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ON OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS OF TWO VARIABLES WITH BOUNDED VARIATION

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Abstract. In this paper, we establish a new generalization of Ostrowski type inequalities for functions of two independent variables with bounded variation and apply it for quadrature formulae. Some connections with the rectangle, the midpoint and Simpson's rule are also given.

1. Introduction

Let $f:[a,b]\rightarrow\mathbb{R}$ be a differentiable mapping on (a,b) whoose derivative $f':(a,b)\rightarrow\mathbb{R}$ is baunded on (a,b), i.e. $\|f'\|_{\infty}:=\sup\limits_{t\in(a,b)}|f'(t)|<\infty$. Then we have the inequality

$$\left|f(x)-\frac{1}{b-a}\int\limits_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_{\infty}, \tag{1.1}$$

for all $x\in[a,b]$[19]. The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality.*

In [11], Dragomir proved following Ostrowski type inequalities related functions of bounded variation:

Theorem 1. *Let $f:[a,b]\rightarrow\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left|\int\limits_a^b f(t)dt-(b-a)f(x)\right|\leq\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f)$$

holds for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best possible.

2. Preliminaries and Lemmas

In 1910, Fréchet [16] has given the following characterization for the double Riemann-Stieltjes integral. Assume that $f(x,y)$ and $\alpha(x,y)$ are defined over the rectangle $Q=[a,b]\times[c,d]$; let R be the divided into rectangular subdivisions, or cells, by the net of straight lines $x=x_i$, $y=y_i$,

$$a=x_0<x_1<...<x_n=b,\text{ and } c=y_0<y_1<...<y_m=d;$$

let ξ_i,η_j be any numbers satisfying $\xi_i\in[x_{i-1},x_i]$, $\eta_j\in[y_{j-1},y_j]$, $(i=1,2,...,n;$ $j=1,2,...,m)$; and for all i,j let

$$\Delta_{11}\alpha(x_i,y_j)=\alpha(x_{i-1},y_{j-1})-\alpha(x_{i-1},y_j)-\alpha(x_i,y_{j-1})+\alpha(x_i,y_j).$$

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Then if the sum

$$S=\sum_{i=1}^{n}\sum_{j=1}^{m}f\left(\xi_i,\eta_j\right)\Delta_{11}\alpha(x_i,y_j)$$

tends to a finite limit as the norm of the subdivisions approaches zero, the integral of f with respect to α is said to exist. We call this limit the restricted integral, and designate it by the symbol

$$\int_a^b\int_c^d f(x,y)d_yd_x\alpha(x,y). \tag{2.1}$$

If in the above formulation S is replaced by the sum

$$S^*=\sum_{i=1}^{n}\sum_{j=1}^{m}f\left(\xi_{ij},\eta_{ij}\right)\Delta_{11}\alpha(x_i,y_j),$$

where ξ_{ij},η_{ij} are numbers satisfying $\xi_{ij}\in[x_{i-1},x_i]$, $\eta_{ij}\in[y_{j-1},y_j]$, we call the limit, when it exist, the unrestricted integral, and designate it by the symbol

$$\int_a^b\int_c^d f(x,y)d_yd_x\alpha(x,y). \tag{2.2}$$

Clearly, the existence of (2.2) implies both the existence of (2.1) and its equality (2.2). On the other hand, Clarkson ([8]) has shown that the existence of (2.1) does not imply the existence of (2.2).

In [7], Clarkson and Adams gave the following definitions of bounded variation for functions of two variables:

2.1. **Definitions.** The function $f(x,y)$ is assumed to be defined in rectangle $R(a\leq x\leq b,\ c\leq y\leq d)$. By the term *net* we shall, unless otherwise specified mean a set of parallels to the axes:

$$\begin{aligned} x &= x_i(i=0,1,2,...,m),\ a=x_0<x_1<...<x_m=b;\\ y &= y_j(j=0,1,2,...,n),\ c=y_0<y_1<...<y_n=d. \end{aligned}$$

Each of the smaller rectangles into which R is devided by a net will be called a *cell.* We employ the notation

$$\Delta_{11}f(x_i,y_j)=f(x_{i+1},y_{j+1})-f(x_{i+1},y_j)-f(x_i,y_{j+1})+f(x_i,y_j),$$
$$\Delta f(x_i,y_j)=f(x_{i+1},y_{j+1})-f(x_i,y_j).$$

The total variation function, $\phi(\overline{x})$ $[\psi(\overline{y})]$, is defined as the total variation of $f(\overline{x},y)$ $[f(x,\overline{y})]$ considered as a function of y $[x]$ alone in interval (c,d) $[(a,b)]$, or as $+\infty$ if $f(\overline{x},y)$ $[f(x,\overline{y})]$ is of unbounded variation.

Definition 1. *(Vitali-Lebesque-Fréchet-de la Vallée Poussin). The function $f(x,y)$ is said tobe of bounded variation if the sum*

$$\sum_{i=0\ ,\ j=0}^{m-1\ ,\ n-1}\left|\Delta_{11}f(x_i,y_j)\right|$$

is bounded for all nets.

Definition 2. *(Fréchet). The function $f(x,y)$ is said tobe of bounded variation if the sum*

$$\sum_{i=0\ ,\ j=0}^{m-1\ ,\ n-1} \epsilon_i \overline{\epsilon_j} \left| \Delta_{11} f(x_i, y_j) \right|$$

is bounded for all nets and all possible choices of $\epsilon_i = \pm 1$ and $\overline{\epsilon_j} = \pm 1$.

Definition 3. *(Hardy-Krause). The function $f(x,y)$ is said tobe of bounded variation if it satisfies the condition of Definition 1 and if in addition $f(\overline{x}, y)$ is of bounded variation in y (i.e. $\phi(\overline{x})$ is finite) for at least one $\overline{x}$ and $f(x, \overline{y})$ is of bounded variation in y (i.e. $\psi(\overline{y})$ is finite) for at least one $\overline{y}$.*

Definition 4. *(Arzelà). Let (x_i, y_i) $(i = 0, 1, 2, ..., m)$ be any set of points satisfiying the conditions*

$$\begin{aligned} a &= x_0 < x_1 < ... < x_m = b; \\ c &= y_0 < y_1 < ... < y_m = d. \end{aligned}$$

Then $f(x,y)$ is said tobe of bounded variation if the sum

$$\sum_{i=1}^{m} \left| \Delta f(x_i, y_i) \right|$$

is bounded for all such sets of points.

Therefore, one can define the consept of total variation of a function of variables, as follows:

Let f be of bounded variation on $Q = [a,b] \times [c,d]$, and let $\sum(P)$ denote the sum $\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{m} \left| \Delta_{11} f(x_i, y_j) \right|$ corresponding to the partition P of Q. The number

$$\bigvee_Q (f) := \bigvee_c^d \bigvee_a^b (f) := \sup \left\{ \sum (P) : P \in P(Q) \right\},$$

is called the total variation of f on Q. Here $P([a,b])$ denotes the family of partitions of $[a,b]$.

In [17], authors proved foolowing Lemmas related double Riemann-Stieltjes integral:

Lemma 1. *(Integrating by parts) If $f \in RS(\alpha)$ on Q, then $\alpha \in RS(f)$ on Q, and we have*

$$\begin{aligned} &\int_c^d \int_a^b f(t,s) d_t d_s \alpha(t,s) + \int_c^d \int_a^b \alpha(t,s) d_t d_s f(t,s) \qquad (2.3) \\ &= f(b,d)\alpha(b,d) - f(b,c)\alpha(b,c) - f(a,d)\alpha(a,d) + f(a,c)\alpha(a,c). \end{aligned}$$

Lemma 2. *Assume that $g \in RS(\alpha)$ on Q and α is of bounded variation on Q, then*

$$\left| \int_c^d \int_a^b g(x,y) d_x d_y \alpha(x,y) \right| \leq \sup_{(x,y) \in Q} |g(x,y)| \bigvee_Q (\alpha). \qquad (2.4)$$

In [17], Jawarneh and Noorani obtained following Ostrowski type inequality for functions of two variables with bounded variation:

Theorem 2. *Let $f : Q \to\to \mathbb{R}$ be mapping of bounded variation on Q. Then for all $(x, y) \in Q$, we have inequality*

$$\left|(b-a)(d-c)f(x,y)-\int_c^d\int_a^b f(t,s)dtds\right| \leq \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right] \times\left[\frac{1}{2}(d-c)+\left|y-\frac{c+d}{2}\right|\right]\bigvee_Q(f) \tag{2.5}$$

where $\bigvee_Q(f)$ denotes the total (double) variation of f on Q.

For more information and recent developments on inequalities for mappings of bounded variation, please refer to([1]-[6],[9]-[15],[17],[18],[20]-[24]).

The aim of this paper is to establish a new generzlization of Ostrowski type inequalities for functions of two independent variables with bounded variation and apply it for quadrature formulae. Some connections with the rectangle, the mid-point and Simpson's rule are also given.

3. Main Results

First, we give the following notations used in main our Theorem;

Let

$$\Delta_{n,m} := \{(x_0,y_0),(x_0,y_1),...,(x_0,y_m),(x_1,y_0),...,(x_1,y_m),...,(x_n,y_0),(x_n,y_1),...,(x_n,y_m)\}$$

is a partition of $Q = [a, b] \times [c, d]$ satisfaying $a = x_0$, $b = x_n$, $y_0 = c$, $y_m = d$ with $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1, ..., n)$, $\alpha_{n+1} = b$ and $\beta_0 = c$, $\beta_j \in [y_{j-1}, y_j]$ $(j = 1, ..., m)$, $\beta_{m+1} = d$.

$$\upsilon(h) := \max\{h_i|\ i = 0, ..., n-1\},\ h_i := x_{i+1} - x_i,$$
$$\upsilon(l) := \max\{l_j|\ j = 0, ..., m-1\},\ l_j := y_{j+1} - y_j.$$

Theorem 3. *If $f : Q \to \mathbb{R}$ is of bounded variatin on Q, then we have the inequality*

$$\left|\sum_{i=0}^{n}\sum_{j=0}^{m}(\alpha_{i+1}-\alpha_i)(\beta_{j+1}-\beta_j)f(x_i,y_j)-\int_a^b\int_c^d f(t,s)dsdt\right| \tag{3.1}$$
$$\leq \left[\frac{1}{2}\upsilon(h)+\max_{i\in\{0,\dots,n-1\}}\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\right]$$
$$\times\left[\frac{1}{2}\upsilon(l)+\max_{j\in\{0,\dots,m-1\}}\left|\beta_{j+1}-\frac{y_j+y_{j+1}}{2}\right|\right]\bigvee_a^b\bigvee_c^d(f)$$
$$\leq \upsilon(h)\upsilon(l)\bigvee_a^b\bigvee_c^d(f)$$

where $\bigvee_a^b\bigvee_c^d(f)$ is the total variation of f on Q.

Proof. Let us consider the mappings K and L given by

$$K(t)=\begin{cases} t-\alpha_1, & t\in[a,x_1) \\ t-\alpha_2, & t\in[x_1,x_2) \\ \vdots & \\ t-\alpha_{n-1}, & t\in[x_{n-2},x_{n-1}) \\ t-\alpha_n, & t\in[x_{n-1},b] \end{cases}, \quad L(s)=\begin{cases} s-\beta_1, & s\in[c,y_1) \\ s-\beta_2, & s\in[y_1,y_2) \\ \vdots & \\ s-\beta_{m-1}, & s\in[y_{m-2},y_{m-1}) \\ s-\beta_m, & s\in[y_{m-1},d]. \end{cases}$$

Integrating by parts using Lemma 1, we obtain

$$\begin{aligned}
\int_a^b\int_c^d K(t)L(s)d_sd_tf(t,s) &= \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} K(t)L(s)d_sd_tf(t,s)\right] \\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} (t-\alpha_{i+1})\left(s-\beta_{j+1}\right)d_sd_tf(t,s)\right] \\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\Big[(x_{i+1}-\alpha_{i+1})\left(y_{j+1}-\beta_{j+1}\right)f(x_{i+1},y_{j+1}) \\
&\quad -(x_{i+1}-\alpha_{i+1})\left(y_j-\beta_{j+1}\right)f(x_{i+1},y_j) \\
&\quad -(x_i-\alpha_{i+1})\left(y_{j+1}-\beta_{j+1}\right)f(x_i,y_{j+1}) \\
&\quad +(x_i-\alpha_{i+1})\left(y_j-\beta_{j+1}\right)f(x_i,y_j)-\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} f(t,s)dsdt\Big] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}(x_i-\alpha_i)\left(y_j-\beta_j\right)f(x_i,y_j) \\
&\quad -\sum_{i=1}^{n}\sum_{j=0}^{m-1}(x_i-\alpha_i)\left(y_j-\beta_{j+1}\right)f(x_i,y_j) \\
&\quad -\sum_{i=0}^{n-1}\sum_{j=1}^{m}(x_i-\alpha_{i+1})\left(y_j-\beta_j\right)f(x_i,y_j) \\
&\quad +\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(x_i-\alpha_{i+1})\left(y_j-\beta_{j+1}\right)f(x_i,y_j)-\int_a^b\int_c^d f(t,s)dsdt.
\end{aligned} \tag{3.2}$$

In last equality, we have

(3.3)
$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{m}(x_i-\alpha_i)\left(y_j-\beta_j\right)f(x_i,y_j)\\
=\;&(b-\alpha_n)(d-\beta_m)f(b,d)+(b-\alpha_n)\sum_{j=1}^{m-1}\left(y_j-\beta_j\right)f(b,y_j)\\
&+(d-\beta_m)\sum_{i=1}^{n-1}(x_i-\alpha_i)f(x_i,d)+\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}(x_i-\alpha_i)\left(y_j-\beta_j\right)f(x_i,y_j).
\end{aligned}$$

Similarly, we have

(3.4)
$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=0}^{m-1}(x_i-\alpha_i)\left(y_j-\beta_{j+1}\right)f(x_i,y_j)\\
=\;&(b-\alpha_n)(c-\beta_1)f(b,c)+(b-\alpha_n)\sum_{j=1}^{m-1}\left(y_j-\beta_{j+1}\right)f(b,y_j)\\
&+(c-\beta_1)\sum_{i=1}^{n-1}(x_i-\alpha_i)f(x_i,c)+\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}(x_i-\alpha_i)\left(y_j-\beta_{j+1}\right)f(x_i,y_j),
\end{aligned}$$

(3.5)
$$\begin{aligned}
&\sum_{i=0}^{n-1}\sum_{j=1}^{m}(x_i-\alpha_{i+1})\left(y_j-\beta_j\right)f(x_i,y_j)\\
=\;&(a-\alpha_1)(d-\beta_m)f(a,d)+(a-\alpha_1)\sum_{j=1}^{m-1}\left(y_j-\beta_j\right)f(a,y_j)\\
&+(d-\beta_m)\sum_{i=1}^{n-1}(x_i-\alpha_{i+1})f(x_i,d)+\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}(x_i-\alpha_{i+1})\left(y_j-\beta_j\right)f(x_i,y_j)
\end{aligned}$$

and

(3.6)
$$\begin{aligned}
&\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(x_i-\alpha_{i+1})\left(y_j-\beta_{j+1}\right)f(x_i,y_j)\\
=\;&(a-\alpha_1)(c-\beta_1)f(a,c)+(a-\alpha_1)\sum_{j=1}^{m-1}\left(y_j-\beta_{j+1}\right)f(a,y_j)\\
&+(c-\beta_1)\sum_{i=1}^{n-1}(x_i-\alpha_{i+1})f(x_i,c)+\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}(x_i-\alpha_{i+1})\left(y_j-\beta_{j+1}\right)f(x_i,y_j).
\end{aligned}$$

Adding (3.3)-(3.6) in last equality of (3.2), we obtain

$$
\begin{aligned}
&\int_a^b\int_c^d K(t)L(s)d_sd_tf(t,s)\\
=\;& (b-\alpha_n)(d-\beta_m)f(b,d)+(b-\alpha_n)(\beta_1-c)f(b,c)\\
&+(\alpha_1-a)(d-\beta_m)f(a,d)+(\alpha_1-a)(\beta_1-c)f(a,c)\\
&+(b-\alpha_n)\sum_{j=1}^{m-1}\left(\beta_{j+1}-\beta_j\right)f(b,y_j)+(\alpha_1-a)\sum_{j=1}^{m-1}\left(\beta_{j+1}-\beta_j\right)f(a,y_j)\\
&+(d-\beta_m)\sum_{i=1}^{n-1}(\alpha_{i+1}-\alpha_i)f(x_i,d)+(\beta_1-c)\sum_{i=1}^{n-1}(\alpha_{i+1}-\alpha_i)f(x_i,c)\\
&\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}(\alpha_{i+1}-\alpha_i)\left(\beta_{j+1}-\beta_j\right)f(x_i,y_j)-\int_a^b\int_c^d f(t,s)dsdt\\
=\;& \sum_{i=0}^{n}\sum_{j=0}^{m}(\alpha_{i+1}-\alpha_i)\left(\beta_{j+1}-\beta_j\right)f(x_i,y_j)-\int_a^b\int_c^d f(t,s)dsdt.
\end{aligned}
$$

On the other hand, we have

(3.7)
$$
\begin{aligned}
\left|\int_a^b\int_c^d K(t)L(s)d_sd_tf(t,s)\right| &= \left|\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}K(t)L(s)d_sd_tf(t,s)\right]\right|\\
&\leq \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left|\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}(t-\alpha_{i+1})\left(s-\beta_{j+1}\right)d_sd_tf(t,s)\right|.
\end{aligned}
$$

Using Lemma 2 in the last part of the (3.7), we have

$$\left|\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}(t-\alpha_{i+1})\left(s-\beta_{j+1}\right)d_sd_tf(t,s)\right| \tag{3.8}$$

$$\begin{aligned}
&\leq \sup_{\substack{t\in[x_i,x_{i+1}]\\ s\in[y_j,y_{j+1}]}}\left[\left|t-\alpha_{i+1}\right|\left|s-\beta_{j+1}\right|\right]\bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f)\\
&= \max\left\{\alpha_{i+1}-x_i,x_{i+1}-\alpha_{i+1}\right\}\max\left\{\beta_{j+1}-y_j,y_{j+1}-\beta_{j+1}\right\}\bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f)\\
&= \left[\frac{1}{2}\left(x_{i+1}-x_i\right)+\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\right]\\
&\quad\times\left[\frac{1}{2}\left(y_{j+1}-y_j\right)+\left|\beta_{j+1}-\frac{y_j+y_{j+1}}{2}\right|\right]\bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f).
\end{aligned}$$

Putting (3.8) in (3.7), we obtain

$$\left|\int_a^b\int_c^d K(t)L(s)d_sd_tf(t,s)\right| \tag{3.9}$$

$$\begin{aligned}
&\leq \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[\frac{1}{2}\left(x_{i+1}-x_i\right)+\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\right]\\
&\quad\times\left[\frac{1}{2}\left(y_{j+1}-y_j\right)+\left|\beta_{j+1}-\frac{y_j+y_{j+1}}{2}\right|\right]\bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f)\\
&\leq \max_{i\in[0,\ldots,n-1]}\left[\frac{1}{2}\left(x_{i+1}-x_i\right)+\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\right]\\
&\quad\times\max_{j\in[0,\ldots,m-1]}\left[\frac{1}{2}\left(y_{j+1}-y_j\right)+\left|\beta_{j+1}-\frac{y_j+y_{j+1}}{2}\right|\right]\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[\bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f)\right]\\
&\leq \left[\frac{1}{2}\upsilon(h)+\max_{i\in[0,\ldots,n-1]}\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\right]\\
&\quad\times\left[\frac{1}{2}\upsilon(l)+\max_{j\in[0,\ldots,m-1]}\left|\beta_{j+1}-\frac{y_j+y_{j+1}}{2}\right|\right]\bigvee_a^b\bigvee_c^d(f)
\end{aligned}$$

which completes the proof of first inequality in (3.1).

In last inequality in (3.9), we get

$$\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\leq\frac{1}{2}h_i \text{ and } \max_{i\in[0,\ldots,n-1]}\left|\alpha_{i+1}-\frac{x_i+x_{i+1}}{2}\right|\leq\frac{1}{2}\upsilon(h), \tag{3.10}$$

and similarly,

$$\max_{j\in[0,\ldots,m-1]}\left|\beta_{j+1}-\frac{y_j+y_{j+1}}{2}\right|\le\frac{1}{2}\upsilon(l). \tag{3.11}$$

If we add (3.10) and (3.11) in (3.9), the proof of theorem is completed. □

Now, using the result of the Theorem 3, we give some applications as follows:

Corollary 1. *With the assumptions of Theorem 3. If we choose*

$$\alpha_0=a,\ \alpha_1=\frac{a+x_1}{2},\ \alpha_2=\frac{x_1+x_2}{2},...,\alpha_{n-1}=\frac{x_{n-2}+x_{n-1}}{2},\ \alpha_n=\frac{x_{n-1}+b}{2},\ \alpha_{n+1}=b$$

and

$$\beta_0=c,\ \beta_1=\frac{c+y_1}{2},\ \beta_2=\frac{y_1+y_2}{2},...,\beta_{n-1}=\frac{y_{m-2}+y_{m-1}}{2},\ \beta_n=\frac{y_{m-1}+d}{2},\ \beta_{m+1}=d$$

in Theorem 3, then we have the inequality

$$\begin{aligned}
&\left|\frac{1}{4}\left[(b-x_{n-1})(d-y_{m-1})f(b,d)+(b-x_{n-1})(y_1-c)f(b,c)\right.\right.\\
&+(x_1-a)(d-y_{m-1})f(a,d)+(x_1-a)(y_1-c)f(a,c)\\
&+(b-x_{n-1})\sum_{j=1}^{m-1}(y_{j+1}-y_{j-1})f(b,y_j)+(x_1-a)\sum_{j=1}^{m-1}(y_{j+1}-y_{j-1})f(a,y_j)\\
&+(d-y_{m-1})\sum_{i=1}^{n-1}(x_{i+1}-x_{i-1})f(x_i,d)+(y_1-c)\sum_{i=1}^{n-1}(x_{i+1}-x_{i-1})f(x_i,c)\\
&\left.\left.+\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}(x_{i+1}-x_{i-1})(x_{i+1}-x_{i-1})f(x_i,y_j)\right]-\int_a^b\int_c^d f(t,s)dsdt\right|\\
\le\ &\frac{1}{4}\upsilon(h)\upsilon(l)\bigvee_a^b\bigvee_c^d(f).
\end{aligned}$$

Corollary 2. *In Corollary 1, if we take* $x_i:=a+(b-a)\frac{i}{n}$ $(i=0,1,...,n)$ *and* $y_j:=c+(d-c)\frac{j}{m}$ $(j=0,1,...,m)$, *then we have the inequality*

$$\begin{aligned}
&\left|\frac{(b-a)(d-c)}{4nm}\left[f(b,d)+f(b,c)+f(a,d)+f(a,c)+2\left[\sum_{j=1}^{m-1}f\left(b,\frac{(m-j)c+jd}{m}\right)\right.\right.\right.\\
&\left.+\sum_{j=1}^{m-1}f\left(a,\frac{(m-j)c+jd}{m}\right)+\sum_{i=1}^{n-1}f\left(\frac{(n-i)a+ib}{n},d\right)+\sum_{i=1}^{n-1}f\left(\frac{(n-i)a+ib}{n},c\right)\right]\\
&\left.\left.+4\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}f\left(\frac{(n-i)a+ib}{n},\frac{(m-j)c+jd}{m}\right)\right]-\int_a^b\int_c^d f(t,s)dsdt\right|\\
\le\ &\frac{(b-a)(d-c)}{4nm}\bigvee_a^b\bigvee_c^d(f).
\end{aligned}$$

Corollary 3. *Under assumption Theorem 3, choosing* $x_0 = a$, $x_1 = b$, $\alpha_0 = a$, $\alpha_1 = \alpha$, $\alpha_2 = b$, $y_0 = c$, $y_1 = d$, $\beta_0 = c$, $\beta_1 = \beta$ *and* $\beta_2 = d$, *we obtain the inequality*

$$\begin{aligned} &|(\alpha-a)(\beta-c)f(a,c)+(\alpha-a)(d-\beta)f(a,d) \\ &+(b-\alpha)(\beta-c)f(b,c)+(b-\alpha)(d-\beta)f(b,d)-\int_a^b\int_c^d f(t,s)dsdt\Bigg| \\ \leq\; &\left[\frac{1}{2}(b-a)+\left|\alpha-\frac{a+b}{2}\right|\right]\left[\frac{1}{2}(d-c)+\left|\beta-\frac{c+d}{2}\right|\right]\bigvee_a^b\bigvee_c^d(f). \end{aligned} \tag{3.12}$$

Remark 1. ***a)*** *If we put* $\alpha = b$ *and* $\beta = d$ *in (3.12), then we have the "left rectangle inequality"*

$$\left|(b-a)(d-c)f(a,c)-\int_a^b\int_c^d f(t,s)dsdt\right| \leq (b-a)(d-c)\bigvee_a^b\bigvee_c^d(f),$$

b) *If take* $\alpha = a$ *and* $\beta = c$ *in (3.12), then we have the "right rectangle inequality"*

$$\left|(b-a)(d-c)f(b,d)-\int_a^b\int_c^d f(t,s)dsdt\right| \leq (b-a)(d-c)\bigvee_a^b\bigvee_c^d(f),$$

c) *Similarliy, if we put* $\alpha = \frac{a+b}{2}$ *and* $\beta = \frac{c+d}{2}$ *in (3.12), then we get the "trapezoid inequality"*

$$\begin{aligned} &\left|\frac{f(b,d)+f(b,c)+f(a,d)+f(a,c)}{4}-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right| \\ \leq\; &\frac{1}{4}\bigvee_a^b\bigvee_c^d(f). \end{aligned}$$

Corollary 4. *Under assumption Theorem 3, taking $a \le x_1 \le b$, $a \le \alpha_1 \le x_1 \le \alpha_2 \le b$, $c \le y_1 \le d$, $c \le \beta_1 \le y_1 \le \beta_2 \le d$ we obtain the inequality*

$$
\begin{aligned}
& \left|(\alpha_1 - a)(\beta_1 - c) f(a,c) + (\alpha_1 - a)(\beta_2 - \beta_1) f(a,y_1)\right. \\
& + (\alpha_1 - a)(d - \beta_2) f(a,d) + (\alpha_2 - \alpha_1)(\beta_1 - c) f(x_1,c) \\
& + (\alpha_2 - \alpha_1)(\beta_2 - \beta_1) f(x_1,y_1) + (\alpha_2 - \alpha_1)(d - \beta_2) f(x_1,d) \\
& + (b - \alpha_2)(\beta_1 - c) f(b,c) + (b - \alpha_2)(\beta_2 - \beta_1) f(b,y_1) \\
& \left. + (b - \alpha_2)(d - \beta_2) f(b,d) - \int_a^b \int_c^d f(t,s)\,ds\,dt \right| \\
\le\ & \frac{1}{4}\left[\frac{1}{2}(b-a) + \left|x_1 - \frac{a+b}{2}\right| + \left|\alpha_1 - \frac{a+x_1}{2}\right| + \left|\alpha_2 - \frac{x_1+b}{2}\right|\right. \\
& \left. + \left|\left|\alpha_1 - \frac{a+x_1}{2}\right| - \left|\alpha_2 - \frac{x_1+b}{2}\right|\right|\right] \\
& \times \left[\frac{1}{2}(d-c) + \left|y_1 - \frac{c+d}{2}\right| + \left|\beta_1 - \frac{c+y_1}{2}\right| + \left|\beta_2 - \frac{y_1+d}{2}\right|\right. \\
& \left. + \left|\left|\beta_1 - \frac{c+y_1}{2}\right| - \left|\beta_2 - \frac{y_1+d}{2}\right|\right|\right] \bigvee_a^b \bigvee_c^d (f) \\
\le\ & \left[\frac{1}{2}(b-a) + \left|x_1 - \frac{a+b}{2}\right|\right]\left[\frac{1}{2}(d-c) + \left|y_1 - \frac{c+d}{2}\right|\right] \bigvee_a^b \bigvee_c^d (f) \\
\le\ & (b-a)(d-c) \bigvee_a^b \bigvee_c^d (f).
\end{aligned}
\tag{3.13}
$$

Remark 2. *If we put $\alpha_1 = a$, $\alpha_2 = b$ and $\beta_1 = c$, $\beta_2 = d$ in (3.13), then we get the inequality*

$$
\begin{aligned}
& \left|(b-a)(d-c) f(x_1,y_1) - \int_a^b \int_c^d f(t,s)\,ds\,dt\right| \\
\le\ & \left[\frac{1}{2}(b-a) + \frac{1}{2}\left|x_1 - \frac{a+b}{2}\right|\right]\left[\frac{1}{2}(d-c) + \frac{1}{2}\left|y_1 - \frac{c+d}{2}\right|\right] \bigvee_a^b \bigvee_c^d (f).
\end{aligned}
$$

Particularly, if we take $x_1 = \frac{a+b}{2}$ and $y_1 = \frac{c+d}{2}$,then we get the "midpoint inequality"

$$
\left|(b-a)(d-c) f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \int_a^b \int_c^d f(t,s)\,ds\,dt\right| \le \frac{(b-a)(d-c)}{4} \bigvee_a^b \bigvee_c^d (f)
$$

which is obtained by Jawarneh and Noorani in [17].

Remark 3. *If we choose* $\alpha_1 = \frac{5a+b}{6}$, $\alpha_2 = \frac{a+5b}{6}$, $x_1 \in \left[\frac{5a+b}{6}, \frac{a+5b}{6}\right]$, $\beta_1 = \frac{5c+d}{6}$, $\beta_2 = \frac{c+5d}{6}$ *and* $y_1 \in \left[\frac{5c+d}{6}, \frac{c+5d}{6}\right]$ *in (3.13), then we have the inequality*

$$\begin{aligned}
&\left|\frac{(b-a)(d-c)}{9}\left[\frac{f(b,d)+f(b,c)+f(a,d)+f(a,c)}{4}\right.\right. \\
&\left.\left.+f(a,y_1)+f(x_1,c)+f(b,y_1)+f(x_1,d)+4f(x_1,y_1)\right]-\int_a^b\int_c^d f(t,s)dsdt\right| \\
\leq\ & \frac{1}{4}\left[\frac{1}{2}(b-a)+\frac{1}{2}\left|x_1-\frac{a+b}{2}\right|+\max\left\{\left|x_1-\frac{2a+b}{3}\right|,\left|\frac{a+2b}{3}-x_1\right|\right\}\right] \\
&\times\left[\frac{1}{2}(d-c)+\frac{1}{2}\left|y_1-\frac{c+d}{2}\right|+\max\left\{\left|y_1-\frac{2c+d}{3}\right|,\left|\frac{c+2d}{3}-y_1\right|\right\}\right]\bigvee_a^b\bigvee_c^d(f).
\end{aligned}$$

Particularly, if we chose $x_1 = \frac{a+b}{2}$ *and* $y_1 = \frac{c+d}{2}$*,then we have the "Simpson's rule inequality"*

$$\begin{aligned}
&\left|(b-a)(d-c)\left[\frac{f(b,d)+f(b,c)+f(a,d)+f(a,c)}{36}\right.\right. \\
&+\frac{f\left(a,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)}{9} \\
&\left.\left.+\frac{4}{9}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]-\int_a^b\int_c^d f(t,s)dsdt\right| \\
\leq\ & \frac{(b-a)(d-c)}{9}\bigvee_a^b\bigvee_c^d(f)
\end{aligned}$$

which is proved by Jawarneh and Noorani in [17].

4. Same Composite Qudrature Formula

Let us consider the arbitrary division $I_n : a = x_0 < x_1 < ... < x_n = b$, and $J_m : c = y_0 < y_1 < ... < y_m = d$, $h_i := x_{i+1} - x_i$, and $l_j := y_{j+1} - y_j$,

$$\upsilon(h) := \max\{h_i|\ i = 0, ..., n-1\},$$

$$\upsilon(l) := \max\{l_j|\ j = 0, ..., m-1\}.$$

Then, the following theorem holds.

Theorem 4. *Let* $f : Q \to \mathbb{R}$ *is of bounded variatin on* Q *and* $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$, $\eta_j \in [y_j, y_{j+1}]$ $(j = 0, ..., m-1)$. *Then we have the quadrature* *formula:*

$$\int_a^b\int_c^d f(t,s)dsdt \tag{4.1}$$
$$= \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(\xi_i - x_i)\left(\eta_j - y_j\right)f(x_i,y_j)$$
$$+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(\xi_i - x_i)\left(y_{j+1} - \eta_j\right)f(x_i,y_{j+1})$$
$$+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(x_{i+1} - \xi_i)\left(\eta_j - y_j\right)f(x_{i+1},y_j)$$
$$+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}(x_{i+1} - \xi_i)\left(y_{j+1} - \eta_j\right)f(x_{i+1},y_{j+1}) + R(\xi,\eta,I_n,J_m,f).$$

The remainder $R(\xi,\eta,I_n,J_m,f)$ satisfies

$$|R(\xi,\eta,I_n,J_m,f)| \tag{4.2}$$
$$\leq \left[\frac{1}{2}\upsilon(h) + \max_{i\in\{0,..,n-1\}}\left\{\left|\xi_i - \frac{x_i+x_{i+1}}{2}\right|\right\}\right]$$
$$\times\left[\frac{1}{2}\upsilon(l) + \max_{j\in\{0,..,m-1\}}\left\{\left|\eta_j - \frac{y_j+y_{j+1}}{2}\right|\right\}\right]\bigvee_a^b\bigvee_c^d(f)$$
$$\leq \upsilon(h)\upsilon(l)\bigvee_a^b\bigvee_c^d(f)$$

for all $\xi_i \in [x_i,x_{i+1}]$ $(i=0,...,n-1)$ and $\eta_j \in [y_j,y_{j+1}]$ $(j=0,...,m-1)$.

Proof. Aplying Corollary 3 on the bidimentional interval $[x_i,x_{i+1}]\times[y_j,y_{j+1}]$, we get

$$\Big|(\xi_i - x_i)\left(\eta_j - y_j\right)f(x_i,y_j) \tag{4.3}$$
$$+(\xi_i - x_i)\left(y_{j+1} - \eta_j\right)f(x_i,y_{j+1}) + (x_{i+1} - \xi_i)\left(\eta_j - y_j\right)f(x_{i+1},y_j)$$
$$+(x_{i+1} - \xi_i)\left(y_{j+1} - \eta_j\right)f(x_{i+1},y_{j+1}) - \int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}f(t,s)dsdt\Bigg|$$
$$\leq \left[\frac{1}{2}h_i + \left|\xi_i - \frac{x_i+x_{i+1}}{2}\right|\right]\left[\frac{1}{2}l_j + \left|\eta_j - \frac{y_j+y_{j+1}}{2}\right|\right]\bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f).$$

Summing the inequality (4.3) over i from 0 to $n-1$ and j from 0 to $m-1$, then we get

$$
\begin{aligned}
&|R(\xi,\eta,I_n,J_m,f)| \\
\leq\ & \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[\frac{1}{2}h_i+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]\left[\frac{1}{2}l_j+\left|\eta_j-\frac{y_j+y_{j+1}}{2}\right|\right]\bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f) \\
\leq\ & \max_{i\in\{0,..,n-1\}}\left\{\frac{1}{2}h_i+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right\} \\
&\times\max_{j\in\{0,..,m-1\}}\left\{\frac{1}{2}l_j+\left|\eta_j-\frac{y_j+y_{j+1}}{2}\right|\right\}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f) \\
\leq\ & \left[\frac{1}{2}\upsilon(h)+\max_{i\in\{0,..,n-1\}}\left\{\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right\}\right] \\
&\times\left[\frac{1}{2}\upsilon(l)+\max_{j\in\{0,..,m-1\}}\left\{\left|\eta_j-\frac{y_j+y_{j+1}}{2}\right|\right\}\right]\bigvee_{a}^{b}\bigvee_{c}^{d}(f)
\end{aligned}
\tag{4.4}
$$

which copletes the proof of first inequality in (4.2).

In last inequality

$$
\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\leq\frac{1}{2}h_i \text{ and } \max_{i\in[0,\dots,n-1]}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\leq\frac{1}{2}\upsilon(h),
\tag{4.5}
$$

and similarly,

$$
\max_{j\in[0,\dots,m-1]}\left|\eta_j-\frac{y_j+y_{j+1}}{2}\right|\leq\frac{1}{2}\upsilon(l).
\tag{4.6}
$$

If we add (4.5) and (4.6) in (4.4), we obtain the required result. □

Corollary 5. *Let f, I_n and J_m be as above.*

1) If we choose $\xi_i=x_{i+1}$ and $\eta_j=y_{j+1}$ in (4.1), then we have the "left rectangle rule"

$$
\int_a^b\int_c^d f(t,s)dsdt=\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}f(x_i,y_j)h_il_j+R_L(I_n,J_m,f).
$$

2) If we choose $\xi_i=x_i$ and $\eta_j=y_j$ in (4.1), then we have the "right rectangle rule"

$$
\int_a^b\int_c^d f(t,s)dsdt=\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}f(x_{i+1},y_{j+1})h_il_j+R_R(I_n,J_m,f).
$$

3) Finally, if we choose $\xi_i = \frac{x_i+x_{i+1}}{2}$ *and* $\eta_j = \frac{x_i+x_{i+1}}{2}$ *in (4.1), then we have the "trapezoid rule"*

$$\begin{aligned} &\int_a^b \int_c^d f(t,s)dsdt \\ = \quad &\frac{1}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f(x_i,y_j)+f(x_i,y_{j+1})+f(x_{i+1},y_j)+f(x_{i+1},y_{j+1})\right]h_i l_j + R_T(I_n,J_m,f). \end{aligned}$$

Theorem 5. *Let* f*,* I_n *and* J_m *be as above and* $x_i \le \alpha_i^{(1)} \le \xi_i \le \alpha_i^{(2)} \le x_{i+1}$*,* $y_j \le \beta_j^{(1)} \le \eta_j \le \beta_j^{(2)} \le y_{j+1}$*. Then we have the quadrature formula*

$$\begin{aligned} &\int_a^b \int_c^d f(t,s)dsdt \\ = \quad &\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left(\alpha_i^{(1)}-x_i\right)\left(\beta_j^{(1)}-y_j\right)f(x_i,y_j) \\ &+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left(\alpha_i^{(1)}-x_i\right)\left(\beta_j^{(2)}-\beta_j^{(1)}\right)f(x_i,\eta_j) \\ &+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left(\alpha_i^{(1)}-x_i\right)\left(y_{j+1}-\beta_j^{(2)}\right)f(x_i,y_{j+1}) \\ &+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left(\alpha_i^{(2)}-\alpha_i^{(1)}\right)\left(\beta_j^{(1)}-y_j\right)f(\xi_i,y_j) \\ &+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left(\alpha_i^{(2)}-\alpha_i^{(1)}\right)\left(\beta_j^{(2)}-\beta_j^{(1)}\right)f(\xi_i,\eta_j) \\ &+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left(\alpha_i^{(2)}-\alpha_i^{(1)}\right)\left(y_{j+1}-\beta_j^{(2)}\right)f(\xi_i,y_{j+1}) \\ &+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left(x_{i+1}-\alpha_i^{(2)}\right)\left(\beta_j^{(1)}-y_j\right)f(x_{i+1},y_j) \\ &+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left(x_{i+1}-\alpha_i^{(2)}\right)\left(\beta_j^{(2)}-\beta_j^{(1)}\right)f(x_{i+1},\eta_j) \\ &+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left(x_{i+1}-\alpha_i^{(2)}\right)\left(y_{j+1}-\beta_j^{(2)}\right)f(x_{i+1},y_{j+1}) \\ &+R(\xi,\eta,\alpha_i^{(1)},\alpha_i^{(2)},\beta_j^{(1)},\beta_j^{(2)}I_n,J_m,f). \end{aligned} \tag{4.7}$$

The remainder $R(\xi,\eta,\alpha_i^{(1)},\alpha_i^{(2)},\beta_j^{(1)},\beta_j^{(2)},I_n,J_m,f)$ *satisfies*

$$\begin{aligned}
&\left|R(\xi,\eta,\alpha_i^{(1)},\alpha_i^{(2)},\beta_j^{(1)},\beta_j^{(2)},I_n,J_m,f)\right| \\
\leq\;& \left[\frac{1}{2}\upsilon(h)+\max_{i\in\{0,..,n-1\}}\left\{\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right\}\right] \\
&\times\left[\frac{1}{2}\upsilon(l)+\max_{j\in\{0,..,m-1\}}\left\{\left|\eta_j-\frac{y_j+y_{j+1}}{2}\right|\right\}\right]\bigvee_a^b\bigvee_c^d(f) \\
\leq\;& \upsilon(h)\upsilon(l)\bigvee_a^b\bigvee_c^d(f).
\end{aligned} \tag{4.8}$$

Proof. Aplying Corollary 4 on the bidimentional interval $[x_i,x_{i+1}]\times[y_j,y_{j+1}]$, we have

$$\begin{aligned}
&\Bigg|\left(\alpha_i^{(1)}-x_i\right)\left(\beta_j^{(1)}-y_j\right)f(x_i,y_j)+\left(\alpha_i^{(1)}-x_i\right)\left(\beta_j^{(2)}-\beta_j^{(1)}\right)f(x_i,\eta_j) \\
&+\left(\alpha_i^{(1)}-x_i\right)\left(y_{j+1}-\beta_j^{(2)}\right)f(x_i,y_{j+1})+\left(\alpha_i^{(2)}-\alpha_i^{(1)}\right)\left(\beta_j^{(1)}-y_j\right)f(\xi_i,y_j) \\
&+\left(\alpha_i^{(2)}-\alpha_i^{(1)}\right)\left(\beta_j^{(2)}-\beta_j^{(1)}\right)f(\xi_i,\eta_j)+\left(\alpha_i^{(2)}-\alpha_i^{(1)}\right)\left(y_{j+1}-\beta_j^{(2)}\right)f(\xi_i,y_{j+1}) \\
&+\left(x_{i+1}-\alpha_i^{(2)}\right)\left(\beta_j^{(1)}-y_j\right)f(x_{i+1},y_j)+\left(x_{i+1}-\alpha_i^{(2)}\right)\left(\beta_j^{(2)}-\beta_j^{(1)}\right)f(x_{i+1},\eta_j) \\
&+\left(x_{i+1}-\alpha_i^{(2)}\right)\left(y_{j+1}-\beta_j^{(2)}\right)f(x_{i+1},y_{j+1})-\int_a^b\int_c^d f(t,s)dsdt.\Bigg| \\
\leq\;& \left[\frac{1}{2}h_i+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]\left[\frac{1}{2}l_j+\left|\eta_j-\frac{y_j+y_{j+1}}{2}\right|\right]\bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f).
\end{aligned} \tag{4.9}$$

Summing the inequality (4.9) over i from 0 to $n-1$ and j from 0 to $m-1$, then we get

$$\begin{aligned}
&\left|R(\xi,\eta,\alpha_i^{(1)},\alpha_i^{(2)},\beta_j^{(1)},\beta_j^{(2)},I_n,J_m,f)\right| \\
\leq\ & \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[\frac{1}{2}h_i+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]\left[\frac{1}{2}l_j+\left|\eta_j-\frac{y_j+y_{j+1}}{2}\right|\right]\bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f) \\
\leq\ & \max_{i\in\{0,..,n-1\}}\left\{\frac{1}{2}h_i+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right\} \\
&\times\max_{j\in\{0,..,m-1\}}\left\{\frac{1}{2}l_j+\left|\eta_j-\frac{y_j+y_{j+1}}{2}\right|\right\}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f) \\
\leq\ & \left[\frac{1}{2}\upsilon(h)+\max_{i\in\{0,..,n-1\}}\left\{\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right\}\right] \\
&\times\left[\frac{1}{2}\upsilon(l)+\max_{j\in\{0,..,m-1\}}\left\{\left|\eta_j-\frac{y_j+y_{j+1}}{2}\right|\right\}\right]\bigvee_a^b\bigvee_c^d(f) \\
\leq\ & \upsilon(h)\upsilon(l)\bigvee_a^b\bigvee_c^d(f).
\end{aligned}$$

This completes the proof of Theorem. □

Corollary 6. *Under assumption of Theorem 5 with* $\alpha_i^{(1)}=x_i$, $\alpha_i^{(2)}=x_{i+1}$, $\xi_i=\frac{x_i+x_{i+1}}{2}$, $\beta_j^{(1)}=y_j$ $\beta_j^{(2)}=y_{j+1}$ *and* $\eta_j=\frac{y_j+y_{j+1}}{2}$ *then we have the "midpoint rule"*

$$\int_a^b\int_c^d f(t,s)dsdt=\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}f\left(\frac{x_i+x_{i+1}}{2},\frac{y_j+y_{j+1}}{2}\right)h_il_j+R_M(I_n,J_m,f)$$

where the remainder satisfies

$$|R_M(I_n,J_m,f)|\leq\frac{1}{4}\upsilon(h)\upsilon(l)\bigvee_a^b\bigvee_c^d(f).$$

References

[1] M. W. Alomari, *A Generalization of Weighted Companion of Ostrowski Integral Inequality for Mappings of Bounded Variation*, RGMIA Research Report Collection, 14(2011), Article 87, 11 pp.

[2] M. W. Alomari and M.A. Latif, *Weighted Companion for the Ostrowski and the Generalized Trapezoid Inequalities for Mappings of Bounded Variation,* RGMIA Research Report Collection, 14(2011), Article 92, 10 pp.

[3] M. W. Alomari, *Approximating the Riemann-Stieltjes integral by a three-point quadrature rule and applications*, Konuralp Journal of Mathematics, Volume 2 No.2, pp. 22–34 (2014).

[4] M. W. Alomari and S. S. Dragomir, *Some Grus type inequalities for the Riemann-Stieltjes integral with lipshitzian integrators*, Konuralp Journal of Mathematics,Volume 2 No. 1 pp. 36–44 (2014)

[5] P. Cerone, W.S. Cheung, and S.S. Dragomir, *On Ostrowski type inequalities for Stieltjes integrals with absolutely continuous integrands and integrators of bounded variation,* Computers and Mathematics with Applications 54 (2007) 183–191.

[6] P. Cerone, S. S. Dragomir, and C. E. M. Pearce, *A generalized trapezoid inequality for functions of bounded variation*, Turk J Math, 24 (2000), 147-163.
[7] J.A. Clarkson and C. R. Adams , *On definitions of bounded variation for functions of two variables*, Bull. Amer. Math. Soc. 35 (1933), 824-854.
[8] J.A. Clarkson, *On double Riemann-Stieltjes integrals*, Bull. Amer. Math. Soc. 39 (1933), 929-936.
[9] S. S. Dragomir, *The Ostrowski integral inequality for mappings of bounded variation*, Bull.Austral. Math. Soc., 60(1) (1999), 495-508.
[10] S. S. Dragomir, *On the midpoint quadrature formula for mappings with bounded variation and applications*, Kragujevac J. Math. 22(2000) 13-19.
[11] S. S. Dragomir, *On the Ostrowski's integral inequality for mappings with bounded variation and applications*, Math. Inequal. Appl. 4 (2001), no. 1, 59–66.
[12] S. S. Dragomir, *A companion of Ostrowski's inequality for functions of bounded variation and applications*, Int. J. Nonlinear Anal. Appl. 5 (2014) No. 1, 89-97
[13] S. S. Dragomir, *Refinements of the generalised trapezoid and Ostrowski inequalities for functions of bounded variation*, Arch. Math. (Basel) 91 (2008), no. 5, 450–460.
[14] S.S. Dragomir and E. Momoniat, *A Three Point Quadrature Rule for Functions of Bounded Variation and Applications*, RGMIA Research Report Collection, 14(2011), Article 33, 16 pp.
[15] S. S. Dragomir, *Some perturbed Ostrowski type inequalities for functions of bounded variation*, Preprint RGMIA Res. Rep. Coll. 16 (2013), Art. 93.
[16] Fréchet, M., *Extension au cas des intégrals multiples d'une définition de l'intégrale due á Stieltjes*, Nouvelles Annales de Math ematiques 10 (1910), 241-256.
[17] Y. Jawarneh and M.S.M Noorani, *Inequalities of Ostrowski and Simpson Type for Mappings of Two Variables with Bounded Variation and Applications*, TJMM, 3 (2011), No. 2, 81-94
[18] W. Liu and Y. Sun, *A Refinement of the Companion of Ostrowski inequality for functions of bounded variation and Applications*, arXiv:1207.3861v1, 2012.
[19] A. M. Ostrowski, *Über die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert*, Comment. Math. Helv. 10(1938), 226-227.
[20] K-L Tseng, G-S Yang, and S. S. Dragomir, *Generalizations of Weighted Trapezoidal Inequality for Mappings of Bounded Variation and Their Applications*, Mathematical and Computer Modelling 40 (2004) 77-84.
[21] K-L Tseng, *Improvements of some inequalites of Ostrowski type and their applications*, Taiwan. J. Math. 12 (9) (2008) 2427–2441.
[22] K-L Tseng, S-R Hwang, G-S Yang, and Y-M Chou, *Improvements of the Ostrowski integral inequality for mappings of bounded variation I*, Applied Mathematics and Computation 217 (2010) 2348–2355.
[23] K-L Tseng, S-R Hwang, G-S Yang, and Y-M Chou, *Weighted Ostrowski Integral Inequality for Mappings of Bounded Variation*, Taiwanese J. of Math., Vol. 15, No. 2, pp. 573-585, April 2011.
[24] K-L Tseng, *Improvements of the Ostrowski integral inequality for mappings of bounded variation II*, Applied Mathematics and Computation 218 (2012) 5841–5847.

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