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ON MONOTONICITY OF RATIOS OF q–KUMMER CONFLUENT HYPERGEOMETRIC AND q–HYPERGEOMETRIC FUNCTIONS AND ASSOCIATED TURÁN TYPES INEQUALITIES

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Abstract. In this paper we prove monotonicity of some ratios of q–Kummer confluent hypergeometric and q–hypergeometric functions. The results are also closely connected with Turán type inequalities. In order to obtain main results we apply methods developed for the case of classical Kummer and Gauss hypergeometric functions in [1]-[2].

Keywords: Kummer functions, Gauss hypergeometric functions, q–Kummer confluent hypergeometric functions, q–hypergeometric functions, Turán type inequalities.

1. Introduction

In 1941 while studying the zeros of Legendre polynomials the Hungarian mathematician Paul Turán discovered the following inequality

$$P_{n-1}(x)P_{n+1}(x) \leq [P_n(x)]^2 ,$$

where $|x| \leq 1$, $n \in \mathbb{N} = \{1, 2, ...\}$ and P_n stands for the classical Legendre polynomial. This inequality was published by P. Turán only in 1950 in [3]. However, since the publication in 1948 by G. Szegő [4] of the above famous Turán inequality for Legendre polynomials many authors have deduced analogous results for classical polynomials and special functions. It has been shown by several researchers that the most important polynomials (e.g. Laguerre, Hermite, Appell, Bernoulli, Jacobi, Jensen, Pollaczek, Lommel, Askey–Wilson, ultraspherical) and special functions (e.g. Bessel, modified Bessel, gamma, polygamma, Riemann zeta) satisfy Turán type inequalities. In 1981 one of the PhD students of P. Turán, L. Alpár [5] in Turáns biography mentioned that the above Turán inequality had a wide–ranging effect, this inequality was dealt with in more than 60 papers. Also Turán type inequalities are closely connected with log–convexity and log–concavity of hypergeometric–like functions, cf. [8]-[9]. A survey of recent results on Turán type inequalities [6] is published in the proceedings of the conference [7] dedicated to Paul Turán's achievements in different areas of mathematics and applications.

Since Turáns inequality was first investigated for orthogonal polynomials in hypergeometric representation afterwards such inequalities were extensively studied for various hypergeometric functions as well, e.g. in [10] Turán type inequalities for the q–Kummer and q–hypergeometric functions were proved.

In [1]-[2] in terms of monotonicity of ratios of Kummer, Gauss and generalized hypergeometric functions the authors presented some new Turán type inequalities. They are connected with problems having some history.

Let us consider the series for the exponential function

$$\exp(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},\ x \geq 0,$$

its section $S_n(x)$ and series remainder $R_n(x)$ in the form

$$S_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!},\ R_n(x) = \exp(x) - S_n(x) = \sum_{k=n+1}^{\infty} \frac{x^k}{k!},\ x \geq 0. \tag{1}$$

Besides simplicity and elementary nature of these functions many mathematicians studied problems for them. G. Szegő proved a remarkable limit distribution for zeroes of sections, accumulated along so–called the Szegő curve ([12]). S. Ramanujan seems was the first who proved the non–trivial inequality for exponential sections in the form ([13], pp. 323–324) : if

$$\frac{e^n}{2} = R_{n-1}(n) + \frac{n^n}{n!}\theta(n)$$

then

$$\frac{1}{3} < \theta(n) = \frac{n!\left(\frac{e^n}{2} - R_{n-1}(n)\right)}{n^n} < \frac{1}{2}.$$

This result is important as it also leads to explicit rational bounds for e^n as it was specially pointed out in ([13], pp. 323–324).

In the preprint [11] in 1993 were thoroughly studied inequalities of the form

$$m(n) \leq f_n(x) = \frac{R_{n-1}(x)R_{n+1}(x)}{[R_n(x)]^2} \leq M(n),\ x \geq 0. \tag{2}$$

The search for the best constants $m(n) = m_{best}(n)$, $M(n) = M_{best}(n)$ has some history. The left–hand side of (2) was first proved by Kesava Menon in [14] with $m(n) = \frac{1}{2}$ (not best) and by Horst Alzer in [15] with

$$m_{best}(n) = \frac{n+1}{n+2} = f_n(0), \tag{3}$$

cf. [11] for the more detailed history. In [11] it was also shown that in fact the inequality (2) with the sharp lower constant (3) is a special case of the stronger inequality proved earlier in 1982 by Walter Gautschi in [16].

It seems that the right–hand side of (2) was first proved by the author in [11] with $M_{best} = 1 = f_n(\infty)$. In [11] dozens of generalizations of inequality (2) and related results were proved. May be in fact it was the first example of so called Turan–type inequality for special case of the Kummer hypergeometric functions.

Obviously the above inequalities are consequences of the next conjecture originally formulated in [11] and recently revived in [18]–[19].

Conjecture 1. *The function $f_n(x)$ in (2) is monotone increasing for $x \in [0;\infty), n \in \mathbb{N}$.* So the next inequality is valid

$$\frac{n+1}{n+2} = f_n(0) \leq f_n(x) < 1 = f_n(\infty). \tag{4}$$

In 1990's we tried to prove this conjecture in the straightforward manner by expanding an inequality $(f_n(x))' \geq 0$ in series and multiplying triple products of hypergeometric functions but failed ([17]–[18]).

Consider a representation via Kummer hypergeometric functions

$$f_n(x) = \frac{n+1}{n+2}\, g_n(x),\;\; g_n(x) = \frac{{}_1F_1(1;n+1;x)\,{}_1F_1(1;n+3;x)}{\left[{}_1F_1(1;n+2;x)\right]^2}. \tag{5}$$

So the conjecture 1 may be reformulated in terms of this function $g_n(x)$ as conjecture 2.

Conjecture 2. *The function $g_n(x)$ in (5) is monotone increasing for $x \in [0;\infty), n \in \mathbb{N}$.*

This leads us to the next more general

Problem 1. *Find monotonicity in x conditions for $x \in [0;\infty)$ for all parameters a,b,c for the function*

$$h(a,b,c,x) = \frac{{}_1F_1(a;b-c;x)\,{}_1F_1(a;b+c;x)}{\left[{}_1F_1(a;b;x)\right]^2}. \tag{6}$$

We may also call (6) mockingly (in Ramanujan way, remember his mock theta–functions!) "*The abc–problem*" for Kummer hypergeometric functions, why not?

Another generalization is to change Kummer hypergeometric functions to higher ones.

Problem 2. *Find monotonicity in x conditions for $x \in [0;\infty)$ for all vector–valued parameters a,b,c for the function*

$$h_{p,q}(a,b,c,x) = \frac{{}_pF_q(a;b-c;x)\,{}_pF_q(a;b+c;x)}{\left[{}_pF_q(a;b;x)\right]^2}, \tag{7}$$

$$a = (a_1,\dots,a_p), b = (b_1,\dots,b_q), c = (c_1,\dots,c_q).$$

This is "*The abc–problem*" for generalized hypergeometric functions. The more complicated problems are obvious and may be considered for pairs or triplets of parameters and also for multivariable hypergeometric functions.

Recently the above problems 1,2 and conjectures 1,2 were proved by the authors [1]–[2]. In this paper we prove q–versions of these results for the classical Kummer and Gauss hypergeometric functions.

Next let us recall the following results which will be used in the sequel.

Lemma 1. *Let (a_n) and (b_n) $(n = 0,1,2...)$ be real numbers, such that $b_n > 0$, $n = 0,1,2,...$ and $\left(\frac{a_n}{b_n}\right)_{n\geq 0}$ is increasing (decreasing), then $\left(\frac{a_0+\dots+a_n}{b_0+\dots+b_n}\right)_n$ is also increasing (decreasing).*

Lemma 2. *(cf. [23]–[24]). Let (a_n) and (b_n) $(n = 0, 1, 2...)$ be real numbers and let the power series $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be convergent if $|x| < r$. If $b_n > 0$, $n = 0, 1, 2, ...$ and if the sequence $\left(\frac{a_n}{b_n}\right)_{n\geq 0}$ is (strictly) increasing (decreasing) , then the function $\frac{A(x)}{B(x)}$ is also (strictly) increasing on $[0, r[$.*

2. Notations and preliminaries

Throughout this paper, we fix $q \in]0, 1[$. We refer to [20], [21] and [22] for the definitions, notations and properies of the q–shifted factorials and q–hypergeometric functions.

2.1. **Basic symbols.** Let $a \in \mathbb{R}$ then q–shifted factorials are defined by

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1}(1 - aq^k), \;\; (a; q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k),$$

and we write

$$(a_1, a_2, ..., a_p; q) = (a_1; q)_n (a_1; q)_n ...(a_p; q)_n, \;\; n = 0, 1, 2, ...$$

Note that for $q \longrightarrow 1$ the expression $\frac{(q^a;q)_n}{(1-q)^n}$ tends to $(a)_n = a(a + 1)...(a + n - 1)$.

2.2. **q-Kummer confluent hypergeometric functions.** The q–Kummer confluent hypergeometric function is defined by

$$\phi(q^a, q^c; q, x) ={}_1 \phi_1(q^a, q^c; q, (1 - q)x) = \sum_{n\geq 0} \frac{(q^a; q)_n (1 - q)^n}{(q^c; q)(q; q)_n} x^n, \tag{8}$$

for all $a, c \in \mathbb{R}$ and $x > 0$, which for $q \longrightarrow 1$ is reduced to the Kummer confluent hypergeometric function

$${}_1F_1(a; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n n!} x^n.$$

2.3. **q- hypergeometric functions.** The q–hypergeometric series or basic hypergeometric series is defined by [21],[22]

$$\begin{aligned} &{}_p\Phi_r(a_1, ..., a_p; b_1, ..., b_r; q; x) = \\ &= \sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_2; q)_n ...(a_p; q)_n}{(b_1; q)_n (b_2; q)_n ...(b_r, q)_n (q; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+r-p} x^n \end{aligned} \tag{9}$$

with $\binom{n}{2} = \frac{n(n-1)}{2}$, $\;a_k, b_k \in \mathbb{R} \in \mathbb{C}$, $b_k \neq q^{-n}$, $k = 1, ..., r$, $n \in \mathbb{N}_0$, $0 < |q| < 1$. The left hand side of (9) represents the q–hypergeometric function ${}_p\phi_r$ where the series converges. Assuming $0 < |q| < 1$, the following conditions are valid for the convergence of (9) (cf.[22]).

- $p < r + 1$: the series converges absolutely for $x \in \mathbb{C}$,
- $p = r + 1$: the series converges for $|x| < 1$,

- $p > r+1$: the series converges only for $x = 0$, unless it terminates.

Since for $q \longrightarrow 1$ the expression $\frac{(q^a;q)_n}{(1-q)^n}$ tends to $(a)_n = a(a+1)...(a+n-1)$, we evaluate

$$\lim_{q\longrightarrow 1} {}_p\Phi_r(q^{a_1},...,q^{a_p};q^{b_1},...,q^{b_r};q;x) = {}_pF_r(a_1,...,a_p;b_1,...,b_r;x) = \sum_{n=0}^{\infty} \frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_r)_n n!} x^n,$$

where ${}_pF_r$ stands for the generalized hypergeometric function.

3. Monotonicity of ratios of q–Kummer hypergeometric functions

In this section we consider the function

$$h(a,b,c,q,x) = \frac{\phi(q^a,q^{b-c},q,x)\phi(q^a,q^{b+c},q,x)}{[\phi(q^a,q^b,q,x)]^2}, \tag{10}$$

for all $a, b \in \mathbb{R}$ and $x > 0$. The following theorem is the q–version of the theorem 1 from ([1]–[2]).

Theorem 1. *Let $q \in]0,1[$, if $a > b > c > 0$, $b > 1$, then the function $x \longmapsto h(a,b,c,q,x)$ is increasing on $[0,\infty[$. In particular, the following Turán type inequality is valid for all $a > b > c > 0$, $b > 1$ and $q \in]0,1[$*

$$\left[\phi(q^a,q^b,q,x)\right]^2 \le \phi(q^a,q^{b-c},q,x)\phi(q^a,q^{b+c},q,x). \tag{11}$$

Proof. For convenience let us write $\phi(q^a,q^b,q,x)$ as

$$\phi(q^a,q^b,q,x) = \sum_{n=0}^{\infty} u_n(a,b,q)x^n,$$

where

$$u_n(a,b,q) = \frac{(q^a;q)_n(1-q)^n}{(q^b;q)_n(q;q)_n}.$$

Then

$$h(a,b,c,q,x) = \frac{\left(\sum_{n=0}^{\infty} u_n(a,b-c,q)x^n\right)\left(\sum_{n=0}^{\infty} u_n(a,b-c,q)x^n\right)}{\left(\sum_{n=0}^{\infty} u_n(a,b,q)x^n\right)^2} = $$

$$= \frac{\sum_{n=0}^{\infty} v_n(a,b,c,q)x^n}{\sum_{n=0}^{\infty} w_n(a,b,q)x^n},$$

with

$$v_n(a,b,c,q) = \sum_{k=0}^{n} u_k(a,b-c,q)u_{n-k}(a,b+c,q)$$

and

$$w_n(a,b,q) = \sum_{k=0}^{n} u_k(a,b,q)u_{n-k}(a,b,q).$$

Let define the sequences $(A_{n,k})_{k\geq 0}$ by

$$A_{n,k}(a,b,c,q)=\frac{u_k(a,b-c,q)u_{n-k}(a,b+c,q)}{u_k(a,b,q)u_{n-k}(a,b,q)}=\frac{(q^b;q)_k(q^b;q)_{n-k}}{(q^{b-c};q)_k(q^{b+c};q)_{n-k}}$$

and evaluate

$$\begin{aligned}\frac{A_{n,k+1}(a,b,c,q)}{A_{n,k}(a,b,c,q)}&=\frac{(q^b;q)_{k+1}(q^b;q)_{n-k-1}(q^{b-c};q)_k(q^{b+c};q)_{n-k}}{(q^{b-c};q)_{k+1}(q^{b+c};q)_{n-k-1}(q^b;q)_k(q^b;q)_{n-k}}=\\&=\left(\frac{(q^b;q)_{k+1}}{(q^b;q)_k}\right)\cdot\left(\frac{(q^{b-c};q)_k}{(q^{b-c};q)_{k+1}}\right)\cdot\left(\frac{(q^b;q)_{n-k-1}}{(q^b;q)_{n-k}}\right)\cdot\left(\frac{(q^{b+c};q)_{n-k}}{(q^{b+c};q)_{n-k}}\right)=\\&=\left(\frac{1-q^{b+k}}{1-q^{b-c+k}}\right)\cdot\left(\frac{1-q^{b+c+n-k-1}}{1-q^{b+n-k-1}}\right).\end{aligned}$$

Since $q\in]0,1[$ and $b>1$ it follows $\frac{A_{n,k+1}(a,b,c,q)}{A_{n,k}(a,b,c,q)}\geq 1$ and consequently the sequence $(A_{n,k}(a,b,c,q))_{k\geq 0}$ is increasing. We conclude that C_n defined by $C_n=\frac{u_n}{v_n}$ is increasing by Lemma 1. Thus the function $x\longmapsto h(a,b,c,q,x)$ is increasing on $[0,\infty[$ by Lemma 2.

Furthermore,

$$\lim_{x\longrightarrow 0}h(a,b,q,x)=1,$$

and the Turán type inequality (11) follows. So the proof of Theorem 1 is complete. ■

Remark 1. *The inequality (11) is interesting as a consequence of monotonicity property we consider. This inequality itself is not new and may be found in* [10].

4. Monotonicity of ratios of q–hypergeometric functions

In this section we consider the function $h_r(a,b,c,q)$ defined by

$$h_r(a,b,c,q)=\frac{\phi(q^{a_1},..,q^{a_{r+1}};q^{b_1-c_1},...,q^{b_r-c_r};q,x)\phi(q^{a_1},..,q^{a_{r+1}};q^{b_1-c_1},...,q^{b_r-c_r};q,x)}{[\phi(q^{a_1},..,q^{a_{r+1}};q^{b_1},...,q^{b_r};q,x)]^2}\tag{12}$$

where $a=(a_1,...,a_{r+1})$ $b=(b_1,...,b_r)$ and $c=(c_1,...,c_r)$ for all $a_k,b_k,c_k\in\mathbb{R},\ b_k\neq q^{-n},\ k=1,...,r,\ n\in\mathbb{N}_0,\ 0<|q|<1.$

Theorem 2. *Let $r\in\mathbb{N}$ $q\in(0,1)$ $a=(a_0,...,a_r),\ b=(b_1,...,b_r)$ $c=(c_1,...,c_r),\ b_i>c_i$ for $i=1,...,r$. If $b_i>1$ for $i=1,...,r$, then the function $h_r(a,b,c,q)$ is strictly increasing on $[0,1[$. Moreover, if $b_i>c_i,b_i>1$, and $q\in(0,1)$, then the next Turán type inequality holds*

$$\left[\phi(q^{a_1},..,q^{a_{r+1}};q^{b_1},...,q^{b_r};q,x)\right]^2<\phi(q^{a_1},..,q^{a_{r+1}};q^{b_1-c_1},...,q^{b_p-c_p};q,x)\phi(q^{a_1},..,q^{a_{p+1}};q^{b_1-c_1},...,q^{b_r-c_r};q,x).\tag{13}$$

Proof. By using the inequality (12), we can write h_r in the form

$$h_r(a,b,q,x) = \frac{\left(\sum_{n=0}^{\infty} \frac{(q^{a_1};q)_n \dots (q^{a_{r+1}};q)_n x^n}{(q^{b_1-c_1};q)_n \dots (q^{b_r-c_r};q)_n (q;q)_n}\right)}{\left(\sum \frac{(q^{a_1};q)_n \dots (q^{a_{r+1}};q)_n x^n}{(q^{b_1};q)_n \dots (q^{b_r};q)_n (q;q)_n}\right)^2} \cdot$$

$$\cdot \left(\sum_{n=0}^{\infty} \frac{(q^{a_1};q)_n \dots (q^{a_{r+1}};q)_n x^n}{(q^{b_{1+c_1}};q)_n \dots (q^{b_r+c_r};q)_n (q;q)_n}\right) = \tag{14}$$

$$= \frac{\sum_{n=0}^{\infty} A_n(a,b,c,q)}{\sum_{n=0}^{\infty} B_n(a,b,c,q)} x^n,$$

with use of the next notations

$$A_n(a,b,c,q) = \sum_{k=0}^{n} U_k(a,b,c,q) = $$
$$= \sum_{k=0}^{n} \frac{\prod_{j=1}^{r+1} (q^{a_j};q)_{n-k} (q^{a_j};q)_k}{(q;q)_k (q;q)_{n-k} \prod_{j=1}^{r} (q^{b_j-c_j};q)_k (q^{b_j+c_j};q)_{n-k}}$$

and

$$B_n(a,b,c,q) = \sum_{k=0}^{n} V_k(a,b,c,q) = $$
$$= \sum_{k=0}^{n} \frac{\prod_{j=1}^{r+1} (q^{a_j};q)_{n-k} (q^{a_j};q)_k}{(q;q)_k (q;q)_{n-k} \prod_{j=1}^{r} (q^{b_j};q)_k (q^{b_j};q)_{n-k}}.$$

For fixed $n \in \mathbb{N}$ we define the sequence $(W_{n,k}(a,b,c,q))_{k\geq 0}$ by

$$W_{n,k}(a,b,c,q) = \frac{U_k(a,b,c,q)}{V_k(a,b,c,q)} = $$
$$= \prod_{j=1}^{r} \frac{(q^{b_j};q)_k (q^{b_j};q)_{n-k}}{(q^{b_j-c_j};q)_k (q^{b_j+c_j};q)_{n-k}}.$$

For $n,k \in \mathbb{N}$ we evaluate

$$\frac{W_{n,k+1}(a,b,c,q)}{W_{n,k}(a,b,c,q)} = \prod_{j=1}^{r} \left[\frac{(q^{b_i};q)_{k+1}}{(q^{b_j};q)_k}\right] \cdot \left[\frac{(q^{b_j};q)_{n-k-1}}{(q^{b_j};q)_{n-k}}\right] \cdot \left[\frac{(q^{b_j-c_j};q)_k}{(q^{b_j};q)_{k+1}}\right] \cdot \left[\frac{(q^{b_j+c_j};q)_{n-k}}{(q^{b_j+c_j};q)_{n-k-1}}\right] = $$
$$= \prod_{j=1}^{r} \left[\frac{1-q^{b_j+k}}{1-q^{b_j-c_j-k}}\right] \cdot \left[\frac{1-q^{b_j+c_j+n-k-1}}{1-q^{b_j+n-k-1}}\right].$$

■

Since $0 < q < 1$ and $b_j > 1$ for $j = 1, \dots, r$ we conclude that $(W_{n,k})_k$ is increasing and consequently $\left(C_n = \frac{A_n}{B_n}\right)_{n\geq 0}$ is increasing too by the Lemma 1. Thus the function

$x \longmapsto h_r(a,b,c,q)$ is increasing on $[0,1[$ by the Lemma 2. Therefore the inequality (13) follows immediately from the monotonicity of the function $h_r(a,b,c,q)$.

There are applications of considered inequalities in the theory of transmutation operators for estimating transmutation kernels and norms ([25]–[27]) and for problems of function expansions by systems of integer shifts of Gaussians ([28]–[29]).

References

[1] K. Mehrez, S.M. Sitnik, Proofs of some conjectures on monotonicity of ratios of Kummer and Gauss hypergeometric functions and related Turán–type inequalities. arXiv:1410.6120v2 [math.CA], 2014, 8 pp.

[2] K. Mehrez, S.M. Sitnik, Inequalities for Sections of Exponential Function Series and Proofs of Some Conjectures on Monotonicity of Ratios of Kummer, Gauss and Generalized Hypergeometric Functions. RGMIA Research Report Collection, 17(2014), Article 132, 8 pp.

[3] P. Turán, On the zeros of the polynomials of Legendre, Casopis Pest. Mat. Fys. 75 (1950) 113122.

[4] G. Szegő, On an inequality of P. Turán concerning Legendre polynomials, Bulletin of the American Mathematical Society 54 (1948), no. 4, 401–405.

[5] L. Alpár, In memory of Paul Turán, J. Number Theory 13 (1981) 271-278.

[6] Csordas G. Turán–type inequalities and the distribution of zeros of entire functions. P. 25–40. In the book [7].

[7] János Pintz, András Biró, Kálmán Győry, Gergely Harcos, Miklós Simonovits, József Szabados (Eds.) Number Theory, Analysis, and Combinatorics. Proceedings of the Paul Turán Memorial Conference held on August 22-26, 2011 in Budapest. De Gruyter, 2014, 418 P.

[8] Karp D.B., Sitnik S.M. Log–convexity and log–concavity of hypergeometric–like functions. Journal of Mathematical Analysis and Applications, Vol. 364, No. 2, 2010, P, 384394.

[9] Karp D.B., Sitnik S.M. Inequalities and monotonicity of ratios for generalized hypergeometric function. Journal of Approximation Theory, 2009 , Vol. 161, P. 337–352.

[10] A.Baricz, K. Raghavendarb, A. Swaminathan, Turán type inequalities for q–hypergeometric functions, J. Approx. Theory, 2013, Vol. 168, 2013, p. 69–79.

[11] Sitnik S.M. Inequalities for the exponential remainder, preprint, Institute of Automation and Control Process, Far Eastern Branch of the Russian Academy of Sciences, Vladivostok, 1993 (in Russian).

[12] Edrei A., Saff E.B., Varga R.S. Zeros of Sections of Power Series. Springer, 1983.

[13] Hardy G.H., Seshu Aiyan P.V., Wilson B.M., (eds). Collected Papers of Srinivasa Ramanujan. Cambridge, 1927.

[14] Kesava Menon P. Some integral inequalities. Math. Student, 1943, Vol. 11, P. 36–38.

[15] Alzer H. An inequality for the exponential function. Arch. Math., 1990, Vol. 55, P. 462-464.

[16] Gautschi W. A note on the successive remainders of the exponential series. Elem. Math., 1982, Vol. 37, P. 46–49.

[17] Sitnik S.M., Zelezko O. Unpublished, 1994.

[18] Sitnik S.M. A conjecture on monotonicity of a ratio of Kummer hypergeometric functions. arXiv:1207.0936, 2012, version 2, 2014. 4 pp. (http://arxiv.org/abs/1207.0936).

[19] Sitnik S.M. Conjectures on Monotonicity of Ratios of Kummer and Gauss Hypergeometric Functions. RGMIA Research Report Collection, 17(2014), Article 107, 4 pp.

[20] Kac, V.G., Cheung, P.: Quantum Calculus. Springer-Verlag, New York (2002)

[21] F.W.J. Olver, D.W. Lozier, R.F. Boisvert, C.W. Clark (Eds.), NIST Handbook of Mathematical Functions, Cambridge University Press, Cambridge, 2010.

[22] G. Gasper, M. Rahman, Basic hypergeometric series, in: Encyclopedia of Mathematics and its Applications, Vol. 35, Cambridge Univ. Press, Cambridge, UK, Second edition, 2004.

[23] Biernacki M., Krzyz J. On the monotonicity of certain functionals in the theory of analytic functions. Ann. Univ. M. Curie–Skłodowska, 2 (1995), 134–145.

[24] S. Ponnusamy, M. Vuorinen, Asymptotic expansions and inequalities for hypergeometric functions, Mathematika 44 (1997) 278301.

[25] Sitnik S.M. Factorization and Estimates of Norms in Weighted Lebesgue Spaces of Buschman-Erdelyi Operators. Soviet Math. Dokl, 1991.

[26] Katrakhov V.V., Sitnik S.M. Boundary value problem for the stationary Schroedinger equation with a singular potential. Akademiia Nauk SSSR, Doklady, 1984.

[27] Sitnik S.M. Buschman–Erdelyi transmutations, classification and applications. In the book: Analytic Methods Of Analysis And Differential Equations: Amade 2012. (Edited by M.V. Dubatovskaya, S.V. Rogosin), Cambridge Scientific Publishers, Cottenham, Cambridge, 2013. P. 171–201.

[28] Zhuravlev M.V., Kiselev E.A., Minin L.A., Sitnik S.M. Jacobi theta-functions and systems of integral shifts of Gaussian functions. Journal of Mathematical Sciences, 2011, V. 173, No. 2, P. 231–241.

[29] Kiselev E.A., Minin L.A., Novikov I.Ya., Sitnik S.M. On the Riesz Constants for Systems of Integer Translates. Mathematical Notes, 2014, Vol. 96, No. 2, pp. 228238.

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