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ABOUT SOME INEQUALITIES FOR POSITIVE OPERATORS ON HILBERT SPACES

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Abstract. In this paper, several inequalities for positive definite operators defined on Hilbert spaces will be presented under suitable assumptions, which improves the well-known inequality of Young. These inequalties will be rewritten then using norms, and other particular important case of these inequalities will be presented as consequences.

1. Introduction

It is necessary to recall the following results which are given in the papers [5] and [7] and will be used below in the demonstration of inequalities from Theorem 2, Consequence 1 and Theorem 3. In these propositions the same method as in the paper [2] will be utilized.

Lemma 1. *([5]) Let a and b be such that $a, b \geq 0$ and $0 \leq \nu \leq 1$. Then the following inequality holds:*

$$\nu a^2 + (1-\nu)b^2 \leq (a^\nu b^{1-\nu})^2 + s_0(a-b)^2,$$

where $s_0 = \max\{\nu, 1-\nu\}$.

As a particular case of a result of H. Kober when $n = 2$, see [6] and [7], we will use in next section an improvement of the inequality between arithmetic and geometric means:

$$r(\sqrt{a}-\sqrt{b})^2 \leq \lambda a + (1-\lambda)b - a^\lambda b^{1-\lambda} \leq (1-r)(\sqrt{a}-\sqrt{b})^2, \tag{1}$$

where a, b are the positive real numbers, $\lambda \in [0,1]$ and $r = \min\{\lambda, 1-\lambda\}$.

Theorem 1. *([7]) For $a, b \geq 1$, and $\lambda \in (0,1)$ we have*

$$r(\sqrt{a}-\sqrt{b})^2 + A_1(\lambda)\log^2\left(\frac{a}{b}\right) \leq \lambda a + (1-\lambda)b - a^\lambda b^{1-\lambda} \leq$$

$$\leq (1-r)(\sqrt{a}-\sqrt{b})^2 + B_1(\lambda)\log^2\left(\frac{a}{b}\right)$$

where $r = \min\{\lambda, 1-\lambda\}$, $A_1(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{r}{4}$ *and* $B_1(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{1-r}{4}$.

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First, it is necessary to recall that for selfadjoint operators $A, B \in B(H)$ we write $A \leq B$ (or $B \geq A$) if $< Ax, x > \leq < Bx, x >$ for every vector $x \in H$. In this paper, we will consider A as being a selfadjoint linear operator on a complex Hilbert space $(H; < .,. >)$ as in [2] and the references therein. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, an the C^*- algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows: For any $f, f \in C(Sp(A))$ and for any $\alpha, \beta \in \mathbf{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;
(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(f) = \Phi(f^*)$;
(iii) $||\Phi(f)|| = ||f|| := \sup_{t \in Sp(A)} |f(t)|$;
(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$ for $t \in Sp(A.)$

Using this notation, as in [2] for example, we define

$$f(A) := \Phi(f) \quad \text{for all} \quad f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A. It is known that if A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a *positive operator* on H. In addition, if and f and g are real valued functions on $Sp(A)$ then the following property holds:

$$f(t) \geq g(t) \quad \text{for any} \quad t \in Sp(A) \quad \text{implies that} \quad f(A) \geq g(A) \tag{2}$$

in the operator order of $B(H)$.

We recall the following inequality given in [1].

Proposition 1. *([1]) Let A and B be two positive definite operators on H. Then we have*

$$\nu < A^2x, x > +(1-\nu) < B^2y, y > \leq < A^{2\nu}x, x >< B^{2(1-\nu)}y, y > +$$
$$+s_0 \left[< A^2x, x > -2 < Ax, x >< By, y > + < B^2y, y >\right],$$

for each $x, y \in H$ with $||x|| = ||y|| = 1$, where $0 \leq \nu \leq 1$ and $s_0 = \max\{\nu, 1-\nu\}$.

2. **Main results**

The following results present several inequalities for positive operators. We start by recaling the following particular cases as applications of Proposition 1. Some of these particular cases were given in [1].

Remark 1. *(i) If we take in inequality from Proposition 1, $y = x$ then we have:*

$$\nu < A^2x, x > +(1-\nu) < B^2x, x > \leq < A^{2\nu}x, x >< B^{2(1-\nu)}x, x > +$$
$$+s_0 \left[< A^2x, x > -2 < Ax, x >< Bx, x > + < B^2x, x >\right]$$

for each $x \in H$ with $||x|| = 1$, where $s_0 = \max\{\nu, 1-\nu\}$.

(ii) If in addition $A = B$ then in inequality from Proposition 1 we obtain:

$$1 - 2s_0 \left[< A^2x, x > -(< Ax, x >)^2\right] \leq < A^{2\nu}x, x >< A^{2(1-\nu)}x, x >$$

for each $x \in H$ with $||x|| = 1$, where $s_0 = \max\{\nu, 1-\nu\}$.

(iii) We notice that under conditions of Proposition 1, the corresponding inequality becomes:

$$\nu||Ax||^2+(1-\nu)||By||^2 \leq ||A^\nu x||^2||B^{1-\nu}y||^2+s_0[||Ax||^2-2||A^{\frac{1}{2}}x||^2||B^{\frac{1}{2}}y||^2+||By||^2].$$

(iv) We see that under conditions of (i), this inequality can be also rewritten like below:

$$\nu||Ax||^2+(1-\nu)||Bx||^2 \leq ||A^\nu x||^2||B^{1-\nu}x||^2+s_0[||Ax||^2-2||A^{\frac{1}{2}}x||^2||B^{\frac{1}{2}}x||^2+||Bx||^2].$$

(v) Under conditions of (ii), this inequality can be rewritten like below:

$$1-2s_0[||Ax||^2-||A^{\frac{1}{2}}x||^4] \leq ||A^\nu x||^2||A^{1-\nu}x||^2.$$

Theorem 2. *Let A and B be two positive definite operators on H. If $\lambda \in [0,1]$ then we have*

$$r\left(< Ax,x > + < By,y > -2 < A^{\frac{1}{2}}x,x >< B^{\frac{1}{2}}y,y >\right) \leq$$

$$\leq \lambda < Ax,x > +(1-\lambda) < By,y > - < A^\lambda x,x >< B^{1-\lambda}y,y > \leq$$
$$\leq (1-r)\left(< Ax,x > + < By,y > -2 < A^{\frac{1}{2}}x,x >< B^{\frac{1}{2}}y,y >\right),$$

for each $x,y \in H$ with $||x|| = ||y|| = 1$, where $r = \min\{\lambda, 1-\lambda\}$.

Proof. The proof will be as in [1], Proposition 3. ∎

Consequence 1. *Let A a positive definite operator on H. If $\lambda \in [0,1]$ then we have*

$$||x||^4\left[1-2(1-r)\left(\frac{< Ax,x >}{||x||^2}-\frac{(< A^{\frac{1}{2}}x.x >)^2}{||x||^4}\right)\right] \leq$$
$$\leq < A^{1-\lambda}x,x >< A^\lambda x,x > \leq$$
$$\leq ||x||^4\left[1-2r\left(\frac{< Ax,x >}{||x||^2}-\frac{(< A^{\frac{1}{2}}x.x >)^2}{||x||^4}\right)\right],$$

for each $x \in H$ with $x \neq 0$, where $r = \min\{\lambda, 1-\lambda\}$.

Proof. Taking in Theorem 2, $A = B$ and $x = y$ we find the following inequality:

$$1-2(1-r)\left[< Ax,x > -(< A^{\frac{1}{2}}x,x >)^2\right] \leq < A^{1-\lambda}x,x >< A^\lambda x,x > \leq$$
$$\leq 1-2r\left[< Ax,x > -(< A^{\frac{1}{2}}x,x >)^2\right],$$

when $||x|| = 1$.

If we replace in this last inequality $x \in H$ for which $||x|| = 1$ by $\frac{x}{||x||}$, $x \in H$ then previous inequality becomes:

$$1-2(1-r)\left[< A\frac{x}{||x||},\frac{x}{||x||} > -\left(< A^{\frac{1}{2}}\frac{x}{||x||},\frac{x}{||x||} >\right)^2\right] \leq$$
$$\leq < A^{1-\lambda}\frac{x}{||x||},\frac{x}{||x||} >< A^\lambda\frac{x}{||x||},\frac{x}{||x||} > \leq$$
$$\leq 1-2r\left[< A\frac{x}{||x||},\frac{x}{||x||} > -\left(< A^{\frac{1}{2}}\frac{x}{||x||},\frac{x}{||x||} >\right)^2\right],$$

or

$$1-2(1-r)\left[\frac{<Ax,x>}{||x||^2}-\frac{(<A^{\frac{1}{2}}x,x>)^2}{||x||^4}\right]\le\frac{<A^{1-\lambda}x,x><A^{\lambda}x,x>}{||x||^4}\le$$

$$\le 1-2r\left[\frac{<Ax,x>}{||x||^2}-\frac{(<A^{\frac{1}{2}}x,x>)^2}{||x||^4}\right].$$

Therefore we get,

$$||x||^4\left[1-2(1-r)\left(\frac{<Ax,x>}{||x||^2}-\frac{(<A^{\frac{1}{2}}x,x>)^2}{||x||^4}\right)\right]\le<A^{1-\lambda}x,x><A^{\lambda}x,x>\le$$

$$\le||x||^4\left[1-2r\left(\frac{<Ax,x>}{||x||^2}-\frac{(<A^{\frac{1}{2}}x,x>)^2}{||x||^4}\right)\right].$$

∎

Now we rewrite above inequality using the norm, obtaining:

Remark 2. *Let A a positive definite operator on H. If $\lambda\in[0,1]$ then we have*

$$||x||^4-2(1-r)\left(||x||^2||A^{\frac{1}{2}}x||^2-||A^{\frac{1}{4}}x||^2\right)\le$$

$$\le||A^{\frac{1-\lambda}{2}}x||^2||A^{\frac{\lambda}{2}}x||^2\le||x||^4-2r\left(||x||^2||A^{\frac{1}{2}}x||^2-||A^{\frac{1}{4}}x||^2\right),$$

for each $x\in H$ with $x\neq 0$, where $r=\min\{\lambda,1-\lambda\}$.

Moreover, if we replace in inequality from Theorem 2 $x\in H$ for which $||x||=1$ by $\frac{x}{||x||}$, $x\in H$ and $y\in H$ for which $||y||=1$ by $\frac{y}{||y||}$, $y\in H$, then doing similar calculus as in the proof of Consequence 1, previous inequality becomes:

Remark 3. *Let A and B be two positive definite operators on H. If $\lambda\in[0,1]$ then we have the inequality*

$$\lambda||y||^2<Ax,x>+(1-\lambda)||x||^2<By,y>-(1-r)(||y||^2<Ax,x>+||x||^2<By,y>-$$

$$-2<A^{\frac{1}{2}}x,x><B^{\frac{1}{2}}y,y>)\le<A^{\lambda}x,x><B^{1-\lambda}y,y>\le \tag{a}$$

$$\le\lambda||y||^2<Ax,x>+(1-\lambda)||x||^2<By,y>-r(||y||^2<Ax,x>+||x||^2<By,y>-$$

$$-2<A^{\frac{1}{2}}x,x><B^{\frac{1}{2}}y,y>),$$

or the following form can also takes place using norm

$$\lambda||y||^2||A^{\frac{1}{2}}x||^2+(1-\lambda)||x||^2||B^{\frac{1}{2}}y||^2-(1-r)(||y||^2||A^{\frac{1}{2}}x||^2+||x||^2||B^{\frac{1}{2}}y||^2-$$

$$-2||A^{\frac{1}{4}}x||^2||B^{\frac{1}{4}}y||^2)\le||A^{\frac{\lambda}{2}}x||^2||B^{\frac{1-\lambda}{2}}y||^2\le \tag{b}$$

$$\le\lambda||y||^2||A^{\frac{1}{2}}x||^2+(1-\lambda)||x||^2||B^{\frac{1}{2}}y||^2-r(||y||^2||A^{\frac{1}{2}}x||^2+||x||^2||B^{\frac{1}{2}}y||^2-$$

$$-2||A^{\frac{1}{4}}x||^2||B^{\frac{1}{4}}y||^2),$$

for each $x,y\in H$ with $x\neq 0$, $y\neq 0$ where $r=\min\{\lambda,1-\lambda\}$.

Theorem 3. *Let A and B be two positive definite operators on H, $\lambda \in [0,1]$ and $r = \min\{\lambda, 1-\lambda\}$. Then for each $x, y \in H$ with $x \neq 0,\ y \neq 0$ the following inequality holds:*

$$\lambda||y||^2 + (1-\lambda)||x||^2 - (1-r)\left[||y||^2 + ||x||^2 - 2\frac{< A^{\frac{1}{2}}x, x >< B^{\frac{1}{2}}y, y >}{(< Ax, x >< By, y >)^{\frac{1}{2}}}\right] \leq$$

$$\leq \frac{< A^{\lambda}x, x >< B^{1-\lambda}y, y >}{(< Ax, x >)^{\lambda}(< By, y >)^{1-\lambda}} \leq \tag{3}$$

$$\leq \lambda||y||^2 + (1-\lambda)||x||^2 - r\left[||y||^2 + ||x||^2 - 2\frac{< A^{\frac{1}{2}}x, x >< B^{\frac{1}{2}}y, y >}{(< Ax, x >< By, y >)^{\frac{1}{2}}}\right].$$

Proof. We consider inequality (1) where a will be replaced by $\frac{a}{<Ax,x>}$ and b by $\frac{b}{<By,y>}$. Thus we get

$$r\left(\frac{a}{< Ax, x >} + \frac{b}{< By, y >} - 2\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{(< Ax, x >)^{\frac{1}{2}}(< By, y >)^{\frac{1}{2}}}\right) \leq$$

$$\leq \lambda\frac{a}{< Ax, x >} + (1-\lambda)\frac{b}{< By, y >} - \frac{a^{\lambda}b^{1-\lambda}}{(< Ax, x >)^{\lambda}(< By, y >)^{1-\lambda}} \leq$$

$$\leq (1-r)\left(\frac{a}{< Ax, x >} + \frac{b}{< By, y >} - 2\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{(< Ax, x >)^{\frac{1}{2}}(< By, y >)^{\frac{1}{2}}}\right),$$

for all a, b real positive numbers and $x, y \in H$ with $x \neq 0, y \neq 0$.

If we fix $a > 0$ and apply the inequality (2), see the property (P) from [2], then we have for each $x \in H,\ x \neq 0$ that

$$r\left(\frac{A}{< Ax, x >} + \frac{bI}{< By, y >} - 2\frac{A^{\frac{1}{2}}b^{\frac{1}{2}}}{(< Ax, x >)^{\frac{1}{2}}(< By, y >)^{\frac{1}{2}}}\right) \leq$$

$$\leq \lambda\frac{A}{< Ax, x >} + (1-\lambda)\frac{bI}{< By, y >} - \frac{A^{\lambda}b^{1-\lambda}}{(< Ax, x >)^{\lambda}(< By, y >)^{1-\lambda}} \leq$$

$$\leq (1-r)\left(\frac{A}{< Ax, x >} + \frac{bI}{< By, y >} - 2\frac{A^{\frac{1}{2}}b^{\frac{1}{2}}}{(< Ax, x >)^{\frac{1}{2}}(< By, y >)^{\frac{1}{2}}}\right),$$

which is equivalent to

$$r\left(1 + \frac{bI < x, x >}{< By, y >} - 2\frac{b^{\frac{1}{2}} < A^{\frac{1}{2}}x, x >}{(< Ax, x >)^{\frac{1}{2}}(< By, y >)^{\frac{1}{2}}}\right) \leq$$

$$\leq \lambda + (1-\lambda)\frac{bI < x, x >}{< By, y >} - \frac{b^{1-\lambda} < A^{\lambda}x, x >}{(< Ax, x >)^{\lambda}(< By, y >)^{1-\lambda}} \leq$$

$$\leq (1-r)\left(1 + \frac{bI < x, x >}{< By, y >} - 2\frac{b^{\frac{1}{2}} < A^{\frac{1}{2}}x, x >}{(< Ax, x >)^{\frac{1}{2}}(< By, y >)^{\frac{1}{2}}}\right)$$

for all $x, y \in H$ with $x \neq 0,\ y \neq 0$.

But, if we apply now the inequality (2), see the property (P) from [2], then for any $y \in H, y \neq 0$ we have:

$$r\left(I+\frac{B<x,x>}{<By,y>}-2\frac{B^{\frac{1}{2}}<A^{\frac{1}{2}}x,x>}{(<Ax,x>)^{\frac{1}{2}}(<By,y>)^{\frac{1}{2}}}\right)\leq$$
$$\leq\lambda I+(1-\lambda)\frac{B<x,x>}{<By,y>}-\frac{B^{1-\lambda}<A^{\lambda}x,x>}{(<Ax,x>)^{\lambda}(<By,y>)^{1-\lambda}}\leq$$
$$\leq(1-r)\left(I+\frac{B<x,x>}{<By,y>}-2\frac{B^{\frac{1}{2}}<A^{\frac{1}{2}}x,x>}{(<Ax,x>)^{\frac{1}{2}}(<By,y>)^{\frac{1}{2}}}\right)$$

and therefore

$$r\left(||y||^2+||x||^2-2\frac{<B^{\frac{1}{2}}y,y><A^{\frac{1}{2}}x,x>}{(<Ax,x>)^{\frac{1}{2}}(<By,y>)^{\frac{1}{2}}}\right)\leq$$
$$\leq\lambda||y||^2+(1-\lambda)||x||^2-\frac{<B^{1-\lambda}y,y><A^{\lambda}x,x>}{(<Ax,x>)^{\lambda}(<By,y>)^{1-\lambda}}\leq$$
$$\leq(1-r)\left(||y||^2+||x||^2-2\frac{<B^{\frac{1}{2}}y,y><A^{\frac{1}{2}}x,x>}{(<Ax,x>)^{\frac{1}{2}}(<By,y>)^{\frac{1}{2}}}\right)$$

for all $x,y\in H$ with $x\neq 0,\ y\neq 0$.

∎

Inequalities from below represent several important special cases of the inequality (3) from Theorem 3.

Consequence 2. *(i) If we take now $x=y$, in the conditions of previous theorem, Theorem 3, then the inequality becomes:*

$$||x||^2-2(1-r)\left[||x||^2-\frac{<A^{\frac{1}{2}}x,x><B^{\frac{1}{2}}x.x>}{(<Ax,x><Bx,x>)^{\frac{1}{2}}}\right]\leq$$
$$\leq\frac{<A^{\lambda}x,x><B^{1-\lambda}x,x>}{(<Ax,x>)^{\lambda}(<Bx,x>)^{1-\lambda}}\leq$$
$$\leq||x||^2-2r\left[||x||^2-\frac{<A^{\frac{1}{2}}x,x><B^{\frac{1}{2}}x.x>}{(<Ax,x><Bx,x>)^{\frac{1}{2}}}\right].$$

(ii) If we take instead $A=B$, in the conditions of previous theorem, Theorem 3, then the inequality becomes:

$$\lambda||y||^2+(1-\lambda)||x||^2-(1-r)\left[||y||^2+||x||^2-2\frac{<A^{\frac{1}{2}}x,x><A^{\frac{1}{2}}y,y>}{<(Ax,x><Ay,y>)^{\frac{1}{2}}}\right]\leq$$
$$\leq\frac{<A^{\lambda}x,x><A^{1-\lambda}x,x>}{<Ax,x>}\leq$$
$$\leq\lambda||y||^2+(1-\lambda)||x||^2-r\left[||y||^2+||x||^2-2\frac{<A^{\frac{1}{2}}x,x><A^{\frac{1}{2}}y,y>}{<(Ax,x><Ay,y>)^{\frac{1}{2}}}\right].$$

(iii) If we consider $A=B$ and $x=y$, in the conditions of Theorem 3, then the inequality becomes:

$$||x||^2-2(1-r)\left[||x||^2-\frac{(<A^{\frac{1}{2}}x,x>)^2}{<Ax,x>}\right]\leq\frac{<A^{\lambda}x,x><A^{1-\lambda}x,x>}{<Ax,x>}\leq$$

$$\leq ||x||^2 - 2r\left[||x||^2 - \frac{(< A^{\frac{1}{2}}x, x >)^2}{< Ax, x >}\right].$$

Remark 4. *Moreover, the inequality from Theorem 3 and inequalities from last consequence, Consequence 2, will be rewritten below, using the norm:*

$$\lambda||y||^2 + (1-\lambda)||x||^2 - (1-r)\left[||y||^2 + ||x||^2 - 2\frac{||A^{\frac{1}{4}}x||^2||B^{\frac{1}{4}}y||^2}{||A^{\frac{1}{2}}x||||B^{\frac{1}{2}}y||}\right] \leq$$

$$(a) \qquad \leq \frac{||A^{\frac{\lambda}{2}}x||^2||B^{\frac{1-\lambda}{2}}y||^2}{||A^{\frac{1}{2}}x||^{2\lambda}||B^{\frac{1}{2}}y||^{2(1-\lambda)}} \leq$$

$$\leq \lambda||y||^2 + (1-\lambda)||x||^2 - r\left[||y||^2 + ||x||^2 - 2\frac{||A^{\frac{1}{4}}x||^2||B^{\frac{1}{4}}y||^2}{||A^{\frac{1}{2}}x||||B^{\frac{1}{2}}y||}\right];$$

$$(b) \qquad ||x||^2 - 2(1-r)\left[||x||^2 - \frac{||A^{\frac{1}{4}}x||^2||B^{\frac{1}{4}}x||^2}{||A^{\frac{1}{2}}x||||B^{\frac{1}{2}}x||}\right] \leq \frac{||A^{\frac{\lambda}{2}}x||^2||B^{\frac{1-\lambda}{2}}x||^2}{||A^{\frac{1}{2}}x||^{2\lambda}||B^{\frac{1}{2}}x||^{2(1-\lambda)}} \leq$$

$$\leq ||x||^2 - 2r\left[||x||^2 - \frac{||A^{\frac{1}{4}}x||^2||B^{\frac{1}{4}}x||^2}{||A^{\frac{1}{2}}x||||B^{\frac{1}{2}}x||}\right];$$

$$\lambda||y||^2 + (1-\lambda)||x||^2 - (1-r)\left[||y||^2 + ||x||^2 - 2\frac{||A^{\frac{1}{4}}x||^2||A^{\frac{1}{4}}y||^2}{||A^{\frac{1}{2}}x||||A^{\frac{1}{2}}y||}\right] \leq$$

$$(c) \qquad \leq \frac{||A^{\frac{\lambda}{2}}x||^2||A^{\frac{1-\lambda}{2}}y||^2}{||A^{\frac{1}{2}}x||^{2\lambda}||A^{\frac{1}{2}}y||^{2(1-\lambda)}} \leq$$

$$\leq \lambda||y||^2 + (1-\lambda)||x||^2 - r\left[||y||^2 + ||x||^2 - 2\frac{||A^{\frac{1}{4}}x||^2||A^{\frac{1}{4}}y||^2}{||A^{\frac{1}{2}}x||||A^{\frac{1}{2}}y||}\right];$$

$$(d) \qquad 1 - 2(1-r)\left[1 - \frac{||A^{\frac{1}{4}}x||^4}{||A^{\frac{1}{2}}x||^2||x||^2}\right] \leq \frac{||A^{\frac{\lambda}{2}}x||^2||A^{\frac{1-\lambda}{2}}x||^2}{||A^{\frac{1}{2}}x||^2||x||^2} \leq$$

$$\leq 1 - 2r\left[1 - \frac{||A^{\frac{1}{4}}x||^4}{||A^{\frac{1}{2}}x||^2||x||^2}\right].$$

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