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INEQUALITIES OF HERMITE-HADAMARD TYPE FOR AH-CONVEX FUNCTIONS

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Abstract. Some inequalities of Hermite-Hadamard type for AH-convex functions defined on convex subsets in real or complex linear spaces are given. The case for functions of one real variable is explored in depth. Applications for special means are provided as well.

1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$(b-a)f\left(\frac{a+b}{2}\right)\leq\int_a^b f(x)dx\leq(b-a)\frac{f(a)+f(b)}{2},\quad a,b\in\mathbb{R},\ a<b. \tag{1.1}$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [41]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [5]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [41]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the Hermite-Hadamard inequality.

For related results, see [1]-[19], [22]-[24], [25]-[34] and [35]-[44].

Let X be a vector space over the real or complex number field $\mathbb{K}$ and $x,y\in X,\ x\neq y$. Define the segment

$$[x,y]:=\{(1-t)x+ty,\ t\in[0,1]\}.$$

We consider the function $f:[x,y]\to\mathbb{R}$ and the associated function

$$g(x,y):[0,1]\to\mathbb{R},\ g(x,y)(t):=f[(1-t)x+ty],\ t\in[0,1].$$

Note that f is convex on $[x,y]$ if and only if $g(x,y)$ is convex on $[0,1]$.

For any convex function defined on a segment $[x,y]\subset X$, we have the *Hermite-Hadamard integral inequality* (see [20, p. 2], [21, p. 2])

$$f\left(\frac{x+y}{2}\right)\leq\int_0^1 f[(1-t)x+ty]dt\leq\frac{f(x)+f(y)}{2}, \tag{1.2}$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x,y):[0,1]\to\mathbb{R}$.

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Let X be a linear space and C a convex subset in X. A function $f:C\rightarrow\mathbb{R}\backslash\{0\}$ is called *AH-convex (concave)* on the convex set C if the following inequality holds

$$\text{(AH)}\qquad f\left((1-\lambda)x+\lambda y\right)\leq(\geq)\frac{1}{(1-\lambda)\frac{1}{f(x)}+\lambda\frac{1}{f(y)}}=\frac{f(x)f(y)}{(1-\lambda)f(y)+\lambda f(x)}$$

for any $x,y\in C$ and $\lambda\in[0,1]$.

An important case which provides many examples is that one in which the function is assumed to be positive for any $x\in C$. In that situation the inequality (AH) is equivalent to

$$(1-\lambda)\frac{1}{f(x)}+\lambda\frac{1}{f(y)}\leq(\geq)\frac{1}{f\left((1-\lambda)x+\lambda y\right)}$$

for any $x,y\in C$ and $\lambda\in[0,1]$.

Therefore we can state the following fact:

Criterion 1. *Let X be a linear space and C a convex subset in X. The function $f:C\rightarrow(0,\infty)$ is AH-convex (concave) on C if and only if $\frac{1}{f}$ is concave (convex) on C in the usual sense.*

If we apply the Hermite-Hadamard inequality (1.2) for the function $\frac{1}{f}$ then we state the following result:

Proposition 1. *Let X be a linear space and C a convex subset in X. If the function $f:C\rightarrow(0,\infty)$ is AH-convex (concave) on C, then*

$$\frac{f(x)+f(y)}{2f(x)f(y)}\leq(\geq)\int_0^1\frac{d\lambda}{f\left((1-\lambda)x+\lambda y\right)}\leq(\geq)\frac{1}{f\left(\frac{x+y}{2}\right)}\tag{1.3}$$

for any $x,y\in C$.

Motivated by the above results, in this paper we establish some new Hermite-Hadamard type inequalities for AH-convex (concave) functions, first in the general setting of linear spaces and then in the particular case of functions of a real variable. Some examples for special means are provided as well.

2. Some Hermite-Hadamard Type Inequalities

The following result holds:

Theorem 1. *Let X be a linear space and C a convex subset in X. If the function $f:C\rightarrow(0,\infty)$ is AH-convex (concave) on C, then for any $x,y\in C$ we have*

$$\int_0^1 f\left((1-\lambda)x+\lambda y\right)d\lambda\leq(\geq)\frac{G^2\left(f(x),f(y)\right)}{L\left(f(x),f(y)\right)},\tag{2.1}$$

where the Logarithmic mean of positive numbers a,b is defined as

$$L(a,b):=\begin{cases}\frac{b-a}{\ln b-\ln a} & \text{if } a\neq b\\ a & \text{if } a=b,\end{cases}$$

and the geometric mean is $G=\sqrt{ab}$.

Proof. Let $x, y \in C$ with $x \neq y$. If $f : C \to (0, \infty)$ is AH-convex (concave) on C, then $\frac{1}{f}$ is concave (convex) on C. This implies that the function

$$\varphi_{x,y} : [0,1] \to (0,\infty), \varphi_{x,y}(t) = \frac{1}{f((1-\lambda)x + \lambda y)}$$

is concave (convex) on $[0,1]$ and therefore continuous on $(0,1)$ with $\varphi_{x,y}(0) = \frac{1}{f(x)}$ and $\varphi_{x,y}(1) = \frac{1}{f(y)}$. The function $[0,1] \ni t \mapsto f((1-t)x + ty)$ is continuous on $(0,1)$ and since $f(x), f(y) > 0$ are finite, then the Lebesgue integral $\int_0^1 f((1-t)x + ty)\,dt$ exists and by (AH) we have

$$\int_0^1 f((1-\lambda)x + \lambda y)\,d\lambda \leq (\geq) f(x) f(y) \int_0^1 \frac{d\lambda}{(1-\lambda) f(y) + \lambda f(x)}. \tag{2.2}$$

If $f(y) = f(x)$, then

$$\int_0^1 \frac{d\lambda}{(1-\lambda) f(y) + \lambda f(x)} = \frac{1}{f(y)}.$$

If $f(y) \neq f(x)$, then by changing the variable $u = \lambda(f(x) - f(y)) + f(y)$ we have

$$\int_0^1 \frac{d\lambda}{(1-\lambda) f(y) + \lambda f(x)} = \frac{\ln f(x) - \ln f(y)}{f(x) - f(y)} = \frac{1}{L(f(x), f(y))}.$$

By the use of (2.2) we get the desired result (2.1). □

Remark 1. *Using the following well known inequalities*

$$H(a,b) \leq G(a,b) \leq L(a,b)$$

we have

$$\int_0^1 f((1-\lambda)x + \lambda y)\,d\lambda \leq \frac{G^2(f(x), f(y))}{L(f(x), f(y))} \leq G(f(x), f(y)) \tag{2.3}$$

for any $x, y \in C$, provided that $f : C \to (0,\infty)$ is AH-convex.

If $f : C \to (0,\infty)$ is AH-concave, then

$$\begin{aligned} \int_0^1 f((1-\lambda)x + \lambda y)\,d\lambda &\geq \frac{G^2(f(x), f(y))}{L(f(x), f(y))} \\ &\geq \frac{G(f(x), f(y))}{L(f(x), f(y))} H(f(x), f(y)) \end{aligned} \tag{2.4}$$

for any $x, y \in C$.

Theorem 2. *Let X be a linear space and C a convex subset in X. If the function $f : C \to (0,\infty)$ is AH-convex (concave) on C, then for any $x, y \in C$ we have*

$$f\left(\frac{x+y}{2}\right) \leq (\geq) \frac{\int_0^1 f((1-\lambda)x + \lambda y) f(\lambda x + (1-\lambda) y)\,d\lambda}{\int_0^1 f((1-\lambda)x + \lambda y)\,d\lambda}. \tag{2.5}$$

Proof. By the definition of AH-convexity (concavity) we have

$$f\left(\frac{u+v}{2}\right) \leq (\geq) \frac{2f(u) f(v)}{f(u) + f(v)} \tag{2.6}$$

for any $u, v \in C$.

Let $x,y\in C$ and $\lambda\in[0,1]$. If we take in (2.6) $u=(1-\lambda)x+\lambda y$ and $v=\lambda x+(1-\lambda)y$, then we get

$$f\left(\frac{x+y}{2}\right)\le(\ge)\frac{2f((1-\lambda)x+\lambda y)f(\lambda x+(1-\lambda)y)}{f((1-\lambda)x+\lambda y)+f(\lambda x+(1-\lambda)y)},$$

which is equivalent to

$$\begin{aligned}\frac{1}{2}f\left(\frac{x+y}{2}\right)&[f((1-\lambda)x+\lambda y)+f(\lambda x+(1-\lambda)y)]\\&\le(\ge)f((1-\lambda)x+\lambda y)f(\lambda x+(1-\lambda)y).\end{aligned}\tag{2.7}$$

Integrating the inequality on $[0,1]$ over $\lambda\in[0,1]$ and taking into account that

$$\int_0^1 f((1-\lambda)x+\lambda y)\,d\lambda=\int_0^1 f(\lambda x+(1-\lambda)y)\,d\lambda$$

we deduce from (2.7) the desired result (2.5). □

Remark 2. *By the Cauchy-Bunyakovsky-Schwarz integral inequality we have*

$$\begin{aligned}&\int_0^1 f((1-\lambda)x+\lambda y)f(\lambda x+(1-\lambda)y)\,d\lambda\\&\le\left[\int_0^1 f^2((1-\lambda)x+\lambda y)\,d\lambda\int_0^1 f^2(\lambda x+(1-\lambda)y)\,d\lambda\right]^{1/2}\\&=\int_0^1 f^2((1-\lambda)x+\lambda y)\,d\lambda\end{aligned}\tag{2.8}$$

for any $x,y\in C$.

If the function $f:C\to(0,\infty)$ is AH-convex on C, then we have

$$\begin{aligned}f\left(\frac{x+y}{2}\right)&\le\frac{\int_0^1 f((1-\lambda)x+\lambda y)f(\lambda x+(1-\lambda)y)\,d\lambda}{\int_0^1 f((1-\lambda)x+\lambda y)\,d\lambda}\\&\le\frac{\int_0^1 f^2((1-\lambda)x+\lambda y)\,d\lambda}{\int_0^1 f((1-\lambda)x+\lambda y)\,d\lambda}.\end{aligned}\tag{2.9}$$

If the function $\psi_{x,y}(t)=f((1-t)x+ty)$, for some given $x,y\in C$ with $x\neq y$, is monotonic nondecreasing on $[0,1]$, then $\chi_{x,y}(t)=f(tx+(1-t)y)$ is monotonic nonincreasing on $[0,1]$ and by Čebyšev's inequality for monotonic opposite functions we have

$$\begin{aligned}&\int_0^1 f((1-\lambda)x+\lambda y)f(\lambda x+(1-\lambda)y)\,d\lambda\\&\le\int_0^1 f((1-\lambda)x+\lambda y)\,d\lambda\int_0^1 f(\lambda x+(1-\lambda)y)\,d\lambda\\&=\left(\int_0^1 f((1-\lambda)x+\lambda y)\,d\lambda\right)^2.\end{aligned}$$

So, for some given $x, y \in C$ with $x \neq y$, $\psi_{x,y}(t) = f((1-t)x + ty)$ is monotonic nondecreasing (nonincreasing) on $[0,1]$ and if the function $f : C \to (0,\infty)$ is AH-convex on C, then we have

$$f\left(\frac{x+y}{2}\right) \leq \frac{\int_0^1 f((1-\lambda)x + \lambda y) f(\lambda x + (1-\lambda)y) d\lambda}{\int_0^1 f((1-\lambda)x + \lambda y) d\lambda} \leq \int_0^1 f((1-\lambda)x + \lambda y) d\lambda. \tag{2.10}$$

If $(X, \|\cdot\|)$ is a normed space, then the function $g : X \to [0,\infty)$, $g(x) = \|x\|^p$, $p \geq 1$ is convex and then the function $f : C \subset X \to (0,\infty)$, $f(x) = \frac{1}{\|x\|^p}$ is *AH-concave* on any convex subset of X which does not contain $\{0\}$.

Utilising (2.1) we have

$$\int_0^1 \frac{d\lambda}{\|(1-\lambda)x + \lambda y\|^p} \geq \frac{1}{L(\|x\|^p, \|y\|^p)}, \tag{2.11}$$

for any linearly independent $x, y \in X$ and $p \geq 1$.

Making use of (2.5) we also have

$$\int_0^1 \frac{d\lambda}{\|(1-\lambda)x + \lambda y\|^p} \geq \left\|\frac{x+y}{2}\right\|^p \int_0^1 \frac{d\lambda}{\|(1-\lambda)x + \lambda y\|^p \|\lambda x + (1-\lambda)y\|^p} \tag{2.12}$$

for any linearly independent $x, y \in X$ and $p \geq 1$.

3. More Results for Scalar Case

If the function f is defined on an interval I and $a, b \in I$ with $a < b$, then

$$\int_0^1 f((1-\lambda)x + \lambda y) d\lambda = \frac{1}{b-a}\int_a^b f(t) dt$$

and the inequalities (1.3), (2.1) and (2.5) can be written as

$$\frac{f(a) + f(b)}{2f(a)f(b)} \leq (\geq) \frac{1}{b-a}\int_a^b \frac{1}{f(t)} dt \leq (\geq) \frac{1}{f\left(\frac{a+b}{2}\right)}, \tag{3.1}$$

$$\frac{1}{b-a}\int_a^b f(t) dt \leq (\geq) \frac{G^2(f(a), f(b))}{L(f(a), f(b))}, \tag{3.2}$$

and

$$f\left(\frac{a+b}{2}\right) \leq (\geq) \frac{\int_a^b f(t) f(a+b-t) dt}{\int_a^b f(t) dt}, \tag{3.3}$$

respectively, where $f : I \to (0,\infty)$ is assumed to be AH-convex (concave) on I.

The following proposition holds:

Proposition 2. *Let $f : I \to (0,\infty)$ be AH-convex (concave) on I. Let $x, y \in \mathring{I}$, the interior of I, then there exists $\varphi(y) \in [f'_-(y), f'_+(y)]$ such that*

$$\frac{f(y)}{f(x)} - 1 \leq (\geq) \frac{\varphi(y)}{f(y)}(y - x) \tag{3.4}$$

holds.

Proof. Let $x, y \in \mathring{I}$. Since the function $\frac{1}{f}$ is concave (convex) then the lateral derivatives $f'_{-}(y), f'_{+}(y)$ exists for $y \in \mathring{I}$ and $\left(\frac{1}{f}\right)'_{-(+)}(y) = -\frac{f'_{-(+)}(y)}{f^2(y)}$.

Since $\frac{1}{f}$ is concave (convex) then we have the *gradient inequality*

$$\frac{1}{f(y)} - \frac{1}{f(x)} \geq (\leq) \lambda(y)(y-x) = -\lambda(y)(x-y)$$

with $\lambda(y) \in \left[-\frac{f'_{+}(y)}{f^2(y)}, -\frac{f'_{-}(y)}{f^2(y)}\right]$, which is equivalent to

$$\frac{1}{f(y)} - \frac{1}{f(x)} \geq (\leq) \frac{\varphi(y)}{f^2(y)}(x-y) \tag{3.5}$$

with $\varphi(y) \in \left[f'_{-}(y), f'_{+}(y)\right]$.

The inequality (3.5) can be also written as

$$1 - \frac{f(y)}{f(x)} \geq (\leq) \frac{\varphi(y)}{f(y)}(x-y)$$

or as

$$\frac{f(y)}{f(x)} - 1 \leq (\geq) \frac{\varphi(y)}{f(y)}(y-x)$$

and the inequality (3.4) is proved. □

Corollary 1. *Let $f : I \to (0, \infty)$ be AH-convex (concave) on I. If f is differentiable on $\mathring{I}$ then for any $x, y \in \mathring{I}$, we have*

$$\frac{f(y)}{f(x)} - 1 \leq (\geq) \frac{f'(y)}{f(y)}(y-x). \tag{3.6}$$

The following result also holds:

Theorem 3. *Let $f : I \to (0, \infty)$ be AH-convex (concave) on I. If $a, b \in I$ with $a < b$, then we have the inequality*

$$\frac{1}{b-a}\int_a^b f^2(t)\,dt \leq (\geq) \left[\frac{b-s}{b-a}f(b) + \frac{s-a}{b-a}f(a)\right] f(s) \tag{3.7}$$

for any $s \in [a, b]$.

In particular, we have

$$\frac{1}{b-a}\int_a^b f^2(t)\,dt \leq (\geq) f\left(\frac{a+b}{2}\right)\frac{f(a)+f(b)}{2} \tag{3.8}$$

and

$$\frac{1}{b-a}\int_a^b f^2(t)\,dt \leq (\geq) f(a) f(b). \tag{3.9}$$

Proof. If the function $f : I \to (0, \infty)$ is AH-convex (concave) on I, then the function f is differentiable almost everywhere on I and we have the inequality

$$\frac{f(t)}{f(s)} - 1 \leq (\geq) \frac{f'(t)}{f(t)}(t-s) \tag{3.10}$$

for every $s \in [a, b]$ and almost every $t \in [a, b]$.

Multiplying (3.10) by $f(t) > 0$ and integrating over $t \in [a, b]$ we have

$$\frac{1}{f(s)}\int_a^b f^2(t)\,dt - \int_a^b f(t)\,dt \leq (\geq) \int_a^b f'(t)(t-s)\,dt. \tag{3.11}$$

Integrating by parts we have

$$\int_a^b f'(t)(t-s)\,dt = f(b)(b-s) + f(a)(s-a) - \int_a^b f(t)\,dt$$

and by (3.11) we get the desired result (3.7).

We observe that (3.8) follows by (3.7) for $s = \frac{a+b}{2}$ while (3.9) follows by (3.7) for either $s = a$ or $s = b$. □

Remark 3. *By the Cauchy-Bunyakovsky-Schwarz integral inequality we have*

$$\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)^2 \le \frac{1}{b-a}\int_a^b f^2(t)\,dt$$

and if we assume that $f : I \to (0,\infty)$ is AH-convex on I, then we have

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le \left(\frac{1}{b-a}\int_a^b f^2(t)\,dt\right)^{1/2} \le \sqrt{f\left(\frac{a+b}{2}\right)\frac{f(a)+f(b)}{2}} \tag{3.12}$$

and

$$\frac{1}{b-a}\int_a^b f(t)\,dt \le \left(\frac{1}{b-a}\int_a^b f^2(t)\,dt\right)^{1/2} \le \sqrt{f(a)\,f(b)}. \tag{3.13}$$

The following result also holds:

Theorem 4. *Let $f : I \to (0,\infty)$ be AH-convex (concave) on I. If $a, b \in I$ with $a < b$, then we have the inequality*

$$\begin{aligned}&\int_a^b \ln f(t)\,dt + \frac{1}{f(s)}\int_a^b f(t)\,dt \\ &\le (\ge)\, b-a+(b-s)\ln f(b) + (s-a)\ln f(a)\end{aligned} \tag{3.14}$$

for any $s \in [a,b]$.

In particular, we have

$$\begin{aligned}&\frac{1}{b-a}\int_a^b \ln f(t)\,dt + \frac{1}{f\left(\frac{a+b}{2}\right)}\frac{1}{b-a}\int_a^b f(t)\,dt \\ &\le (\ge)\, 1 + \ln\sqrt{f(b)\,f(a)}\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}&\frac{1}{b-a}\int_a^b \ln f(t)\,dt + \left[\frac{f(b)+f(a)}{2f(a)\,f(b)}\right]\frac{1}{b-a}\int_a^b f(t)\,dt \\ &\le (\ge)\, 1 + \ln\sqrt{f(b)\,f(a)}.\end{aligned} \tag{3.16}$$

Proof. Integrating the inequality (3.10) over $t \in [a,b]$ we have

$$\frac{1}{f(s)}\int_a^b f(t)\,dt - (b-a) \le (\ge) \int_a^b \frac{f'(t)}{f(t)}(t-s)\,dt. \tag{3.17}$$

Observe that

$$\int_a^b \frac{f'(t)}{f(t)}(t-s)\,dt = \int_a^b (\ln f(t))'(t-s)\,dt$$
$$= (t-s)\ln f(t)|_a^b - \int_a^b \ln f(t)\,dt$$
$$= (b-s)\ln f(b) + (s-a)\ln f(a) - \int_a^b \ln f(t)\,dt$$

and by (3.17) we get

$$\frac{1}{f(s)}\int_a^b f(t)\,dt - (b-a)$$
$$\leq (\geq)\,(b-s)\ln f(b) + (s-a)\ln f(a) - \int_a^b \ln f(t)\,dt,$$

which is equivalent to

$$\int_a^b \ln f(t)\,dt + \frac{1}{f(s)}\int_a^b f(t)\,dt$$
$$\leq (\geq)\, b-a + (b-s)\ln f(b) + (s-a)\ln f(a)$$

for any $s \in [a,b]$.

If we take in (3.14) $s = \frac{a+b}{2}$ then we get the desired result (3.15).

If we take in (3.14) $s = a$ and $s = b$ we get

$$\int_a^b \ln f(t)\,dt + \frac{1}{f(a)}\int_a^b f(t)\,dt \leq (\geq)\, b-a + (b-a)\ln f(b)$$

and

$$\int_a^b \ln f(t)\,dt + \frac{1}{f(b)}\int_a^b f(t)\,dt \leq (\geq)\, b-a + (b-a)\ln f(a),$$

which by addition produces

$$2\int_a^b \ln f(t)\,dt + \frac{1}{f(a)}\int_a^b f(t)\,dt + \frac{1}{f(b)}\int_a^b f(t)\,dt$$
$$\leq (\geq)\, 2(b-a) + (b-a)\ln f(b) + (b-a)\ln f(a)$$

and then

$$\int_a^b \ln f(t)\,dt + \left[\frac{f(b)+f(a)}{2f(a)f(b)}\right]\int_a^b f(t)\,dt$$
$$\leq (\geq)\, b-a + (b-a)\ln\sqrt{f(b)f(a)},$$

which is equivalent to (3.16). □

Remark 4. *We observe that*

$$(b-s)\ln f(b) + (s-a)\ln f(a) = 0$$

iff

$$s = \frac{b\ln f(b) - a\ln f(a)}{\ln f(b) - \ln f(a)} = \frac{L(f(a), f(b))}{L\left([f(a)]^a, [f(b)]^b\right)}.$$

If

$$s=\frac{L\left(f\left(a\right),f\left(b\right)\right)}{L\left(\left[f\left(a\right)\right]^{a},\left[f\left(b\right)\right]^{b}\right)}\in I$$

then from (3.14) we have

$$\frac{1}{b-a}\int_{a}^{b}\ln f\left(t\right)dt+\frac{1}{f\left(\frac{L\left(f\left(a\right),f\left(b\right)\right)}{L\left(\left[f\left(a\right)\right]^{a},\left[f\left(b\right)\right]^{b}\right)}\right)}\frac{1}{b-a}\int_{a}^{b}f\left(t\right)dt\leq\left(\geq\right)1. \tag{3.18}$$

Let $(X,\|\cdot\|)$ be a normed space and $x,y\in X$ two linearly independent vectors on X. Since the function $g:[0,1]\to(0,\infty)$, $g\left(t\right)=\|(1-t)\,x+ty\|^{p}$, $p\geq1$ is convex on $[0,1]$, then the function $f:[0,1]\to(0,\infty)$, $g\left(t\right)=\frac{1}{\|(1-t)x+ty\|^{p}}$ is AH-concave on $[0,1]$.

Making use of the inequalities (3.8) and (3.9) we get

$$\left\|\frac{x+y}{2}\right\|^{p}\int_{0}^{1}\frac{1}{\|(1-t)\,x+ty\|^{2p}}dt\geq\frac{\|x\|^{p}+\|y\|^{p}}{2\|x\|^{p}\|y\|^{p}} \tag{3.19}$$

and

$$\int_{0}^{1}\frac{1}{\|(1-t)\,x+ty\|^{2p}}dt\geq\frac{1}{\|x\|^{p}\|y\|^{p}}. \tag{3.20}$$

4. Applications for Special Means

Let us recall the following means:

a) The *arithmetic mean*

$$A\left(a,b\right):=\frac{a+b}{2},\quad a,b>0,$$

b) The *geometric mean*

$$G\left(a,b\right):=\sqrt{ab};\quad a,b\geq0,$$

c) The *harmonic mean*

$$H\left(a,b\right):=\frac{2}{\frac{1}{a}+\frac{1}{b}};\quad a,b>0,$$

d) The *identric mean*

$$I\left(a,b\right):=\begin{cases}\frac{1}{e}\left(\frac{b^{b}}{a^{a}}\right)^{\frac{1}{b-a}} & \text{if}\quad b\neq a\\ a & \text{if}\quad b=a\end{cases};\quad a,b>0$$

e) The *logarithmic mean*

$$L\left(a,b\right):=\begin{cases}\frac{b-a}{\ln b-\ln a} & \text{if}\quad b\neq a\\ a & \text{if}\quad b=a\end{cases};\quad a,b>0$$

f) The $p-$*logarithmic mean*

$$L_p(a,b) := \begin{cases} \left(\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}} & \text{if} \quad b \neq a, \;\; p \in \mathbb{R}\backslash\{-1,0\} \\ & \qquad\qquad\qquad\qquad ; \;\; a,b>0. \\ a & \text{if} \quad b=a \end{cases}$$

It is well known that, if $L_{-1} := L$ and $L_0 := I$, then the function $\mathbb{R} \ni p \mapsto L_p$ is monotonically strictly increasing. In particular, we have

$$H(a,b) \leq G(a,b) \leq L(a,b) \leq I(a,b) \leq A(a,b).$$

Consider the function

$$f(t) = t^p = \frac{1}{t^{-p}}$$

if $-p > 1$ or $-p < 0$, i.e. $p \in (-\infty,-1) \cup (0,\infty)$ then the function $f(t) = t^p, t > 0$ is AH-concave. If $p \in (-1,0)$ then the function $f(t) = t^p, t > 0$ is AH-convex.

Now, if we write the inequality (3.2) for the function $f(t) = t^p$ and $0 < a < b$ we get

$$\frac{1}{b-a}\int_a^b t^p dt \leq (\geq) \frac{G^2(a^p,b^p)}{L(a^p,b^p)}, \tag{4.1}$$

where $p \in (-1,0)\,(p \in (-\infty,-1) \cup (0,\infty))$.

Now, observe that

$$\frac{1}{b-a}\int_a^b t^p dt = \frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)} = L_p^p(a,b),$$

$$\begin{aligned} L(a^p,b^p) &= \frac{b^p-a^p}{\ln b^p - \ln a^p} = \frac{b^p-a^p}{p(b-a)}\frac{b-a}{\ln b - \ln a} \\ &= L_{p-1}^{p-1}(a,b)\,L(a,b)\,, \;\; p \in \mathbb{R}\backslash\{0,1\} \end{aligned}$$

and

$$G^2(a^p,b^p) = G^{2p}(a,b).$$

Then by (4.1) we get

$$L_p^p(a,b)\,L_{p-1}^{p-1}(a,b)\,L(a,b) \leq (\geq)\, G^{2p}(a,b), \tag{4.2}$$

where $p \in (-1,0)\,(p \in (-\infty,-1) \cup (0,\infty) \setminus \{1\})$.

If we write the inequality (3.8) for the function $f(t) = t^p$ and $0 < a < b$ we get

$$\frac{1}{b-a}\int_a^b t^{2p} dt \leq (\geq) \left(\frac{a+b}{2}\right)^p \frac{a^p+b^p}{2} \tag{4.3}$$

where $p \in (-1,0)\,(p \in (-\infty,-1) \cup (0,\infty))$.

Since

$$\frac{1}{b-a}\int_a^b t^{2p} dt = L_{2p}^{2p}(a,b)\,, \;\; p \in \mathbb{R}\backslash\left\{-\frac{1}{2},0\right\},$$

$$\left(\frac{a+b}{2}\right)^p = A^p(a,b)\,, \;\; \frac{a^p+b^p}{2} = A(a^p,b^p)\,,$$

then by (4.3) we have

$$L_{2p}^{2p}(a,b) \leq (\geq)\, A^p(a,b)\,A(a^p,b^p) \tag{4.4}$$

where $p \in (-1,0) \setminus \left\{-\frac{1}{2}\right\}(p \in (-\infty,-1) \cup (0,\infty))$.

Now consider the function $f(t) = \ln t$, $t > 1$. The function

$$g(t) := \frac{1}{\ln t},\ t > 1$$

is convex on $(1, \infty)$. If we apply the inequality (3.2) for the AH-concave function $f(t) = \ln t$, $t > 1$ on $[a, b] \subset (1, \infty)$, then we get

$$\ln I(a, b) \geq \frac{G^2(\ln a, \ln b)}{L(\ln a, \ln b)}. \tag{4.5}$$

References

[1] M. Alomari and M. Darus, The Hadamard's inequality for s-convex function. *Int. J. Math. Anal.* (Ruse) **2** (2008), no. 13-16, 639–646.

[2] M. Alomari and M. Darus, Hadamard-type inequalities for s-convex functions. *Int. Math. Forum* **3** (2008), no. 37-40, 1965–1975.

[3] G. A. Anastassiou, Univariate Ostrowski inequalities, revisited. *Monatsh. Math.,* **135** (2002), no. 3, 175–189.

[4] N. S. Barnett, P. Cerone, S. S. Dragomir, M. R. Pinheiro,and A. Sofo, Ostrowski type inequalities for functions whose modulus of the derivatives are convex and applications. *Inequality Theory and Applications*, **Vol. 2** (Chinju/Masan, 2001), 19–32, Nova Sci. Publ., Hauppauge, NY, 2003. Preprint: *RGMIA Res. Rep. Coll.* **5** (2002), No. 2, Art. 1 [Online http://rgmia.org/papers/v5n2/Paperwapp2q.pdf].

[5] E. F. Beckenbach, Convex functions, *Bull. Amer. Math. Soc.* **54**(1948), 439–460.

[6] M. Bombardelli and S. Varošanec, Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities. *Comput. Math. Appl.* **58** (2009), no. 9, 1869–1877.

[7] W. W. Breckner, Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen. (German) *Publ. Inst. Math.* (Beograd) (N.S.) **23(37)** (1978), 13–20.

[8] W. W. Breckner and G. Orbán, Continuity properties of rationally s-convex mappings with values in an ordered topological linear space. Universitatea "Babeş-Bolyai", Facultatea de Matematica, Cluj-Napoca, 1978. viii+92 pp.

[9] P. Cerone and S. S. Dragomir, Midpoint-type rules from an inequalities point of view, Ed. G. A. Anastassiou, *Handbook of Analytic-Computational Methods in Applied Mathematics*, CRC Press, New York. 135-200.

[10] P. Cerone and S. S. Dragomir, New bounds for the three-point rule involving the Riemann-Stieltjes integrals, in *Advances in Statistics Combinatorics and Related Areas*, C. Gulati, et al. (Eds.), World Science Publishing, 2002, 53-62.

[11] P. Cerone, S. S. Dragomir and J. Roumeliotis, Some Ostrowski type inequalities for n-time differentiable mappings and applications, *Demonstratio Mathematica*, **32**(2) (1999), 697—712.

[12] G. Cristescu, Hadamard type inequalities for convolution of h-convex functions. *Ann. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convexity* **8** (2010), 3–11.

[13] S. S. Dragomir, Ostrowski's inequality for monotonous mappings and applications, *J. KSIAM,* **3**(1) (1999), 127-135.

[14] S. S. Dragomir, The Ostrowski's integral inequality for Lipschitzian mappings and applications, *Comp. Math. Appl.,* **38** (1999), 33-37.

[15] S. S. Dragomir, On the Ostrowski's inequality for Riemann-Stieltjes integral, *Korean J. Appl. Math.,* **7** (2000), 477-485.

[16] S. S. Dragomir, On the Ostrowski's inequality for mappings of bounded variation and applications, *Math. Ineq. & Appl.,* **4**(1) (2001), 33-40.

[17] S. S. Dragomir, On the Ostrowski inequality for Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ where f is of Hölder type and u is of bounded variation and applications, *J. KSIAM,* **5**(1) (2001), 35-45.

[18] S. S. Dragomir, Ostrowski type inequalities for isotonic linear functionals, *J. Inequal. Pure & Appl. Math.,* **3**(5) (2002), Art. 68.

[19] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products. *J. Inequal. Pure Appl. Math.* **3** (2002), no. 2, Article 31, 8 pp.

[20] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure Appl. Math.* **3** (2002), No. 2, Article 31.

[21] S. S. Dragomir, An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure Appl. Math.* **3** (2002), No.3, Article 35.

[22] S. S. Dragomir, An Ostrowski like inequality for convex functions and applications, *Revista Math. Complutense*, **16**(2) (2003), 373-382.

[23] S. S. Dragomir, *Operator Inequalities of Ostrowski and Trapezoidal Type.* Springer Briefs in Mathematics. Springer, New York, 2012. x+112 pp. ISBN: 978-1-4614-1778-1

[24] S. S. Dragomir, P. Cerone, J. Roumeliotis and S. Wang, A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis, *Bull. Math. Soc. Sci. Math. Romanie,* **42(90)** (4) (1999), 301-314.

[25] S.S. Dragomir and S. Fitzpatrick, The Hadamard inequalities for s-convex functions in the second sense. *Demonstratio Math.* **32** (1999), no. 4, 687–696.

[26] S.S. Dragomir and S. Fitzpatrick,The Jensen inequality for s-Breckner convex functions in linear spaces. *Demonstratio Math.* **33** (2000), no. 1, 43–49.

[27] S. S. Dragomir and B. Mond, On Hadamard's inequality for a class of functions of Godunova and Levin. *Indian J. Math.* **39** (1997), no. 1, 1–9.

[28] S. S. Dragomir and C. E. M. Pearce, On Jensen's inequality for a class of functions of Godunova and Levin. *Period. Math. Hungar.* **33** (1996), no. 2, 93–100.

[29] S. S. Dragomir and C. E. M. Pearce, Quasi-convex functions and Hadamard's inequality, *Bull. Austral. Math. Soc.* **57** (1998), 377-385.

[30] S. S. Dragomir, J. Pečarić and L. Persson, Some inequalities of Hadamard type. *Soochow J. Math.* **21** (1995), no. 3, 335–341.

[31] S. S. Dragomir and Th. M. Rassias (Eds), *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publisher, 2002.

[32] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_1-norm and applications to some special means and to some numerical quadrature rules, *Tamkang J. of Math.*, **28** (1997), 239-244.

[33] S. S. Dragomir and S. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Appl. Math. Lett.*, **11** (1998), 105-109.

[34] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_p-norm and applications to some special means and to some numerical quadrature rules, *Indian J. of Math.,* **40**(3) (1998), 245-304.

[35] A. El Farissi, Simple proof and refeinment of Hermite-Hadamard inequality, *J. Math. Ineq.* **4** (2010), No. 3, 365–369.

[36] E. K. Godunova and V. I. Levin, Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions. (Russian) *Numerical mathematics and mathematical physics* (Russian), 138–142, 166, Moskov. Gos. Ped. Inst., Moscow, 1985

[37] H. Hudzik and L. Maligranda, Some remarks on s-convex functions. *Aequationes Math.* **48** (1994), no. 1, 100–111.

[38] E. Kikianty and S. S. Dragomir, Hermite-Hadamard's inequality and the p-HH-norm on the Cartesian product of two copies of a normed space, *Math. Inequal. Appl.* (in press)

[39] U. S. Kirmaci, M. Klaričić Bakula, M. E Özdemir and J. Pečarić, Hadamard-type inequalities for s-convex functions. *Appl. Math. Comput.* **193** (2007), no. 1, 26–35.

[40] M. A. Latif, On some inequalities for h-convex functions. *Int. J. Math. Anal.* (Ruse) **4** (2010), no. 29-32, 1473–1482.

[41] D. S. Mitrinović and I. B. Lacković, Hermite and convexity, *Aequationes Math.* **28** (1985), 229–232.

[42] D. S. Mitrinović and J. E. Pečarić, Note on a class of functions of Godunova and Levin. *C. R. Math. Rep. Acad. Sci. Canada* **12** (1990), no. 1, 33–36.

[43] C. E. M. Pearce and A. M. Rubinov, P-functions, quasi-convex functions, and Hadamard-type inequalities. *J. Math. Anal. Appl.* **240** (1999), no. 1, 92–104.

[44] J. E. Pečarić and S. S. Dragomir, On an inequality of Godunova-Levin and some refinements of Jensen integral inequality. *Itinerant Seminar on Functional Equations, Approximation and Convexity* (Cluj-Napoca, 1989), 263–268, Preprint, 89-6, Univ. "Babeş-Bolyai", Cluj-Napoca, 1989.

[45] J. Pečarić and S. S. Dragomir, A generalization of Hadamard's inequality for isotonic linear functionals, *Radovi Mat.* (Sarajevo) **7** (1991), 103–107.

[46] M. Radulescu, S. Radulescu and P. Alexandrescu, On the Godunova-Levin-Schur class of functions. *Math. Inequal. Appl.* **12** (2009), no. 4, 853–862.

[47] M. Z. Sarikaya, A. Saglam, and H. Yildirim, On some Hadamard-type inequalities for h-convex functions. *J. Math. Inequal.* **2** (2008), no. 3, 335–341.

[48] E. Set, M. E. Özdemir and M. Z. Sarıkaya, New inequalities of Ostrowski's type for s-convex functions in the second sense with applications. *Facta Univ. Ser. Math. Inform.* **27** (2012), no. 1, 67–82.

[49] M. Z. Sarikaya, E. Set and M. E. Özdemir, On some new inequalities of Hadamard type involving h-convex functions. *Acta Math. Univ. Comenian.* (N.S.) **79** (2010), no. 2, 265–272.

[50] M. Tunç, Ostrowski-type inequalities via h-convex functions with applications to special means. *J. Inequal. Appl.* **2013**, 2013:326.

[51] S. Varošanec, On h-convexity. *J. Math. Anal. Appl.* **326** (2007), no. 1, 303–311.

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