lambda y)(x-y)\right\|_Y\\&+\lambda\left\|f(y)-f((1-\lambda)x+\lambda y)+(1-\lambda)f'((1-\lambda)x+\lambda y)(x-y)\right\|_Y\end{aligned}\tag{4.10}$$

for any $x,y\in C$ and $\lambda\in[0,1]$.

Making use of (4.9) and (4.10) we deduce the desired result (4.4). □

Corollary 2. *With the assumptions from Theorem 4 we have the inequalities*

$$\left\|\frac{f(x)+f(y)}{2}-\int_0^1 f((1-\lambda)x+\lambda y)\,d\lambda\right\|_Y \le \frac{1}{12}\sup_{z\in C}\|f''(z)\|_{\mathcal{L}(X^2;Y)}\|x-y\|_X^2 \tag{4.11}$$

and

$$\left\|\int_0^1 f((1-\lambda)x+\lambda y)\,d\lambda-f\left(\frac{x+y}{2}\right)\right\|_Y \le \frac{1}{24}\sup_{z\in C}\|f''(z)\|_{\mathcal{L}(X^2;Y)}\|x-y\|_X^2 \tag{4.12}$$

for any $x,y\in C$.

The constants $\frac{1}{12}$ and $\frac{1}{24}$ are best possible.

5. Related Inequalities

We have the following result as well:

Theorem 5. *Let $(X;\|\cdot\|_X)$ and $(Y;\|\cdot\|_Y)$ be two normed linear spaces over the complex number field $\mathbb{C}$ with Y complete. Assume that the mapping $F:C\subset X\to Y$ is continuous on the convex set C in the norm topology. If $F\in\mathcal{BN}_K(C)$ for some $K>0$, then we have*

$$\left\|\int_0^1 F(uy+(1-u)x)\,du-\frac{1}{2\lambda-1}\int_{1-\lambda}^{\lambda}F(sx+(1-s)y)\,ds\right\|_F \le \frac{1}{6}K\lambda(1-\lambda)\|y-x\|_X^2 \tag{5.1}$$

for any $\lambda\in[0,1]$, $\lambda\ne\frac{1}{2}$ and $x,y\in C$.

Proof. Since $F \in \mathcal{BN}_K(C)$ for $K > 0$, then

$$\|(1-\lambda)F(u) + \lambda F(v) - F((1-\lambda)u + \lambda v)\|_Y \leq \frac{1}{2}K\lambda(1-\lambda)\|u-v\|_X^2 \tag{5.2}$$

for any $u, v \in C$ and $\lambda \in [0,1]$.

Let $t \in [0,1]$ and for $x, y \in C$, take

$$u = (1-t)((1-\lambda)x + \lambda y) + ty, \;\; v = tx + (1-t)((1-\lambda)x + \lambda y) \in C$$

in (5.2) to get

$$\begin{aligned} &\|(1-\lambda)F((1-t)((1-\lambda)x + \lambda y) + ty) \\ &\qquad + \lambda F(tx + (1-t)((1-\lambda)x + \lambda y)) \\ &-F((1-\lambda)[(1-t)((1-\lambda)x + \lambda y) + ty] + \lambda[tx + (1-t)((1-\lambda)x + \lambda y)])\|_Y \\ &\leq \frac{1}{2}K\lambda(1-\lambda)\|(1-t)((1-\lambda)x + \lambda y) + ty - [tx + (1-t)((1-\lambda)x + \lambda y)]\|_X^2 . \end{aligned} \tag{5.3}$$

Observe that

$$\begin{aligned} &(1-\lambda)[(1-t)((1-\lambda)x + \lambda y) + ty] + \lambda[tx + (1-t)((1-\lambda)x + \lambda y)] \\ &= (1-\lambda)(1-t)((1-\lambda)x + \lambda y) + (1-\lambda)ty \\ &+ \lambda tx + \lambda(1-t)((1-\lambda)x + \lambda y) \\ &= (1-t)((1-\lambda)x + \lambda y) + (1-\lambda)ty + \lambda tx \\ &= [(1-t)(1-\lambda) + \lambda t]x + [(1-t)\lambda + (1-\lambda)t]y \end{aligned}$$

and

$$\begin{aligned} &(1-t)((1-\lambda)x + \lambda y) + ty - [tx + (1-t)((1-\lambda)x + \lambda y)] \\ &= (1-t)(1-\lambda)x + (1-t)\lambda y + ty - tx - (1-t)(1-\lambda)x - (1-t)\lambda y \\ &= t(y-x). \end{aligned}$$

Then by (5.3) we have

$$\begin{aligned} &\|(1-\lambda)F((1-t)((1-\lambda)x + \lambda y) + ty) \\ &\qquad + \lambda F(tx + (1-t)((1-\lambda)x + \lambda y)) \\ &\quad -F([(1-t)(1-\lambda) + \lambda t]x + [(1-t)\lambda + (1-\lambda)t]y)\|_Y \\ &\qquad\qquad \leq \frac{1}{2}K\lambda(1-\lambda)t^2\|y-x\|_X^2 , \end{aligned} \tag{5.4}$$

for any $t, \lambda \in [0,1]$ and $x, y \in C$.

Integrating the inequality (5.4) over t on $[0,1]$ and using the generalized triangle inequality for norms and integrals, we get

$$\begin{aligned} &\Bigg\|(1-\lambda)\int_0^1 F((1-t)((1-\lambda)x + \lambda y) + ty)\,dt \\ &\qquad + \lambda\int_0^1 F(tx + (1-t)((1-\lambda)x + \lambda y))\,dt \\ &\quad - \int_0^1 F([(1-t)(1-\lambda) + \lambda t]x + [(1-t)\lambda + (1-\lambda)t]y)\,dt\Bigg\|_Y \\ &\qquad\qquad \leq \frac{1}{6}K\lambda(1-\lambda)\|y-x\|_X^2 , \end{aligned} \tag{5.5}$$

for any $\lambda \in [0,1]$ and $x, y \in C$.

Observe that

$$\int_0^1 F[(1-t)(\lambda y+(1-\lambda)x)+ty]\,dt = \int_0^1 F[((1-t)\lambda+t)y+(1-t)(1-\lambda)x]\,dt \tag{5.6}$$

and

$$\int_0^1 F(tx+(1-t)((1-\lambda)x+\lambda y))\,dt = \int_0^1 F((1-t)x+t((1-\lambda)x+\lambda y))\,dt = \int_0^1 F[t\lambda y+(1-\lambda t)x]\,dt. \tag{5.7}$$

If we make the change of variable $u := (1-t)\lambda + t$ then we have $1-u = (1-t)(1-\lambda)$ and $du = (1-\lambda)\,du$. Then

$$\int_0^1 F[((1-t)\lambda+t)y+(1-t)(1-\lambda)x]\,dt = \frac{1}{1-\lambda}\int_\lambda^1 F[uy+(1-u)x]\,du.$$

If we make the change of variable $u := \lambda t$ then we have $du = \lambda dt$ and

$$\int_0^1 F[t\lambda y+(1-\lambda t)x]\,dt = \frac{1}{\lambda}\int_0^\lambda F[uy+(1-u)x]\,du.$$

Therefore

$$(1-\lambda)\int_0^1 F[(1-t)(\lambda y+(1-\lambda)x)+ty]\,dt + \lambda\int_0^1 F[t(\lambda y+(1-\lambda)x)+(1-t)x]\,dt = \int_\lambda^1 F[uy+(1-u)x]\,du + \int_0^\lambda F[uy+(1-u)x]\,du = \int_0^1 F[uy+(1-u)x]\,du,$$

and we have the simple equality

$$(1-\lambda)\int_0^1 F((1-t)((1-\lambda)x+\lambda y)+ty)\,dt + \lambda\int_0^1 F(tx+(1-t)((1-\lambda)x+\lambda y))\,dt = \int_0^1 F[uy+(1-u)x]\,du \tag{5.8}$$

for any $\lambda \in [0,1]$ and $x, y \in C$.

Consider now the integral

$$\int_0^1 F([(1-t)(1-\lambda)+\lambda t]x+[(1-t)\lambda+(1-\lambda)t]y)\,dt.$$

Put

$$s = (1-t)(1-\lambda) + \lambda t = 1 - \lambda + (2\lambda - 1)t.$$

Then

$$1 - s = (1-t)\lambda + (1-\lambda)t.$$

If $\lambda \neq \frac{1}{2}$, then $s = 1 - \lambda + (2\lambda - 1)t$ is a change of variable with $dt = \frac{1}{2\lambda-1}$ and we have

$$\int_0^1 F([(1-t)(1-\lambda) + \lambda t]x + [(1-t)\lambda + (1-\lambda)t]y)\,dt$$
$$= \frac{1}{2\lambda-1}\int_{1-\lambda}^{\lambda} F(sx + (1-s)y)\,ds.$$

Now, making use of (5.5) we get the desired result (5.1). □

Remark 1. *We observe that for $\lambda \to \frac{1}{2}$ we recapture from (5.1) the inequality (2.9).*

If we take in (5.1) $\lambda = \frac{3}{4}$, then we get

$$\left\| \int_0^1 F[uy + (1-u)x]\,du - 2\int_{1/4}^{3/4} F(sx + (1-s)y)\,ds \right\|_F \leq \frac{1}{32}K\|y-x\|_X^2. \tag{5.9}$$

Let $f(z) = \sum_{n=0}^{\infty}\alpha_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. For any x, y in the Banach algebra $\mathcal{B}$ with $\|x\|, \|y\| \leq M < R$, $M > 0$ we have that

$$\left\| \int_0^1 f(uy + (1-u)x)\,du - \frac{1}{2\lambda-1}\int_{1-\lambda}^{\lambda} f(sx + (1-s)y)\,ds \right\| \leq \frac{1}{6}f_a''(M)\lambda(1-\lambda)\|y-x\|^2 \tag{5.10}$$

for any $\lambda \in [0,1]$, $\lambda \neq \frac{1}{2}$.

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed linear spaces, with Y complete, C an open convex subset of X and $f : C \to Y$ a twice-differentiable mapping on C. Then for any $x, y \in C$ and $\lambda \in [0,1]$, $\lambda \neq \frac{1}{2}$, we have

$$\left\| \int_0^1 f(uy + (1-u)x)\,du - \frac{1}{2\lambda-1}\int_{1-\lambda}^{\lambda} f(sx + (1-s)y)\,ds \right\| \leq \frac{1}{6}\sup_{z\in C}\|f''(z)\|_{\mathcal{L}(X^2;Y)}\lambda(1-\lambda)\|y-x\|^2. \tag{5.11}$$

6. Applications for Gâteaux Differentiable Functions

Following [11, p. 59], let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed linear spaces, Ω an open subset of X and $f : \Omega \to Y$. If $a \in \Omega$, $u \in X \setminus \{0\}$ and if the limit

$$\lim_{t\to 0}\frac{1}{t}[f(a+tu) - f(a)]$$

exists, then we denote this derivative $\partial_u f(a)$. It is called the directional derivative of f at a in the direction u. If the directional derivative is defined in all directions and there is a continuous linear mapping Φ from X into Y such that for all $u \in X$

$$\partial_u f(a) = \Phi(u),$$

then we say that f is Gâteaux-differentiable at a and that Φ is the Gâteaux differential of f at a. If a mapping f is differentiable at a point a, then clearly all its directional derivatives exist and we have

$$\partial_u f(a) = f'(a) u, \; u \in X.$$

Thus f is Gâteaux-differentiable at a. However, the Gâteaux differential may exist without the differential existing. The existence of directional derivatives at a point does not imply that the mapping is Gâteaux-differentiable. To distinguish the differential from the Gâteaux differential, the differential is often referred as the Fréchet differential.

Theorem 6. *Let $(X; \|\cdot\|_X)$ and $(Y; \|\cdot\|_Y)$ be two normed linear spaces over the complex number field $\mathbb{C}$. Assume that the mapping $F : C \subset X \to Y$ is defined on the open convex set C and $F \in \mathcal{BN}_K(C)$ for some $K > 0$. If $x_k \in C$, $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^n p_k = 1$ and F is Gâteaux-differentiable at $\sum_{k=1}^n p_k x_k \in C$, then for any $y_j \in C$ and $q_j \geq 0$ for $j \in \{1, ..., m\}$ with $\sum_{j=1}^m q_j = 1$ and $\sum_{j=1}^m q_j y_j = \sum_{k=1}^n p_k x_k$ we have*

$$\left\| \sum_{j=1}^m q_j F(y_j) - F\left(\sum_{k=1}^n p_k x_k \right) \right\|_Y \leq \frac{1}{2} K \sum_{j=1}^m q_j \left\| y_j - \sum_{k=1}^n p_k x_k \right\|_X^2 . \tag{6.1}$$

In particular, we have

$$\left\| \sum_{j=1}^n p_j F(x_j) - F\left(\sum_{k=1}^n p_k x_k \right) \right\|_Y \leq \frac{1}{2} K \sum_{j=1}^n p_j \left\| x_j - \sum_{k=1}^n p_k x_k \right\|_X^2 . \tag{6.2}$$

Proof. Since $F \in \mathcal{BN}_K(C)$ then we have

$$\| \lambda [F(y) - F(x)] + F(x) - F((1-\lambda) x + \lambda y) \|_Y \leq \frac{1}{2} K \lambda (1-\lambda) \|x - y\|_X^2$$

for any $x, y \in C$ and $\lambda \in [0, 1]$.

This implies that

$$\left\| F(y) - F(x) - \frac{F(x + \lambda(y - x)) - F(x)}{\lambda} \right\|_Y \leq \frac{1}{2} K (1-\lambda) \|x - y\|_X^2 \tag{6.3}$$

for any $x, y \in C$ and $\lambda \in (0, 1)$.

If we assume that F is Gâteaux-differentiable at x, then by taking the limit over $\lambda \to 0+$ in (6.3) we get

$$\| F(y) - F(x) - \partial_{y-x} F(x) \|_Y \leq \frac{1}{2} K \|x - y\|_X^2 \tag{6.4}$$

for any $x, y \in C$.

Now, if F is Gâteaux-differentiable at $\sum_{k=1}^{n} p_k x_k \in C$, then

$$\left\| F(y) - F\left(\sum_{k=1}^{n} p_k x_k\right) - \partial_{y-\sum_{k=1}^{n} p_k x_k} F\left(\sum_{k=1}^{n} p_k x_k\right) \right\|_Y \leq \frac{1}{2} K \left\| \sum_{k=1}^{n} p_k x_k - y \right\|_X^2 \tag{6.5}$$

for any $y \in C$.

If $y_j \in C$ and $q_j \geq 0$ for $j \in \{1, ..., m\}$ with $\sum_{j=1}^{m} q_j = 1$, then by (6.5) we have

$$\sum_{j=1}^{m} q_j \left\| F(y_j) - F\left(\sum_{k=1}^{n} p_k x_k\right) - \partial_{y_j-\sum_{k=1}^{n} p_k x_k} F\left(\sum_{k=1}^{n} p_k x_k\right) \right\|_Y \leq \frac{1}{2} K \sum_{j=1}^{m} q_j \left\| \sum_{k=1}^{n} p_k x_k - y_j \right\|_X^2 . \tag{6.6}$$

By the generalized triangle inequality we have

$$\left\| \sum_{j=1}^{m} q_j F(y_j) - F\left(\sum_{k=1}^{n} p_k x_k\right) - \partial_{\sum_{j=1}^{m} q_j y_j-\sum_{k=1}^{n} p_k x_k} F\left(\sum_{k=1}^{n} p_k x_k\right) \right\|_Y \leq \sum_{j=1}^{m} q_j \left\| F(y_j) - F\left(\sum_{k=1}^{n} p_k x_k\right) - \partial_{y_j-\sum_{k=1}^{n} p_k x_k} F\left(\sum_{k=1}^{n} p_k x_k\right) \right\|_Y \tag{6.7}$$

and by (6.6) and (6.7) we have the following inequality of interest

$$\left\| \sum_{j=1}^{m} q_j F(y_j) - F\left(\sum_{k=1}^{n} p_k x_k\right) - \partial_{\sum_{j=1}^{m} q_j y_j-\sum_{k=1}^{n} p_k x_k} F\left(\sum_{k=1}^{n} p_k x_k\right) \right\|_Y \leq \frac{1}{2} K \sum_{j=1}^{m} q_j \left\| \sum_{k=1}^{n} p_k x_k - y_j \right\|_X^2 . \tag{6.8}$$

If we take $\sum_{j=1}^{m} q_j y_j = \sum_{k=1}^{n} p_k x_k$ in (6.8), then we get the desired inequality (6.1).

The inequality (6.2) follows by (6.1) on taking $m = n$ and $q_j = p_j$, $j \in \{1, ..., n\}$. □

Remark 2. *If $(X; \langle \cdot, \cdot \rangle)$ is an inner product space, then*

$$\sum_{j=1}^{n} p_j \left\| x_j - \sum_{k=1}^{n} p_k x_k \right\|_X^2 = \sum_{j=1}^{n} p_j \|x_j\|_X^2 - \left\| \sum_{k=1}^{n} p_k x_k \right\|_X^2$$

and by (6.2) we have

$$\left\| \sum_{j=1}^{n} p_j F(x_j) - F\left(\sum_{k=1}^{n} p_k x_k\right) \right\|_Y \leq \frac{1}{2} K \left[\sum_{j=1}^{n} p_j \|x_j\|_X^2 - \left\| \sum_{k=1}^{n} p_k x_k \right\|_X^2 \right] . \tag{6.9}$$

Corollary 3. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed linear spaces, C an open convex subset of X and $F : C \to Y$ a twice-differentiable mapping on C. If $x_k \in C$, $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$, then*

$$\left\| \sum_{j=1}^{n} p_j F(x_j) - F\left(\sum_{k=1}^{n} p_k x_k\right) \right\|_Y \leq \frac{1}{2} \sup_{z \in C} \|F''(z)\|_{\mathcal{L}(X^2;Y)} \sum_{j=1}^{n} p_j \left\| x_j - \sum_{k=1}^{n} p_k x_k \right\|_X^2 . \tag{6.10}$$

We also have:

Theorem 7. *Let $(X; \|\cdot\|_X)$ and $(Y; \|\cdot\|_Y)$ be two normed linear spaces over the complex number field $\mathbb{C}$. Assume that the mapping $F : C \subset X \to Y$ is defined on the open convex set C and $F \in \mathcal{BN}_K(C)$ for some $K > 0$. Let $x_k \in C$, $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$ and F is Gâteaux-differentiable at x_k for any $k \in \{1, ..., n\}$. If there exists $z \in C$ such that*

$$\sum_{k=1}^{n} p_k \partial_z F(x_k) = \sum_{k=1}^{n} p_k \partial_{x_k} F(x_k), \tag{6.11}$$

then we have

$$\left\| F(z) - \sum_{k=1}^{n} p_k F(x_k) \right\|_Y \leq \frac{1}{2} K \sum_{k=1}^{n} p_k \|x_k - z\|_X^2 . \tag{6.12}$$

Proof. From (6.4) we have

$$\|F(y) - F(x_k) - \partial_{y-x_k} F(x_k)\|_Y \leq \frac{1}{2} K \|x_k - y\|_X^2 \tag{6.13}$$

for any $y \in C$ and for any $k \in \{1, ..., n\}$.

If we multiply (6.13) by $p_k \geq 0$ for $k \in \{1, ..., n\}$ and sum, we get

$$\sum_{k=1}^{n} p_k \|F(y) - F(x_k) - \partial_{y-x_k} F(x_k)\|_Y \leq \frac{1}{2} K \sum_{k=1}^{n} p_k \|x_k - y\|_X^2 \tag{6.14}$$

for any $y \in C$.

By the generalized triangle inequality we get

$$\begin{aligned} &\sum_{k=1}^{n} p_k \|F(y) - F(x_k) - \partial_{y-x_k} F(x_k)\|_Y \\ &\geq \left\| \sum_{k=1}^{n} p_k [F(y) - F(x_k) - \partial_{y-x_k} F(x_k)] \right\|_Y \\ &= \left\| F(y) - \sum_{k=1}^{n} p_k F(x_k) - \sum_{k=1}^{n} p_k \partial_{y-x_k} F(x_k) \right\|_Y . \end{aligned} \tag{6.15}$$

By the linearity of the Gâteaux differential we have

$$\sum_{k=1}^{n} p_k \partial_{y-x_k} F(x_k) = \sum_{k=1}^{n} p_k \partial_y F(x_k) - \sum_{k=1}^{n} p_k \partial_{x_k} F(x_k)$$

and by (6.14) and (6.15) we have the inequality of interest

$$\left\| F(y) - \sum_{k=1}^{n} p_k F(x_k) - \sum_{k=1}^{n} p_k \partial_y F(x_k) + \sum_{k=1}^{n} p_k \partial_{x_k} F(x_k) \right\|_Y \leq \frac{1}{2} K \sum_{k=1}^{n} p_k \|x_k - y\|_X^2 \tag{6.16}$$

for any $y \in C$.

Now, if $z \in C$ is such that (6.11) holds, then by (6.16) we get the desired result (6.12). □

Remark 3. *Let $x_k \in C$, $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$ and F is differentiable at x_k for any $k \in \{1, ..., n\}$. If there exists $z \in C$ such that*

$$\sum_{k=1}^{n} p_k F'(x_k) z = \sum_{k=1}^{n} p_k F(x_k) x_k, \tag{6.17}$$

then we have the inequality (6.12).

Moreover, if the operator $\sum_{k=1}^{n} p_k F'(x_k)$ is invertible and

$$\left(\sum_{k=1}^{n} p_k F'(x_k) \right)^{-1} \left(\sum_{k=1}^{n} p_k F(x_k) x_k \right) \in C, \tag{6.18}$$

then we have the inequality

$$\left\| F\left(\left(\sum_{k=1}^{n} p_k F'(x_k) \right)^{-1} \left(\sum_{k=1}^{n} p_k F(x_k) x_k \right) \right) - \sum_{k=1}^{n} p_k F(x_k) \right\|_Y \leq \frac{1}{2} K \sum_{k=1}^{n} p_k \left\| x_k - \left(\sum_{k=1}^{n} p_k F'(x_k) \right)^{-1} \left(\sum_{k=1}^{n} p_k F(x_k) x_k \right) \right\|_X^2 . \tag{6.19}$$

Corollary 4. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed linear spaces, C an open convex subset of X and $F : C \to Y$ a twice-differentiable mapping on C. If $x_k \in C$, $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$ and there exists $z \in C$ such that (6.17) holds, then we have the inequality (6.12) with $K = \sup_{z \in C} \|F''(z)\|_{\mathcal{L}(X^2;Y)}$. Moreover, if the operator $\sum_{k=1}^{n} p_k F'(x_k)$ is invertible and the condition (6.18) holds, then we have the inequality (6.19) with $K = \sup_{z \in C} \|F''(z)\|_{\mathcal{L}(X^2;Y)}$.*

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