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DISCRETE INEQUALITIES OF JENSEN TYPE FOR λ-CONVEX FUNCTIONS ON LINEAR SPACES

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ABSTRACT. Some discrete inequalities of Jensen type for λ-convex functions defined on convex subsets in real or complex linear spaces are given. Applications for norm inequalities are provided as well.

1. INTRODUCTION

We recall here some concepts of convexity that are well known in the literature. Let I be an interval in $\mathbb{R}$.

Definition 1 ([39]). *We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0,1)$ we have*

$$f(tx + (1-t)y) \leq \frac{1}{t} f(x) + \frac{1}{1-t} f(y). \tag{1.1}$$

Some further properties of this class of functions can be found in [29], [30], [32], [45], [48] and [49]. Among others, its has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

The above concept can be extended for functions $f : C \subseteq X \to [0,\infty)$ where C is a convex subset of the real or complex linear space X and the inequality (1.1) is satisfied for any vectors $x, y \in C$ and $t \in (0,1)$. If the function $f : C \subseteq X \to \mathbb{R}$ is non-negative and convex, then is of Godunova-Levin type.

Definition 2 ([32]). *We say that a function $f : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0,1]$ we have*

$$f(tx + (1-t)y) \leq f(x) + f(y). \tag{1.2}$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contains all nonnegative monotone, convex and *quasi convex functions,* i. e. nonnegative functions satisfying

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\} \tag{1.3}$$

for all $x, y \in I$ and $t \in [0,1]$.

For some results on P-functions see [32] and [46] while for quasi convex functions, the reader can consult [31].

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If $f : C \subseteq X \to [0, \infty)$, where C is a convex subset of the real or complex linear space X, then we say that it is of P-type (or quasi-convex) if the inequality (1.2) (or (1.3)) holds true for $x, y \in C$ and $t \in [0, 1]$.

Definition 3 ([7])**.** *Let s be a real number, $s \in (0, 1]$. A function $f : [0, \infty) \to [0, \infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0, \infty)$ and $t \in [0, 1]$.

For some properties of this class of functions see [1], [2], [7], [8], [27], [28], [40], [42] and [51].

The concept of Breckner s-convexity can be similarly extended for functions defined on convex subsets of linear spaces.

It is well known that if $(X, \|\cdot\|)$ is a normed linear space, then the function $f(x) = \|x\|^p, p \geq 1$ is convex on X.

Utilising the elementary inequality $(a+b)^s \leq a^s + b^s$ that holds for any $a, b \geq 0$ and $s \in (0, 1]$, we have for the function $g(x) = \|x\|^s$ that

$$\begin{aligned} g(tx + (1-t)y) &= \|tx + (1-t)y\|^s \leq (t\|x\| + (1-t)\|y\|)^s \\ &\leq (t\|x\|)^s + [(1-t)\|y\|]^s \\ &= t^s g(x) + (1-t)^s g(y) \end{aligned}$$

for any $x, y \in X$ and $t \in [0, 1]$, which shows that g is Breckner s-convex on X.

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}$, $(0, 1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 4 ([54])**.** *Let $h : J \to [0, \infty)$ with h not identical to 0. We say that $f : I \to [0, \infty)$ is an h-convex function if for all $x, y \in I$ we have*

$$f(tx + (1-t)y) \leq h(t) f(x) + h(1-t) f(y) \tag{1.4}$$

for all $t \in (0, 1)$.

For some results concerning this class of functions see [54], [6], [43], [52], [50] and [53].

This concept can be extended for functions defined on convex subsets of linear spaces in the same way as above replacing the interval I be the corresponding convex subset C of the linear space X.

We can introduce now another class of functions.

Definition 5. *We say that the function $f : C \subseteq X \to [0, \infty)$ is of s-Godunova-Levin type, with $s \in [0, 1]$, if*

$$f(tx + (1-t)y) \leq \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y), \tag{1.5}$$

for all $t \in (0, 1)$ and $x, y \in C$.

We observe that for $s = 0$ we obtain the class of P-functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(C)$ the class of s-Godunova-Levin functions defined on C, then we obviously have

$$P(C) = Q_0(C) \subseteq Q_{s_1}(C) \subseteq Q_{s_2}(C) \subseteq Q_1(C) = Q(C)$$

for $0 \leq s_1 \leq s_2 \leq 1$.

For different inequalities related to these classes of functions, see [1]-[4], [6], [9]-[38], [41]-[43] and [46]-[53].

A function $h : J \to \mathbb{R}$ is said to be *supermultiplicative* if

$$h(ts) \geq h(t)h(s) \text{ for any } t, s \in J. \tag{1.6}$$

If the inequality (1.6) is reversed, then h is said to be *submultiplicative*. If the equality holds in (1.6) then h is said to be a multiplicative function on J.

In [54] it has been noted that if $h : [0, \infty) \to [0, \infty)$ with $h(t) = (x + c)^{p-1}$, then for $c = 0$ the function h is multiplicative. If $c \geq 1$, then for $p \in (0, 1)$ the function h is supermultiplicative and for $p > 1$ the function is submultiplicative.

We observe that, if h, g are nonnegative and supermultiplicative, the same is their product. In particular, if h is supermultiplicative then its product with a power function $\ell_r(t) = t^r$ is also supermultiplicative.

The case of h-convex function with h supermultiplicative is of interest due to several Jensen type inequalities one can derive.

The following results were obtained in [54] for functions of a real variable. However, with similar proofs they can be extended to h-convex function defined on convex subsets in linear spaces.

Theorem 1. *Let $h : J \to [0, \infty)$ be a supermultiplicative function on J. If the function $f : C \subseteq X \to [0, \infty)$ is h-convex on the convex subset C of the linear space X, then for any $w_i \geq 0$, $i \in \{1, ..., n\}$, $n \geq 2$ with $W_n := \sum_{i=1}^n w_i > 0$ we have*

$$f\left(\frac{1}{W_n}\sum_{i=1}^n w_i x_i\right) \leq \sum_{i=1}^n h\left(\frac{w_i}{W_n}\right) f(x_i). \tag{1.7}$$

In particular, we have the unweighted inequality

$$f\left(\frac{1}{n}\sum_{i=1}^n x_i\right) \leq h\left(\frac{1}{n}\right)\sum_{i=1}^n f(x_i). \tag{1.8}$$

Corollary 1 ([28]). *If the function $f : C \subseteq X \to [0, \infty)$ is Breckner s-convex on the convex subset C of the linear space X with $s \in (0, 1)$, then for any $x_i \in C$, $w_i \geq 0$, $i \in \{1, ..., n\}$, $n \geq 2$ with $W_n := \sum_{i=1}^n w_i > 0$ we have*

$$f\left(\frac{1}{W_n}\sum_{i=1}^n w_i x_i\right) \leq \frac{1}{W_n^s}\sum_{i=1}^n w_i^s f(x_i). \tag{1.9}$$

If $(X, \|\cdot\|)$ is a normed linear space, then for $s \in (0, 1)$, $x_i \in X$, $w_i \geq 0$, $i \in \{1, ..., n\}$, $n \geq 2$ with $W_n := \sum_{i=1}^n w_i > 0$ we have the norm inequality

$$\left\|\sum_{i=1}^n w_i x_i\right\|^s \leq \sum_{i=1}^n w_i^s \|x_i\|^s. \tag{1.10}$$

Corollary 2. *If the function $f : C \subseteq X \to [0, \infty)$ is of s-Godunova-Levin type, with $s \in [0, 1]$, on the convex subset C of the linear space X, then for any $x_i \in C$, $w_i > 0$, $i \in \{1, ..., n\}$, $n \geq 2$ we have*

$$f\left(\frac{1}{W_n}\sum_{i=1}^n w_i x_i\right) \leq W_n^s \sum_{i=1}^n \frac{1}{w_i^s} f(x_i). \tag{1.11}$$

This result generalizes the Jensen type inequality obtained in [45] for $s=1$.

Let K be a finite non-empty set of positive integers. We can define the index set function, see also [54]

$$J(K) := \sum_{i\in K} h(w_i) f(x_i) - h(W_K) f\left(\frac{1}{W_K}\sum_{i\in K} w_i x_i\right), \tag{1.12}$$

where $W_K := \sum_{i\in K} w_i > 0$, $x_i \in C$, $i \in K$.

We notice that if $h : [0,\infty) \to [0,\infty)$ is a supermultiplicative function on $[0,\infty)$ and the function $f : C \subseteq X \to [0,\infty)$ is h-convex on the convex subset C of the linear space X, then

$$J(K) \geq h(W_K)\left[\sum_{i\in K} h\left(\frac{w_i}{W_K}\right) f(x_i) - f\left(\frac{1}{W_K}\sum_{i\in K} w_i x_i\right)\right] \geq 0. \tag{1.13}$$

Theorem 2. *Assume that $h : [0,\infty) \to [0,\infty)$ is a supermultiplicative function on $[0,\infty)$ and the function $f : C \subseteq X \to [0,\infty)$ is h-convex on the convex subset C of the linear space X. Let M and K be finite non-empty sets of positive integers, $w_i > 0$, $x_i \in C$, $i \in K \cup M$. Then*

$$J(K\cup M) \geq J(K) + J(M) \geq 0, \tag{1.14}$$

i.e., J is a superadditive index set functional.

This results was proved in an equivalent form in [54] for functions of a real variable. The proof is similar for functions defined on convex sets in linear spaces.

Corollary 3. *With the assumptions of Theorem 2 and if we note $M_k := \{1,...,k\}$, then*

$$J(M_n) \geq J(M_{n-1}) \geq ... \geq J(M_2) \geq 0 \tag{1.15}$$

and

$$\begin{aligned} &J(M_n) \\ &\geq \max_{1\leq i<j\leq n}\left\{h(w_i) f(x_i) + h(w_j) f(x_j) - h(w_i+w_j) f\left(\frac{w_i x_i + w_j x_j}{w_i+w_j}\right)\right\} \\ &\geq 0. \end{aligned} \tag{1.16}$$

If we consider the functional

$$J_s(K) := \sum_{i\in K} w_i^s \|x_i\|^s - \left\|\sum_{i\in K} w_i x_i\right\|^s$$

for $s\in(0,1)$, then we have the norm inequalities

$$\begin{aligned} \sum_{i=1}^{n} w_i^s \|x_i\|^s - \left\|\sum_{i=1}^{n} w_i x_i\right\|^s &\geq \sum_{i=1}^{n-1} w_i^s \|x_i\|^s - \left\|\sum_{i=1}^{n-1} w_i x_i\right\|^s \\ &\geq ... \geq \sum_{i=1}^{2} w_i^s \|x_i\|^s - \left\|\sum_{i=1}^{2} w_i x_i\right\|^s \geq 0 \end{aligned} \tag{1.17}$$

and

$$\sum_{i=1}^{n} w_i^s \|x_i\|^s - \left\|\sum_{i=1}^{n} w_i x_i\right\|^s \geq \max_{1\leq i<j\leq n} \left\{w_i^s \|x_i\|^s + w_j^s \|x_j\|^s - \|w_i x_i + w_j x_j\|^s\right\} \geq 0 \tag{1.18}$$

where $w_i \geq 0$, $x_i \in X$, $i \in \{1, ..., n\}$, $n \geq 2$.

2. λ-Convex Functions

We start with the following definition (see also [25]):

Definition 6. *Let $\lambda : [0, \infty) \to [0, \infty)$ be a function with the property that $\lambda(t) > 0$ for all $t > 0$. A mapping $f : C \to \mathbb{R}$ defined on convex subset C of a linear space X is called λ-convex on C if*

$$f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) \leq \frac{\lambda(\alpha) f(x) + \lambda(\beta) f(y)}{\lambda(\alpha + \beta)} \tag{2.1}$$

for all $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ and $x, y \in C$.

We observe that if $f : C \to \mathbb{R}$ is λ-convex on C, then f is h-convex on C with $h(t) = \frac{\lambda(t)}{\lambda(1)}$, $t \in [0, 1]$.

If $f : C \to [0, \infty)$ is h-convex function with h supermultiplicative on $[0, \infty)$, then f is λ-convex with $\lambda = h$.

Indeed, if $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ and $x, y \in C$ then

$$\begin{aligned} f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) &\leq h\left(\frac{\alpha}{\alpha + \beta}\right) f(x) + h\left(\frac{\beta}{\alpha + \beta}\right) f(y) \\ &\leq \frac{h(\alpha) f(x) + h(\beta) f(y)}{h(\alpha + \beta)}. \end{aligned}$$

The following proposition contain some properties of λ-convex functions [25].

Proposition 1. *Let $f : C \to \mathbb{R}$ be a λ-convex function on C.*

(i) If $\lambda(0) > 0$, then we have $f(x) \geq 0$ for all $x \in C$;

(ii) If there exists $x_0 \in C$ so that $f(x_0) > 0$, then

$$\lambda(\alpha + \beta) \leq \lambda(\alpha) + \lambda(\beta)$$

for all $\alpha, \beta > 0$, i.e. the mapping λ is subadditive on $(0, \infty)$.

(iii) If there exists $x_0, y_0 \in C$ with $f(x_0) > 0$ and $f(y_0) < 0$, then

$$\lambda(\alpha + \beta) = \lambda(\alpha) + \lambda(\beta)$$

for all $\alpha, \beta > 0$, i.e. the mapping λ is additive on $(0, \infty)$.

We have the following result providing many examples of subadditive functions $\lambda : [0, \infty) \to [0, \infty)$.

Theorem 3 ([25])**.** *Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ a power series with nonnegative coefficients $a_n \geq 0$ for all $n \in \mathbb{N}$ and convergent on the open disk $D(0, R)$ with $R > 0$ or $R = \infty$. If $r \in (0, R)$ then the function $\lambda_r : [0, \infty) \to [0, \infty)$ given by*

$$\lambda_r(t) := \ln\left[\frac{h(r)}{h(r\exp(-t))}\right] \tag{2.2}$$

is nonnegative, increasing and subadditive on $[0, \infty)$.

We have the following fundamental examples of power series with positive coefficients

$$h(z)=\sum_{n=0}^{\infty} z^n=\frac{1}{1-z},\ z\in D(0,1) \tag{2.3}$$
$$h(z)=\sum_{n=0}^{\infty} \frac{1}{n!}z^n=\exp(z) \qquad z\in\mathbb{C},$$
$$h(z)=\sum_{n=0}^{\infty} \frac{1}{(2n)!}z^{2n}=\cosh z,\ z\in\mathbb{C};$$
$$h(z)=\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}z^{2n+1}=\sinh z,\ z\in\mathbb{C};$$
$$h(z)=\sum_{n=1}^{\infty} \frac{1}{n}z^n=\ln\frac{1}{1-z},\ z\in D(0,1).$$

Other important examples of functions as power series representations with positive coefficients are:

$$h(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}=\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right), \qquad z\in D(0,1); \tag{2.4}$$
$$h(z)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}z^{2n+1}=\sin^{-1}(z), \qquad z\in D(0,1);$$
$$h(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}=\tanh^{-1}(z), \qquad z\in D(0,1);$$
$$h(z)={}_2F_1(\alpha,\beta,\gamma,z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}z^n,\alpha,\beta,\gamma>0,$$
$$z\in D(0,1);$$

where Γ is *Gamma function.*

Remark 1. *Now, if we take* $h(z)=\frac{1}{1-z}$, $z\in D(0,1)$, *then*

$$\lambda_r(t)=\ln\left[\frac{1-r\exp(-t)}{1-r}\right] \tag{2.5}$$

is nonnegative, increasing and subadditive on $[0,\infty)$ *for any* $r\in(0,1)$.
If we take $h(z)=\exp(z)$, $z\in\mathbb{C}$ *then*

$$\lambda_r(t)=r[1-\exp(-t)] \tag{2.6}$$

is nonnegative, increasing and subadditive on $[0,\infty)$ *for any* $r>0$.

Corollary 4 ([25])**.** *Let* $h(z)=\sum_{n=0}^{\infty}a_nz^n$ *be a power series with nonnegative coefficients* $a_n\geq 0$ *for all* $n\in\mathbb{N}$ *and convergent on the open disk* $D(0,R)$ *with* $R>0$ *or* $R=\infty$ *and* $r\in(0,R)$. *For a mapping* $f:C\to\mathbb{R}$ *defined on convex subset* C *of a linear space* X, *the following statements are equivalent:*

(i) The function f is λ_r-convex with $\lambda_r : [0,\infty) \to [0,\infty)$,

$$\lambda_r(t) := \ln\left[\frac{h(r)}{h(r\exp(-t))}\right];$$

(ii) We have the inequality

$$\left[\frac{h(r)}{h(r\exp(-\alpha-\beta))}\right]^{f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)} \le \left[\frac{h(r)}{h(r\exp(-\alpha))}\right]^{f(x)}\left[\frac{h(r)}{h(r\exp(-\beta))}\right]^{f(y)} \tag{2.7}$$

for any $\alpha,\beta\ge 0$ with $\alpha+\beta>0$ and $x,y\in C$.

(iii) We have the inequality

$$\frac{[h(r\exp(-\alpha))]^{f(x)}[h(r\exp(-\beta))]^{f(y)}}{[h(r\exp(-\alpha-\beta))]^{f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)}} \le [h(r)]^{f(x)+f(y)-f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)} \tag{2.8}$$

for any $\alpha,\beta\ge 0$ with $\alpha+\beta>0$ and $x,y\in C$.

Remark 2. *We observe that, in the case when*

$$\lambda_r(t) = r[1-\exp(-t)],\ t\ge 0,$$

then the function f is λ_r-convex on convex subset C of a linear space X iff

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) \le \frac{[1-\exp(-\alpha)]f(x)+[1-\exp(-\beta)]f(y)}{1-\exp(-\alpha-\beta)} \tag{2.9}$$

for any $\alpha,\beta\ge 0$ with $\alpha+\beta>0$ and $x,y\in C$.

We observe that this definition is independent of $r>0$.

The inequality (2.9) is equivalent with

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) \le \frac{\exp(\beta)[\exp(\alpha)-1]f(x)+\exp(\alpha)[\exp(\beta)-1]f(y)}{\exp(\alpha+\beta)-1} \tag{2.10}$$

for any $\alpha,\beta\ge 0$ with $\alpha+\beta>0$ and $x,y\in C$.

We can give now more examples of subadditive functions that can be used to define λ-convex mappings on linear spaces.

Let $I=(0,\infty)$ or $[0,\infty)$. A function $h: I\to\mathbb{R}$ is called *superadditive (subadditive)* on I if

(iii) $h(t+s)\ge(\le)\,h(t)+h(s)$ for any $t,s\in I$

and *nonnegative (strictly positive)* on I if, obviously, it satisfies

(iv) $h(t)\ge(>)\,0$ for each $t\in I$.

The following result holds:

Theorem 4. *If $h: I\to[0,\infty)$ is a superadditive (subadditive) function on I and $p\ge 1$ $(0<p<1)$ then the function*

$$\Psi_p : I\to[0,\infty),\ \Psi_p(t) = t^{1-\frac{1}{p}}h(t) \tag{2.11}$$

is superadditive (subadditive) on I.

Proof. First of all we observe that the following elementary inequality holds:

$$(\alpha+\beta)^p \ge (\le)\,\alpha^p+\beta^p \tag{2.12}$$

for any $\alpha,\beta\ge 0$ and $p\ge 1$ $(0<p<1)$.

Indeed, if we consider the function $f_p:[0,\infty)\to\mathbb{R}$, $f_p(t)=(t+1)^p-t^p$ we have $f_p'(t)=p\left[(t+1)^{p-1}-t^{p-1}\right]$. Observe that for $p>1$ and $t>0$ we have that

$f_p'(t) > 0$ showing that f_p is strictly increasing on the interval $[0,\infty)$. Now for $t = \frac{\alpha}{\beta}$ $(\beta > 0, \alpha \geq 0)$ we have $f_p(t) > f_p(0)$ giving that $\left(\frac{\alpha}{\beta}+1\right)^p - \left(\frac{\alpha}{\beta}\right)^p > 1$, i.e., the desired inequality (2.12).

For $p \in (0,1)$ we have that f_p is strictly decreasing on $[0,\infty)$ which proves the second case in (2.12).

Now, if h is superadditive (subadditive) and $p \geq 1$ $(0 < p < 1)$ then we have by (2.12) that

$$h^p(t+s) \geq (\leq) [h(t)+h(s)]^p \geq (\leq) h^p(t) + h^p(s) \tag{2.13}$$

for all $t, s \in I$.

Utilising (2.13) we have for any $t, s \in I$ that

$$\begin{aligned} \frac{h^p(t+s)}{t+s} &\geq (\leq) \frac{h^p(t)+h^p(s)}{t+s} = \frac{t\cdot\frac{h^p(t)}{t} + s\cdot\frac{h^p(s)}{s}}{t+s} \\ &= \frac{t\cdot\left[\frac{h(t)}{t^{1/p}}\right]^p + s\cdot\left[\frac{h(s)}{s^{1/p}}\right]^p}{t+s} =: I. \end{aligned} \tag{2.14}$$

Since for $p \geq 1$ $(0 < p < 1)$ the power function $g(t) = t^p$ is convex (concave), then

$$I \geq (\leq) \left[\frac{t\cdot\frac{h(t)}{t^{1/p}} + s\cdot\frac{h(s)}{s^{1/p}}}{t+s}\right]^p = \left[\frac{h(t)\,t^{1-1/p} + h(s)\,s^{1-1/p}}{t+s}\right]^p \tag{2.15}$$

for any $t, s \in I$.

By combining (2.14) with (2.5) we get

$$\frac{h^p(t+s)}{t+s} \geq (\leq) \left[\frac{h(t)\,t^{1-1/p} + h(s)\,s^{1-1/p}}{t+s}\right]^p,$$

which is equivalent with

$$\frac{h(t+s)}{(t+s)^{1/p}} \geq (\leq) \frac{h(t)\,t^{1-1/p} + h(s)\,s^{1-1/p}}{t+s}$$

i.e., by multiplying with $t+s$,

$$\Psi_p(t+s) \geq (\leq) \Psi_p(t) + \Psi_p(s)$$

for any $t, s \in I$ and the proof is complete. □

Corollary 5. *If $h : I \to [0,\infty)$ is a superadditive (subadditive) function on I and $p, q \geq 1$ $(0 < p, q < 1)$ then the two parameter function*

$$\Psi_{p,q} : I \to [0,\infty), \Psi_{p,q}(t) = t^{q\left(1-\frac{1}{p}\right)} h^q(t) \tag{2.16}$$

is superadditive (subadditive) on I.

Proof. Observe that $\Psi_{p,q}(t) = [\Psi_p(t)]^q$ for $t \in I$. Therefore, by Theorem 4 and the inequality (2.12) for $q \geq 1$ $(0 < q < 1)$ we have that

$$\begin{aligned} \Psi_{p,q}(t+s) &= [\Psi_p(t+s)]^q \geq (\leq) [\Psi_p(t) + \Psi_p(s)]^q \\ &\geq (\leq) [\Psi_p(t)]^q + [\Psi_p(s)]^q = \Psi_{p,q}(t) + \Psi_{p,q}(s) \end{aligned}$$

for any $t, s \in I$ and the statement is proved. □

Remark 3. *If we consider the function* $\psi_p(t) := t^{p-1}h^p(t)$ *then for* $p \geq 1$ $(0 < p < 1)$ *and* $h : I \to [0, \infty)$ *a superadditive (subadditive) function on* I, *the function* ψ_p *is also superadditive (subadditive) on* I.

The following result also holds:

Theorem 5. *If* $h : I \to (0, \infty)$ *is a superadditive function on* I *and* $0 < m < 1$, *then the function*

$$\Phi_p : I \to [0, \infty), \Phi_p(t) = \frac{t^{1-\frac{1}{m}}}{h(t)} \tag{2.17}$$

is subadditive on I.

Proof. Let $m := -p \in [-1, 0)$. For $m < 0$ we have the following inequality

$$(\alpha + \beta)^m \leq \alpha^m + \beta^m \tag{2.18}$$

for any $\alpha, \beta > 0$.

Indeed, by the convexity of the function $f_s(t) = t^m$ on $(0, \infty)$ with $m < 0$ we have that

$$(\alpha + \beta)^m \leq 2^{m-1}(\alpha^m + \beta^m)$$

for any $\alpha, \beta > 0$ and since, obviously, $2^{m-1}(\alpha^m + \beta^m) \leq \alpha^m + \beta^m$, then (2.18) holds true.

Taking into account that h is superadditive, then by (2.18) we have

$$h^m(t + s) \leq [h(t) + h(s)]^m \leq h^m(t) + h^m(s) \tag{2.19}$$

for any $t, s \in I$.

By (2.18) we have that

$$\begin{aligned} \frac{h^m(t+s)}{t+s} &\leq \frac{h^m(t) + h^m(s)}{t+s} \\ &= \frac{t \cdot \left[\frac{h(t)}{t^{1/m}}\right]^m + s \cdot \left[\frac{h(s)}{s^{1/m}}\right]^m}{t+s} \\ &= \frac{t \cdot \left[\frac{t^{1/m}}{h(t)}\right]^{-m} + s \cdot \left[\frac{s^{1/m}}{h(s)}\right]^{-m}}{t+s} =: J. \end{aligned} \tag{2.20}$$

By the concavity of the function $g(t) = t^{-m}$ with $m \in [-1, 0)$ we also have

$$J \leq \left[\frac{t \cdot \frac{t^{1/m}}{h(t)} + s \cdot \frac{s^{1/m}}{h(s)}}{t+s}\right]^{-m}. \tag{2.21}$$

Making use of (2.20) and (2.21) we get

$$\frac{h^m(t+s)}{t+s} \leq \left[\frac{t \cdot \frac{t^{1/m}}{h(t)} + s \cdot \frac{s^{1/m}}{h(s)}}{t+s}\right]^{-m}$$

for any $t, s \in I$, which is equivalent to

$$\frac{h^{-1}(t+s)}{(t+s)^{-1/m}} \leq \frac{\frac{t^{1+1/m}}{h(t)} + \frac{s^{1+1/m}}{h(s)}}{t+s}$$

and, with

$$\frac{(t+s)^{1+1/m}}{h(t+s)} \leq \frac{t^{1+1/m}}{h(t)} + \frac{s^{1+1/m}}{h(s)}$$

for any $t, s \in I$.

This completes the proof. □

The following result may be stated as well:

Corollary 6. *If $h : I \to [0, \infty)$ is a superadditive function on I and $0 < p, q < 1$ then the two parameter function*

$$\Phi_{p,q} : I \to [0, \infty), \Phi_{p,q}(t) = \frac{t^{q\left(1-\frac{1}{p}\right)}}{h^q(t)} \tag{2.22}$$

is subadditive on I.

Proof. Observe that $\Phi_{p,q}(t) = [\Phi_p(t)]^q$ for $t \in I$. Therefore, by Theorem 5 and the inequality (2.12) for $0 < q < 1$ we have that

$$\begin{aligned}\Phi_{p,q}(t+s) &= [\Phi_p(t+s)]^q \leq [\Phi_p(t) + \Phi_p(s)]^q \\ &\leq [\Phi_p(t)]^q + [\Phi_p(s)]^q = \Phi_{p,q}(t) + \Phi_{p,q}(s)\end{aligned}$$

for any $t, s \in I$ and the statement is proved. □

Remark 4. *If we consider the function $\varphi_p(t) := \frac{t^{p-1}}{h^p(t)}$ then for $0 < p < 1$ and $h : I \to [0, \infty)$ a superadditive function on I, the function ψ_p is subadditive on I.*

3. Jensen's Type Inequalities

The following inequality of Jensen's type holds:

Theorem 6. *Let $\lambda : [0, \infty) \to [0, \infty)$ be a function with the property that $\lambda(t) > 0$ for all $t > 0$ and a mapping $f : C \to \mathbb{R}$ defined on convex subset C of a linear space X. The following statements are equivalent:*

(i) f is λ-convex on C;

(ii) For all $x_i \in C$ and $p_i \geq 0$ with $i \in \{1, ..., n\}$, $n \geq 2$ so that $P_n > 0$ we have the inequality

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{\lambda(P_n)}\sum_{i=1}^{n} \lambda(p_i) f(x_i). \tag{3.1}$$

Proof. "$(ii) \Rightarrow (i)$". Follows for $n = 2$.

"$(i) \Rightarrow (ii)$". For $n = 2$ the inequality (2.12) follows by the Definition 6.

Assume that the inequality (3.1) is true for $2, ..., n-1$ ($n \geq 3$) and let prove it for n.

Let $p_i \geq 0$ with $i \in \{1, ..., n\}$, $n \geq 3$ so that $P_n > 0$. If $P_{n-1} = 0$, then $p_1 = ... = p_{n-1} = 0$ and $p_n > 0$ and the inequality (3.1) becomes

$$f(x_n) \leq \frac{\lambda(0)(f(x_1) + ... + f(x_{n-1})) + \lambda(p_n) f(x_n)}{\lambda(p_n)},$$

which is equivalent to

$$\lambda(0)(f(x_1) + ... + f(x_{n-1})) \geq 0. \tag{3.2}$$

Since f is λ-convex on C then for $\beta > 0$ and $x \in C$ we have

$$f\left(\frac{0x+\beta y}{0+\beta}\right) \le \frac{\lambda(0) f(x) + \lambda(\beta) f(y)}{\lambda(\beta)}$$

from where we get

$$\frac{\lambda(0) f(x)}{\lambda(\beta)} \ge 0$$

and since $\lambda(\beta) > 0$ we get $\lambda(0) f(x) \ge 0$. This implies that the inequality (3.2) is true for any $x_1, ..., x_{n-1} \in C$.

Now, let assume that $P_{n-1} > 0$. Then we have

$$\begin{aligned} f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) &= f\left(\frac{P_{n-1}\cdot \frac{1}{P_{n-1}}\sum_{i=1}^{n-1} p_i x_i + p_n x_n}{P_{n-1}+p_n}\right) \\ &\le \frac{\lambda(P_{n-1}) f\left(\frac{1}{P_{n-1}}\sum_{i=1}^{n-1} p_i x_i\right) + \lambda(p_n) f(x_n)}{\lambda(P_n)}. \end{aligned}$$

By the induction hypothesis we have

$$f\left(\frac{1}{P_{n-1}}\sum_{i=1}^{n-1} p_i x_i\right) \le \frac{1}{\lambda(P_{n-1})}\sum_{i=1}^{n-1}\lambda(p_i) f(x_i)$$

and thus, by the above inequality, we can state that

$$\begin{aligned} f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) &\le \frac{\lambda(P_{n-1}) \frac{1}{\lambda(P_{n-1})}\sum_{i=1}^{n-1}\lambda(p_i) f(x_i) + \lambda(p_n) f(x_n)}{\lambda(P_n)} \\ &= \frac{1}{\lambda(P_n)}\sum_{i=1}^{n}\lambda(p_i) f(x_i), \end{aligned}$$

and the theorem is thus proved. □

Corollary 7. *Let $f : C \to \mathbb{R}$ be a λ-convex function on C and $\alpha_i \in [0,1]$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} \alpha_i = 1$. Then for any $x_i \in C$ with $i \in \{1, ..., n\}$ we have the inequality*

$$f\left(\sum_{i=1}^{n}\alpha_i x_i\right) \le \frac{1}{\lambda(1)}\sum_{i=1}^{n}\lambda(\alpha_i) f(x_i). \tag{3.3}$$

In particular, we have

$$f\left(\frac{x_1+...+x_n}{n}\right) \le c(n)\frac{f(x_1)+...+f(x_n)}{n} \tag{3.4}$$

where

$$c(n) := \frac{n\lambda\left(\frac{1}{n}\right)}{\lambda(1)}, \ n \ge 2.$$

We have the following version of Jensen's inequality:

Corollary 8. *Let $f : C \to \mathbb{R}$ be a λ-convex function on C and $x_i \in C$ and $p_i \ge 0$ with $i \in \{1, ..., n\}$, $n \ge 2$ so that $P_n > 0$. Then we have the inequality*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \le \frac{1}{\lambda(1)}\sum_{i=1}^{n}\lambda\left(\frac{p_i}{P_n}\right) f(x_i). \tag{3.5}$$

The proof follows by (3.3) for $\alpha_i = \frac{p_i}{P_n}$, $i \in \{1, ..., n\}$.

Corollary 9. *Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ a power series with nonnegative coefficients $a_n \geq 0$ for all $n \in \mathbb{N}$ and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$. For a mapping $f : C \to \mathbb{R}$ defined on convex subset C of a linear space X, the following statements are equivalent:*

(i) The function f is λ_r-convex with $\lambda_r : [0,\infty) \to [0,\infty)$

$$\lambda_r(t) := \ln\left[\frac{h(r)}{h(r\exp(-t))}\right]$$

on C;

(ii) We have the inequality

$$\left[\frac{h(r)}{h(r\exp(-P_n))}\right]^{f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)} \leq \prod_{i=1}^{n}\left[\frac{h(r)}{h(r\exp(-p_i))}\right]^{f(x_i)} \tag{3.6}$$

for any $x_i \in C$ and $p_i \geq 0$ with $i \in \{1, ..., n\}$, $n \geq 2$ so that $P_n > 0$.

Now, let define the mapping:

$$J(I,p,x,f) := \sum_{i\in I}\lambda(p_i) f(x_i) - \lambda(P_I) f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right),$$

where $p := (p_i)_{i\in\mathbb{N}} \geq 0$, $I \in \mathcal{F}(\mathbb{N}) := \{I \subset \mathbb{N} | \ I \text{ is finite}\}$, $x := (x_i)_{i\in\mathbb{N}} \subset C$ and $P_I := \sum_{i\in I} p_i > 0$.

Theorem 7. *Assume that $f : C \to \mathbb{R}$ is a λ-convex function on C and p, x are as above. Then*

(i) For all $I, K \in \mathcal{F}(\mathbb{N}) \setminus \{\emptyset\}$ with $I \cap K = \emptyset$ we have the inequality

$$J(I \cup K, p, x, f) \geq J(I,p,x,f) + J(K,p,x,f) \geq 0, \tag{3.7}$$

i.e. the mapping $J(\cdot,p,x,f)$ is superadditive as an index set map on $\mathcal{F}(\mathbb{N})$;

(ii) For all $I, K \in \mathcal{F}(\mathbb{N}) \setminus \{\emptyset\}$ with $K \subsetneq I$ one has the inequality

$$J(I,p,x,f) \geq J(K,p,x,f) \geq 0, \tag{3.8}$$

i.e. the mapping $J(\cdot,p,x,f)$ is monotonic nondecreasing as an index set map on $\mathcal{F}(\mathbb{N})$.

Proof. (i) Let $I, K \in \mathcal{F}(\mathbb{N}) \setminus \{\emptyset\}$ with $I \cap K = \emptyset$, then

$$\begin{aligned}
&J(I \cup K, p, x, f) \\
&= \sum_{i\in I}\lambda(p_i) f(x_i) + \sum_{j\in K}\lambda(p_j) f(x_j) \\
&- \lambda(P_I + P_K) f\left[\frac{1}{P_I + P_K}\left(\sum_{i\in I} p_i x_i + \sum_{j\in K} p_j x_j\right)\right] \\
&= \sum_{i\in I}\lambda(p_i) f(x_i) + \sum_{j\in K}\lambda(p_j) f(x_j) \\
&- \lambda(P_I + P_K) f\left[\frac{P_I}{P_I + P_K}\left(\frac{\sum_{i\in I} p_i x_i}{P_I}\right) + \frac{P_K}{P_I + P_K}\left(\frac{\sum_{j\in K} p_j x_j}{P_K}\right)\right].
\end{aligned}$$

As f is λ-convex function on C, then

$$\begin{aligned} & f\left[\frac{P_I}{P_I+P_K}\left(\frac{\sum_{i\in I}p_ix_i}{P_I}\right)+\frac{P_K}{P_I+P_K}\left(\frac{\sum_{j\in K}p_jx_j}{P_K}\right)\right] \\ & \leq \frac{\lambda\left(P_I\right)f\left(\frac{\sum_{i\in I}p_ix_i}{P_I}\right)+\lambda\left(P_K\right)f\left(\frac{\sum_{j\in K}p_jx_j}{P_K}\right)}{\lambda\left(P_I+P_K\right)}. \end{aligned}$$

Therefore

$$\begin{aligned} J\left(I\cup K,p,x,f\right) &\geq \sum_{i\in I}\lambda\left(p_i\right)f\left(x_i\right)+\sum_{j\in K}\lambda\left(p_j\right)f\left(x_j\right) \\ & \quad -\lambda\left(P_I\right)f\left(\frac{\sum_{i\in I}p_ix_i}{P_I}\right)-\lambda\left(P_K\right)f\left(\frac{\sum_{j\in K}p_jx_j}{P_K}\right) \\ &= J\left(I,p,x,f\right)+J\left(K,p,x,f\right) \end{aligned}$$

and the inequality (3.7) is proved.

(ii) By the use of the inequality (3.7) we have

$$\begin{aligned} J\left(I,p,x,f\right) &= J\left(K\cup\left(I\setminus K\right),p,x,f\right)\geq J\left(K,p,x,f\right)+J\left(I\setminus K,p,x,f\right) \\ &\geq J\left(K,p,x,f\right) \end{aligned}$$

since $J\left(I\setminus K,p,x,f\right)\geq 0$, and the inequality (3.8) is proved. □

With the above assumptions, and if $p:=\left(p_i\right)_{i\in\mathbb{N}}>0$ we can consider the sequence

$$J_n\left(p,x,f\right):=\sum_{i=1}^{n}\lambda\left(p_i\right)f\left(x_i\right)-\lambda\left(P_n\right)f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right),\ n\geq 2.$$

Corollary 10. *Assume that $f:C\to\mathbb{R}$ is a λ-convex function on C, then*

$$J_n\left(p,x,f\right)\geq J_{n-1}\left(p,x,f\right)\geq ...\geq J_2\left(p,x,f\right)\geq 0 \tag{3.9}$$

and we have the inequality

$$\begin{aligned} & J_n\left(p,x,f\right) \\ & \geq \max_{1\leq i<j\leq n}\left\{\lambda\left(p_i\right)f\left(x_i\right)+\lambda\left(p_j\right)f\left(x_j\right)-\lambda\left(p_i+p_j\right)f\left(\frac{p_ix_i+p_jx_j}{p_i+p_j}\right)\right\} \\ & \geq 0 \end{aligned} \tag{3.10}$$

for all $n\geq 2$.

For a function f that is λ_r-convex on C with $\lambda_r:[0,\infty)\to[0,\infty)$ and

$$\lambda_r\left(t\right):=\ln\left[\frac{h\left(r\right)}{h\left(r\exp\left(-t\right)\right)}\right],$$

we can consider the functional

$$Q\left(I,p,x,f\right):=\frac{\prod_{i\in I}\left[\frac{h(r)}{h(r\exp(-p_i))}\right]^{f(x_i)}}{\left[\frac{h(r)}{h(r\exp(-P_I))}\right]^{f\left(\frac{1}{P_I}\sum_{i\in I}p_ix_i\right)}},$$

where $p:=\left(p_i\right)_{i\in\mathbb{N}}\geq 0$, $I\in\mathcal{F}\left(\mathbb{N}\right):=\left\{I\subset\mathbb{N}|\ I\text{ is finite}\right\}$, $x:=\left(x_i\right)_{i\in\mathbb{N}}\subset C$ and $P_I:=\sum_{i\in I}p_i>0$.

Corollary 11. *Assume that $f : C \to \mathbb{R}$ is a λ_r-convex function on C and p, x are as above. Then*

(i) For all $I, K \in \mathcal{F}(\mathbb{N}) \setminus \{\emptyset\}$ with $I \cap K = \emptyset$ we have the inequality

$$Q(I \cup K, p, x, f) \geq Q(I, p, x, f) Q(K, p, x, f), \tag{3.11}$$

i.e. the mapping $Q(\cdot, p, x, f)$ is supermultiplicative as an index set map on $\mathcal{F}(\mathbb{N})$;

(ii) For all $I, K \in \mathcal{F}(\mathbb{N}) \setminus \{\emptyset\}$ with $K \subsetneq I$ one has the inequality

$$Q(I, p, x, f) \geq Q(K, p, x, f) \geq 1. \tag{3.12}$$

The proof follows by Theorem 7 on observing that

$$\ln Q(I, p, x, f) = J(I, p, x, f)$$

for $\lambda = \lambda_r$.

In particular, if we consider the sequence

$$Q_n(p, x, f) := \frac{\prod_{i=1}^{n} \left[\frac{h(r)}{h(r\exp(-p_i))}\right]^{f(x_i)}}{\left[\frac{h(r)}{h(r\exp(-P_n))}\right]^{f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)}}, \ n \geq 2$$

then by Corollary 10 we have that

$$Q_n(p, x, f) \geq Q_{n-1}(p, x, f) \geq \dots \geq Q_2(p, x, f) \geq 1 \tag{3.13}$$

and

$$Q_n(p, x, f) \geq \max_{1 \leq i < j \leq n} \left\{ \frac{\left[\frac{h(r)}{h(r\exp(-p_i))}\right]^{f(x_i)} \left[\frac{h(r)}{h(r\exp(-p_j))}\right]^{f(x_j)}}{\left[\frac{h(r)}{h(r\exp(-p_i-p_j))}\right]^{f\left(\frac{1}{p_i+p_j}(p_i x_i + p_j x_j)\right)}} \right\} \geq 1. \tag{3.14}$$

Remark 5. *If the function $f : C \to \mathbb{R}$ is a λ-convex function on C with*

$$\lambda_r(t) = 1 - \exp(-t), \ t \geq 0,$$

then for any $x_i \in C$ and $p_i \geq 0$ with $i \in \{1, ..., n\}$, $n \geq 2$ so that $P_n > 0$ we have the Jensen's type inequality

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{1 - \exp(-P_n)} \sum_{i=1}^{n} [1 - \exp(-p_i)] f(x_i). \tag{3.15}$$

If $\alpha_i \in [0, 1]$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} \alpha_i = 1$, then for any $x_i \in C$ with $i \in \{1, ..., n\}$ we also have the inequality

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \frac{e}{e-1} \sum_{i=1}^{n} [1 - \exp(-\alpha_i)] f(x_i). \tag{3.16}$$

Finally, if $p_i \geq 0$ with $i \in \{1, ..., n\}$, $n \geq 2$ so that $P_n > 0$, then for any $x_i \in C$ with $i \in \{1, ..., n\}$ we have the inequality:

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{e}{e-1} \sum_{i=1}^{n} \left[1 - \exp\left(-\frac{p_i}{P_n}\right)\right] f(x_i). \tag{3.17}$$

4. Inequalities for Double Sums

We have the following result:

Theorem 8. *Let $f : C \to \mathbb{R}$ be a λ-convex function on C and $x_i \in C$ and $p_i \geq 0$ with $i \in \{1, ..., n\}$, $n \geq 2$ so that $P_n > 0$. For $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ we have the inequalities*

$$\begin{aligned}\left[\frac{\lambda(\alpha)}{\lambda(\alpha+\beta)} + \frac{\lambda(\beta)}{\lambda(\alpha+\beta)}\right] \frac{1}{\lambda(P_n)} \sum_{i=1}^{n} \lambda(p_i) f(x_i) \frac{1}{\lambda(P_n)} \sum_{i=1}^{n} \lambda(p_i) \\ \geq \frac{1}{\lambda^2(P_n)} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda(p_i) \lambda(p_j) f\left(\frac{\alpha x_i + \beta x_j}{\alpha+\beta}\right) \\ \geq \frac{1}{\lambda(P_n)} \sum_{i=1}^{n} \lambda(p_i) f\left(\frac{\alpha x_i + \beta \frac{1}{P_n}\sum_{j=1}^{n} p_j x_j}{\alpha+\beta}\right) \geq f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right).\end{aligned} \tag{4.1}$$

Proof. From the λ-convexity of the function f on C we have

$$\frac{\lambda(\alpha) f(x_i) + \lambda(\beta) f(x_j)}{\lambda(\alpha+\beta)} \geq f\left(\frac{\alpha x_i + \beta x_j}{\alpha+\beta}\right) \tag{4.2}$$

for any $i, j \in \{1, ..., n\}$.

If we multiply (4.2) by

$$\frac{\lambda(p_i) \lambda(p_j)}{\lambda^2(P_n)} \geq 0, \ i, j \in \{1, ..., n\}$$

and sum over i and j from 1 to n we get

$$\begin{aligned}\sum_{i=1}^{n} \sum_{j=1}^{n} \left[\frac{\lambda(\alpha)}{\lambda(\alpha+\beta)} f(x_i) + \frac{\lambda(\beta)}{\lambda(\alpha+\beta)} f(x_j)\right] \frac{\lambda(p_i) \lambda(p_j)}{\lambda^2(P_n)} \\ \geq \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\lambda(p_i) \lambda(p_j)}{\lambda^2(P_n)} f\left(\frac{\alpha x_i + \beta x_j}{\alpha+\beta}\right).\end{aligned} \tag{4.3}$$

Since

$$\begin{aligned}&\sum_{i=1}^{n} \sum_{j=1}^{n} \left[\frac{\lambda(\alpha)}{\lambda(\alpha+\beta)} f(x_i) + \frac{\lambda(\beta)}{\lambda(\alpha+\beta)} f(x_j)\right] \frac{\lambda(p_i) \lambda(p_j)}{\lambda^2(P_n)} \\ &= \frac{\lambda(\alpha)}{\lambda(\alpha+\beta)} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\lambda(p_i) \lambda(p_j)}{\lambda^2(P_n)} f(x_i) + \frac{\lambda(\beta)}{\lambda(\alpha+\beta)} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\lambda(p_i) \lambda(p_j)}{\lambda^2(P_n)} f(x_j) \\ &= \frac{\lambda(\alpha)}{\lambda(\alpha+\beta)} \frac{1}{\lambda^2(P_n)} \sum_{i=1}^{n} \lambda(p_i) f(x_i) \sum_{j=1}^{n} \lambda(p_j) \\ &+ \frac{\lambda(\beta)}{\lambda(\alpha+\beta)} \frac{1}{\lambda^2(P_n)} \sum_{j=1}^{n} \lambda(p_j) f(x_j) \sum_{i=1}^{n} \lambda(p_i) \\ &= \left[\frac{\lambda(\alpha)}{\lambda(\alpha+\beta)} + \frac{\lambda(\beta)}{\lambda(\alpha+\beta)}\right] \frac{1}{\lambda(P_n)} \sum_{i=1}^{n} \lambda(p_i) f(x_i) \frac{1}{\lambda(P_n)} \sum_{i=1}^{n} \lambda(p_i),\end{aligned}$$

then by (4.3) we get the first inequality in (4.1).

By the Jensen inequality we have the inequality

$$\begin{aligned}\frac{1}{\lambda(P_n)}\sum_{j=1}^{n}\lambda(p_j)f\left(\frac{\alpha x_i+\beta x_j}{\alpha+\beta}\right)&\geq f\left(\frac{1}{P_n}\sum_{j=1}^{n}p_j\left(\frac{\alpha x_i+\beta x_j}{\alpha+\beta}\right)\right)\\&=f\left(\frac{\alpha x_i+\beta\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j}{\alpha+\beta}\right)\end{aligned}$$

for all $i\in\{1,...,n\}$.

If we multiply this inequality by $\frac{\lambda(p_i)}{\lambda(P_n)}$ and sum over i from 1 to n we get

$$\begin{aligned}&\frac{1}{\lambda^2(P_n)}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda(p_i)\lambda(p_j)f\left(\frac{\alpha x_i+\beta x_j}{\alpha+\beta}\right)\\&\geq\frac{1}{\lambda(P_n)}\sum_{i=1}^{n}\lambda(p_i)f\left(\frac{\alpha x_i+\beta\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j}{\alpha+\beta}\right)\end{aligned}$$

and the second inequality in (4.1) is proved.

If we apply Jensen inequality again we get

$$\begin{aligned}&\frac{1}{\lambda(P_n)}\sum_{i=1}^{n}\lambda(p_i)f\left(\frac{\alpha x_i+\beta\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j}{\alpha+\beta}\right)\\&\geq f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i\left(\frac{\alpha x_i+\beta\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j}{\alpha+\beta}\right)\right)=f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right)\end{aligned}$$

and the last part of (4.1) is proved. □

Corollary 12. *Let $f:C\to\mathbb{R}$ be a λ-convex function on C and $x_i\in C$ and $p_i\geq 0$ with $i\in\{1,...,n\}$, $n\geq 2$ so that $P_n>0$. We have the inequalities*

$$\begin{aligned}(4.4)\qquad&\inf_{\alpha>0}\left(\frac{2\lambda(\alpha)}{\lambda(2\alpha)}\right)\frac{1}{\lambda(P_n)}\sum_{i=1}^{n}\lambda(p_i)f(x_i)\frac{1}{\lambda(P_n)}\sum_{i=1}^{n}\lambda(p_i)\\&\geq\frac{1}{\lambda^2(P_n)}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda(p_i)\lambda(p_j)f\left(\frac{x_i+x_j}{2}\right)\\&\geq\frac{1}{\lambda(P_n)}\sum_{i=1}^{n}\lambda(p_i)f\left(\frac{x_i+\frac{1}{P_n}\sum_{j=1}^{n}p_jx_j}{2}\right)\geq f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right).\end{aligned}$$

We have the following result as well:

Theorem 9. *Let $f:C\to\mathbb{R}$ be a λ-convex function on C and $x_i\in C$ and $p_i\geq 0$ with $i\in\{1,...,n\}$, $n\geq 2$ so that $P_n>0$. For $\alpha,\beta\geq 0$ with $\alpha+\beta>0$ we have the inequalities*

$$\begin{aligned}(4.5)\qquad&\left[\frac{\lambda(\alpha)}{\lambda(\alpha+\beta)}+\frac{\lambda(\beta)}{\lambda(\alpha+\beta)}\right]\frac{1}{\lambda(P_n^2)}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda(p_ip_j)f(x_i)\\&\geq\frac{1}{\lambda(P_n^2)}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda(p_ip_j)f\left(\frac{\alpha x_i+\beta x_j}{\alpha+\beta}\right)\geq f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\right).\end{aligned}$$

Proof. From the λ-convexity of the function f on C we have

$$\frac{\lambda(\alpha) f(x_i) + \lambda(\beta) f(x_j)}{\lambda(\alpha+\beta)} \geq f\left(\frac{\alpha x_i + \beta x_j}{\alpha+\beta}\right) \tag{4.6}$$

for any $i, j \in \{1, ..., n\}$.

If we multiply (4.6) by

$$\frac{\lambda(p_i p_j)}{\lambda(P_n^2)} \geq 0, \ \ i, j \in \{1, ..., n\}$$

and sum over i and j from 1 to n we get

$$\begin{aligned} &\frac{1}{\lambda(P_n^2)} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda(p_i p_j) \left[\frac{\lambda(\alpha) f(x_i) + \lambda(\beta) f(x_j)}{\lambda(\alpha+\beta)}\right] \\ &\geq \frac{1}{\lambda(P_n^2)} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda(p_i p_j) f\left(\frac{\alpha x_i + \beta x_j}{\alpha+\beta}\right). \end{aligned} \tag{4.7}$$

We have

$$\begin{aligned} &\sum_{i=1}^{n} \sum_{j=1}^{n} \lambda(p_i p_j) \left[\frac{\lambda(\alpha) f(x_i) + \lambda(\beta) f(x_j)}{\lambda(\alpha+\beta)}\right] \\ &= \frac{\lambda(\alpha)}{\lambda(\alpha+\beta)} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda(p_i p_j) f(x_i) + \frac{\lambda(\beta)}{\lambda(\alpha+\beta)} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda(p_i p_j) f(x_j) \end{aligned}$$

and since

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \lambda(p_i p_j) f(x_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda(p_i p_j) f(x_j)$$

then we get from (4.7) the first inequality in (4.5).

By Jensen's inequality we have

$$\begin{aligned} &\frac{1}{\lambda\left(\sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j\right)} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda(p_i p_j) f\left(\frac{\alpha x_i + \beta x_j}{\alpha+\beta}\right) \\ &\geq f\left(\frac{1}{\sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j} \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j \left(\frac{\alpha x_i + \beta x_j}{\alpha+\beta}\right)\right) \\ &= f\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i\right) \end{aligned}$$

and the last part of (4.5) is thus proved. □

Corollary 13. *Let $f : C \to \mathbb{R}$ be a λ-convex function on C and $x_i \in C$ and $p_i \geq 0$ with $i \in \{1, ..., n\}$, $n \geq 2$ so that $P_n > 0$. We have the inequalities*

$$\begin{aligned} &\inf_{\alpha>0} \left(\frac{2\lambda(\alpha)}{\lambda(2\alpha)}\right) \frac{1}{\lambda(P_n^2)} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda(p_i p_j) f(x_i) \\ &\geq \frac{1}{\lambda(P_n^2)} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda(p_i p_j) f\left(\frac{x_i + x_j}{2}\right) \geq f\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i\right). \end{aligned} \tag{4.8}$$

It is known that if $(X, \|\cdot\|)$ is a normed linear space, then the function $f(x) = \|x\|^s$, $s \in (0,1)$ is Breckner s-convex on X.

If $x_i \in X$ and $p_i \geq 0$ with $i \in \{1, ..., n\}$, $n \geq 2$ so that $P_n > 0$, then from (4.4) we have

$$
\begin{aligned}
(4.9)\qquad & 2^{1-s} \frac{1}{P_n^s} \sum_{i=1}^n p_i^s \frac{1}{P_n^s} \sum_{i=1}^n p_i^s \|x_i\|^s \\
& \geq \frac{1}{P_n^{2s}} \sum_{i=1}^n \sum_{j=1}^n p_i^s p_j^s \left\| \frac{x_i + x_j}{2} \right\|^s \\
& \geq \frac{1}{P_n^s} \sum_{i=1}^n p_i^s \left\| \frac{x_i + \frac{1}{P_n} \sum_{j=1}^n p_j x_j}{2} \right\|^s \geq \left\| \frac{1}{P_n} \sum_{i=1}^n p_i x_i \right\|^s .
\end{aligned}
$$

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