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ON GENERALIZATION OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS OF TWO VARIABLES WITH BOUNDED VARIATION

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Abstract. In this paper, we establish a generalization of the Ostrowski type integral inequalities for functions of two independent variables with bounded variation and we give some applications for general quadrature formulae.

1. Introduction

Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whoose derivative $f':(a,b)\to\mathbb{R}$ is baunded on (a,b), i.e. $\|f'\|_\infty:=\sup_{t\in(a,b)}|f'(t)|<\infty$. Then we have the inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty, \tag{1.1}$$

for all $x\in[a,b]$[18]. The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality.*

In [10], Dragomir proved following Ostrowski type inequalities related functions of bounded variation:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left|\int_a^b f(t)dt-(b-a)f(x)\right|\le\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f)$$

holds for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best possible.

In a recent years, many authors studied the well-known Ostrowski inequality in one variable for variant types of functions such as, Lipschitzian, absolutely continuous and n-differentiable functions as well as the functions of bounded variables. However, a small attention and a few works have been considered for functions of two variables (see, [3], [6], [7], [16]). Among others, in particular, Dragomir and his group studied a very interesting inequalities for functions of one variable. For more information and recent developments on inequalities for mappings of bounded variation, please refer to([1], [2], [5], [8]-[14], [17], [19]-[24]).

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2. Preliminaries and Lemmas

In 1910, Fréchet [15] has given the following characterization for the double Riemann-Stieltjes integral. Assume that $f(x,y)$ and $\alpha(x,y)$ are defined over the rectangle $Q=[a,b]\times[c,d]$; let R be the divided into rectangular subdivisions, or cells, by the net of straight lines $x=x_i$, $y=y_i$,

$$a=x_0<x_1<...<x_n=b, \text{ and } c=y_0<y_1<...<y_m=d;$$

let ξ_i,η_j be any numbers satisfying $\xi_i\in[x_{i-1},x_i]$, $\eta_j\in[y_{j-1},y_j]$, :$(i=1,2,...,n;$ $j=1,2,...,m)$; and for all i,j let

$$\Delta_{11}\alpha(x_i,y_j)=\alpha(x_{i-1},y_{j-1})-\alpha(x_{i-1},y_j)-\alpha(x_i,y_{j-1})+\alpha(x_i,y_j).$$

Then if the sum

$$S=\sum_{i=1}^{n}\sum_{j=1}^{m}f\left(\xi_i,\eta_j\right)\Delta_{11}\alpha(x_i,y_j)$$

tends to a finite limit as the norm of the subdivisions approaches zero, the integral of f with respect to α is said to exist. We call this limit the restricted integral, and designate it by the symbol

$$\int_a^b\int_c^d f(x,y)d_yd_x\alpha(x,y). \tag{2.1}$$

If in the above formulation S is replaced by the sum

$$S^*=\sum_{i=1}^{n}\sum_{j=1}^{m}f\left(\xi_{ij},\eta_{ij}\right)\Delta_{11}\alpha(x_i,y_j),$$

where ξ_{ij},η_{ij} are numbers satisfying $\xi_{ij}\in[x_{i-1},x_i]$, $\eta_{ij}\in[y_{j-1},y_j]$, we call the limit, when it exist, the unrestricted integral, and designate it by the symbol

$$\int_a^b\int_c^d f(x,y)d_yd_x\alpha(x,y). \tag{2.2}$$

Clearly, the existence of (2.2) implies both the existence of (2.1) and its equality (2.2). On the other hand, Clarkson ([7]) has shown that the existence of (2.1) does not imply the existence of (2.2).

In [6], Clarkson and Adams gave the following definitions of bounded variation for functions of two variables:

2.1. **Definitions.** The function $f(x,y)$ is assumed to be defined in rectangle $R(a\leq x\leq b,\ c\leq y\leq d)$. By the term *net* we shall, unless otherwise specified mean a set of parallels to the axes:

$$\begin{aligned} x &= x_i(i=0,1,2,...,m),\ a=x_0<x_1<...<x_m=b; \\ y &= y_j(j=0,1,2,...,n),\ c=y_0<y_1<...<y_n=d. \end{aligned}$$

Each of the smaller rectangles into which R is devided by a net will be called a *cell*. We employ the notation

$$\Delta_{11}f(x_i,y_j)=f(x_{i+1},y_{j+1})-f(x_{i+1},y_j)-f(x_i,y_{j+1})+f(x_i,y_j),$$
$$\Delta f(x_i,y_j)=f(x_{i+1},y_{j+1})-f(x_i,y_j).$$

The total variation function, $\phi(\overline{x})$ $[\psi(\overline{y})]$, is defined as the total variation of $f(\overline{x}, y)$ $[f(x, \overline{y})]$ considered as a function of y $[x]$ alone in interval (c, d) $[(a, b)]$, or as $+\infty$ if $f(\overline{x}, y)$ $[f(x, \overline{y})]$ is of unbounded variation.

Definition 1. *(Vitali-Lebesque-Fréchet-de la Vallée Poussin). The function $f(x, y)$ is said tobe of bounded variation if the sum*

$$\sum_{i=0\ ,\ j=0}^{m-1\ ,\ n-1} |\Delta_{11} f(x_i, y_j)|$$

is bounded for all nets.

Definition 2. *(Fréchet). The function $f(x, y)$ is said tobe of bounded variation if the sum*

$$\sum_{i=0\ ,\ j=0}^{m-1\ ,\ n-1} \epsilon_i \overline{\epsilon_j}\, |\Delta_{11} f(x_i, y_j)|$$

is bounded for all nets and all possible choices of $\epsilon_i = \pm 1$ and $\overline{\epsilon_j} = \pm 1$.

Definition 3. *(Hardy-Krause). The function $f(x, y)$ is said tobe of bounded variation if it satisfies the condition of Definition 1 and if in addition $f(\overline{x}, y)$ is of bounded variation in y (i.e. $\phi(\overline{x})$ is finite) for at least one $\overline{x}$ and $f(x, \overline{y})$ is of bounded variation in y (i.e. $\psi(\overline{y})$ is finite) for at least one $\overline{y}$.*

Definition 4. *(Arzelà). Let (x_i, y_i) $(i = 0, 1, 2, ..., m)$ be any set of points satisfiying the conditions*

$$\begin{aligned} a &= x_0 < x_1 < ... < x_m = b; \\ c &= y_0 < y_1 < ... < y_m = d. \end{aligned}$$

Then $f(x, y)$ is said tobe of bounded variation if the sum

$$\sum_{i=1}^{m} |\Delta f(x_i, y_i)|$$

is bounded for all such sets of points.

Therefore, one can define the consept of total variation of a function of variables, as follows:

Let f be of bounded variation on $Q = [a, b] \times [c, d]$, and let $\sum(P)$ denote the sum $\sum_{i=1}^{n} \sum_{j=1}^{m} |\Delta_{11} f(x_i, y_j)|$ corresponding to the partition P of Q. The number

$$\bigvee_{Q}(f) := \bigvee_{c}^{d} \bigvee_{a}^{b}(f) := \sup\left\{\sum(P) : P \in P(Q)\right\},$$

is called the total variation of f on Q. Here $P([a, b])$ denotes the family of partitions of $[a, b]$.

In [16], authors proved foolowing Lemmas related double Riemann-Stieltjes integral:

Lemma 1. *(Integrating by parts) If $f \in RS(\alpha)$ on Q, then $\alpha \in RS(f)$ on Q, and we have*

$$\int_c^d \int_a^b f(t,s) d_t d_s \alpha(t,s) + \int_c^d \int_a^b \alpha(t,s) d_t d_s f(t,s) \tag{2.3}$$
$$= f(b,d)\alpha(b,d) - f(b,c)\alpha(b,c) - f(a,d)\alpha(a,d) + f(a,c)\alpha(a,c).$$

Lemma 2. *Assume that $g \in RS(\alpha)$ on Q and α is of bounded variation on Q, then*

$$\left| \int_c^d \int_a^b g(x,y) d_x d_y \alpha(x,y) \right| \leq \sup_{(x,y)\in Q} |g(x,y)| \bigvee_Q (\alpha). \tag{2.4}$$

In [16], Jawarneh and Noorani obtained following Ostrowski type inequality for functions of two variables with bounded variation:

Theorem 2. *Let $f : Q \to\to \mathbb{R}$ be mapping of bounded variation on Q. Then for all $(x,y) \in Q$, we have inequality*

$$\left| (b-a)(d-c) f(x,y) - \int_c^d \int_a^b f(t,s) dt ds \right| \leq \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \times \left[\frac{1}{2}(d-c) + \left| y - \frac{c+d}{2} \right| \right] \bigvee_Q (f) \tag{2.5}$$

where $\bigvee_Q (f)$ denotes the total (double) variation of f on Q.

The aim of this paper is to establish a generalization of the Ostrowski type integral inequalities for functions of two independent variables with bounded variation and we give some applications for general quadrature formulae.

3. Main Results

We first prove the following theorem:

Theorem 3. *Let $f : Q \to \mathbb{R}$ be mapping of bounded variation on Q. Then for all $(x,y) \in Q$, we have inequality*

$$\begin{aligned} &\Bigg| (b-a)(d-c) \Bigg[(1-\lambda)(1-\eta) f(x,y) + \frac{(1-\lambda)\eta}{2} [f(a,y) + f(b,y)] \\ &+ \frac{\lambda(1-\eta)}{2} [f(x,c) + f(x,d)] + \frac{\lambda\eta}{4} [f(a,c) + f(a,d) + f(b,c) + f(b,d)] \Bigg] - \int_a^b \int_c^d f(t,s) ds dt \Bigg| \\ &\leq \max \left\{ \lambda \frac{b-a}{2}, \left(x - \frac{(2-\lambda)a + \lambda b}{2} \right), \left(\frac{(2-\lambda)b + \lambda a}{2} - x \right) \right\} \\ &\times \max \left\{ \eta \frac{d-c}{2}, \left(y - \frac{(2-\eta)c + \eta d}{2} \right), \left(\frac{(2-\eta)d + \eta c}{2} - y \right) \right\} \bigvee_a^b \bigvee_c^d (f) \end{aligned} \tag{3.1}$$

for any $\lambda, \eta \in [0,1]$ *and* $a+\lambda\frac{b-a}{2} \le x \le b-\lambda\frac{b-a}{2}$, $c+\eta\frac{d-c}{2} \le y \le d-\eta\frac{d-c}{2}$, *where* $\bigvee_a^b \bigvee_c^b (f)$ *denotes he total variation of* f *on* Q.

Proof. Applying Lemma 1, we have

$$\int_a^x \int_c^y \left(t-\left(a+\lambda\frac{b-a}{2}\right)\right)\left(s-\left(c+\eta\frac{d-c}{2}\right)\right) d_s d_t f(t,s) \tag{3.2}$$
$$= \left(x-a-\lambda\frac{b-a}{2}\right)\left(y-c-\eta\frac{d-c}{2}\right) f(x,y)$$
$$+\left(x-a-\lambda\frac{b-a}{2}\right)\left(\eta\frac{d-c}{2}\right) f(x,c)$$
$$+\left(\lambda\frac{b-a}{2}\right)\left(y-c-\eta\frac{d-c}{2}\right) f(a,y)$$
$$+\left(\lambda\frac{b-a}{2}\right)\left(\eta\frac{d-c}{2}\right) f(a,c) - \int_a^x \int_c^y f(t,s)dsdt,$$

and similarly

$$\int_a^x \int_y^d \left(t-\left(a+\lambda\frac{b-a}{2}\right)\right)\left(s-\left(d-\eta\frac{d-c}{2}\right)\right) d_s d_t f(t,s) \tag{3.3}$$
$$= \left(x-a-\lambda\frac{b-a}{2}\right)\left(\eta\frac{d-c}{2}\right) f(x,d)$$
$$+\left(x-a-\lambda\frac{b-a}{2}\right)\left(d-y-\eta\frac{d-c}{2}\right) f(x,y)$$
$$+\left(\lambda\frac{b-a}{2}\right)\left(\eta\frac{d-c}{2}\right) f(a,d)$$
$$+\left(\lambda\frac{b-a}{2}\right)\left(d-y-\eta\frac{d-c}{2}\right) f(a,y) - \int_a^x \int_y^d f(t,s)dsdt,$$

$$\int_x^b \int_c^y \left(t-\left(b-\lambda\frac{b-a}{2}\right)\right)\left(s-\left(c+\eta\frac{d-c}{2}\right)\right)d_s d_t f(t,s) \tag{3.4}$$
$$= \left(\lambda\frac{b-a}{2}\right)\left(y-c-\eta\frac{d-c}{2}\right)f(b,y)$$
$$+\left(\lambda\frac{b-a}{2}\right)\left(\eta\frac{d-c}{2}\right)f(a,c)$$
$$+\left(b-x-\lambda\frac{b-a}{2}\right)\left(y-c-\eta\frac{d-c}{2}\right)f(x,y)$$
$$+\left(b-x-\lambda\frac{b-a}{2}\right)\left(\eta\frac{d-c}{2}\right)f(x,c)-\int_x^b\int_c^y f(t,s)dsdt,$$

$$\int_x^b \int_y^d \left(t-\left(b-\lambda\frac{b-a}{2}\right)\right)\left(s-\left(d-\eta\frac{d-c}{2}\right)\right)d_s d_t f(t,s) \tag{3.5}$$
$$= \left(\lambda\frac{b-a}{2}\right)\left(\eta\frac{d-c}{2}\right)f(b,d)$$
$$+\left(\lambda\frac{b-a}{2}\right)\left(d-y-\eta\frac{d-c}{2}\right)f(b,y)$$
$$+\left(b-x-\lambda\frac{b-a}{2}\right)\left(\eta\frac{d-c}{2}\right)f(x,d)$$
$$+\left(b-x-\lambda\frac{b-a}{2}\right)\left(d-y-\eta\frac{d-c}{2}\right)f(x,y)-\int_x^b\int_y^d f(t,s)dsdt.$$

Summing (3.2)-(3.5), we have

$$\int_a^b\int_c^d P(x,t;y,s)d_s d_t f(t,s) \tag{3.6}$$
$$= (b-a)(d-c)\left[(1-\lambda)(1-\eta)f(x,y)+\frac{(1-\lambda)\eta}{2}\left[f(a,y)+f(b,y)\right]\right.$$
$$\left.+\frac{\lambda(1-\eta)}{2}\left[f(x,c)+f(x,d)\right]+\frac{\lambda\eta}{4}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right]\right]-\int_a^b\int_c^d f(t,s)dsdt$$

where

$$P(x,t;y,s)=\begin{cases}\left(t-\left(a+\lambda\frac{b-a}{2}\right)\right)\left(s-\left(c+\eta\frac{d-c}{2}\right)\right) & ,(t,s)\in[a,x]\times[c,y]\\ \left(t-\left(a+\lambda\frac{b-a}{2}\right)\right)\left(s-\left(d-\eta\frac{d-c}{2}\right)\right) & ,(t,s)\in[a,x]\times(y,d]\\ \left(t-\left(b-\lambda\frac{b-a}{2}\right)\right)\left(s-\left(c+\eta\frac{d-c}{2}\right)\right) & ,(t,s)\in(x,b]\times[c,y]\\ \left(t-\left(b-\lambda\frac{b-a}{2}\right)\right)\left(s-\left(d-\eta\frac{d-c}{2}\right)\right) & ,(t,s)\in(x,b]\times(y,d]\end{cases}$$

for any $\lambda,\eta\in[0,1]$ and $a+\lambda\frac{b-a}{2}\leq x\leq b-\lambda\frac{b-a}{2}$, $c+\eta\frac{d-c}{2}\leq y\leq d-\eta\frac{d-c}{2}$.

Now, taking the modulus in (3.6), we have

$$\begin{aligned}&\left|\int_a^b\int_c^d P(x,t;y,s)d_sd_tf(t,s)\right|\\ =\ &\left|(b-a)(d-c)\left[(1-\lambda)(1-\eta)f(x,y)+\frac{(1-\lambda)\eta}{2}\left[f(a,y)+f(b,y)\right]\right.\right.\\ &\left.\left.+\frac{\lambda(1-\eta)}{2}\left[f(x,c)+f(x,d)\right]+\frac{\lambda\eta}{4}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right]\right]-\int_a^b\int_c^d f(t,s)dsdt\right|.\end{aligned}$$

On the other hand, using Lemma 2 it follows that

$$\begin{aligned}&\left|\int_a^b\int_c^d P(x,t;y,s)d_sd_tf(t,s)\right|\\ \leq\ &\sup_{(t,s)\in Q}|P(x,y;t,s)|\bigvee_a^b\bigvee_c^d(f)\\ =\ &\max\left\{\max\left\{\lambda\frac{b-a}{2},\left(x-\frac{(2-\lambda)a+\lambda b}{2}\right)\right\}\max\left\{\eta\frac{d-c}{2},\left(y-\frac{(2-\eta)c+\eta d}{2}\right)\right\},\right.\\ &\max\left\{\lambda\frac{b-a}{2},\left(x-\frac{(2-\lambda)a+\lambda b}{2}\right)\right\}\max\left\{\eta\frac{d-c}{2},\left(\frac{(2-\eta)d+\eta c}{2}-y\right)\right\},\\ &\max\left\{\lambda\frac{b-a}{2},\left(\frac{(2-\lambda)b+\lambda a}{2}-x\right)\right\}\max\left\{\eta\frac{d-c}{2},\left(y-\frac{(2-\eta)c+\eta d}{2}\right)\right\},\\ &\left.\max\left\{\lambda\frac{b-a}{2},\left(\frac{(2-\lambda)b+\lambda a}{2}-x\right)\right\}\max\left\{\eta\frac{d-c}{2},\left(\frac{(2-\eta)d+\eta c}{2}-y\right)\right\}\right\}\bigvee_a^b\bigvee_c^d(f)\\ \leq\ &\max\left\{\lambda\frac{b-a}{2},\left(x-\frac{(2-\lambda)a+\lambda b}{2}\right),\left(\frac{(2-\lambda)b+\lambda a}{2}-x\right)\right\}\\ &\times\max\left\{\eta\frac{d-c}{2},\left(y-\frac{(2-\eta)c+\eta d}{2}\right),\left(\frac{(2-\eta)d+\eta c}{2}-y\right)\right\}\bigvee_a^b\bigvee_c^d(f).\end{aligned}$$

This completes the proof of Theorem. $\square$

Remark 1. *Under the assumptions of Theorem 3 with $\lambda = 0$ and $\eta = 0$, the inequality (3.1) reduces inequality (2.5).*

Remark 2. *If we take $\lambda = 1$ and $\eta = 1$ in Theorem 3, we have the trapezoid inequality*

(3.7)
$$\left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt \right| \leq \frac{1}{4} \bigvee_a^b \bigvee_c^d (f)$$

which proved by Jawarneh and Noorani in [16]. *The constant $\frac{1}{4}$ is the best possible.*

Proof. For proof of the sharpness of the constant, assume that (3.7) holds with a constant $A > 0$,that is,

(3.8)
$$\left| \frac{f(b,d)+f(a,d)+f(b,c)+f(a,c)}{4} - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt \right| \leq A \bigvee_a^b \bigvee_c^d (f).$$

If we choose $f : Q \to \mathbb{R}$ with

$$f(x,y) = \begin{cases} 1 \text{ if } & x = a,b \text{ and } y = c,d \\ 0 \text{ if } & (x,y) \in (a,b) \times (c,d) \end{cases}$$

then f is of bounded variation on Q, and

$$\frac{f(b,d)+f(a,d)+f(b,c)+f(a,c)}{4} = 1, \quad \int_a^b \int_c^d f(t,s)dsdt = 0, \text{ and } \bigvee_Q (f) = 4,$$

giving in (3.8), $1 \leq 4A$, thus $A \geq \frac{1}{4}$. □

Corollary 1. *Under the assumptions of Theorem 3 with $\lambda = \frac{1}{3}$ and $\eta = \frac{1}{3}$, we have the inequality*

(3.9)
$$\begin{aligned} & \left| (b-a)(d-c) \left[\frac{4}{9} f(x,y) + \frac{f(a,y)+f(b,y)+f(x,c)+f(x,d)}{9} \right. \right. \\ & \left. \left. + + \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{36} \right] - \int_a^b \int_c^d f(t,s)dsdt \right| \\ \leq \quad & \max \left\{ \frac{b-a}{6}, \left(x - \frac{5a+b}{6} \right), \left(\frac{5b+a}{6} - x \right) \right\} \\ & \times \max \left\{ \frac{d-c}{6}, \left(y - \frac{5c+d}{6} \right), \left(\frac{5d+c}{6} - y \right) \right\} \bigvee_a^b \bigvee_c^d (f) \end{aligned}$$

for $\frac{5a+b}{6} \leq x \leq \frac{5b+a}{6}$ and $\frac{5c+d}{6} \leq y \leq \frac{5d+c}{6}$.

Remark 3. *If we choose $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in Corollary 1, then we have the "Simpson's rule inequality "*

(3.10)

$$\begin{aligned}
&\left|(b-a)(d-c)\left[\frac{f(b,d)+f(b,c)+f(a,d)+f(a,c)}{36}\right.\right. \\
&+\frac{f\left(a,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)}{9} \\
&\left.\left.+\frac{4}{9}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]-\int_a^b\int_c^d f(t,s)dsdt\right| \\
\leq\ & \frac{1}{9}(b-a)(d-c)\bigvee_a^b\bigvee_c^d(f)
\end{aligned}$$

which is proved by Jawarneh and Noorani in [16].

Corollary 2. *Under the assumptions of Theorem 3 with $\lambda = \frac{1}{2}$ and $\eta = \frac{1}{2}$, we have the inequality*

(3.11)

$$\begin{aligned}
&\left|\frac{(b-a)(d-c)}{4}\left[f(x,y)+\frac{f(a,y)+f(b,y)+f(x,c)+f(x,d)}{2}\right.\right. \\
&\left.\left.++\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right]-\int_a^b\int_c^d f(t,s)dsdt\right| \\
\leq\ & \max\left\{\frac{b-a}{4},\left(x-\frac{3a+b}{4}\right),\left(\frac{3b+a}{4}-x\right)\right\} \\
&\times\max\left\{\frac{d-c}{4},\left(y-\frac{3c+d}{4}\right),\left(\frac{3d+c}{4}-y\right)\right\}\bigvee_a^b\bigvee_c^d(f)
\end{aligned}$$

for $\frac{3a+b}{4} \leq x \leq \frac{3b+a}{4}$ and $\frac{3c+d}{4} \leq y \leq \frac{3d+c}{4}$.

Corollary 3. *If we take $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in Corollary 2, then we get*

$$\begin{aligned}
&\left|\frac{(b-a)(d-c)}{4}\left[\frac{f(b,d)+f(b,c)+f(a,d)+f(a,c)}{4}\right.\right. \\
&+\frac{f\left(a,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},c\right)+f\left(b,\frac{c+d}{2}\right)+f\left(\frac{a+b}{2},d\right)}{2} \\
&\left.\left.+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]-\int_a^b\int_c^d f(t,s)dsdt\right| \\
\leq\ & \frac{(b-a)(d-c)}{16}\bigvee_a^b\bigvee_c^d(f).
\end{aligned}$$

4. Same Composite Qudrature Formula

Let us consider the arbitrary division $I_n : a = x_0 < x_1 < ... < x_n = b$, and $J_m : c = y_0 < y_1 < ... < y_m = d$, $h_i := x_{i+1} - x_i$, and $l_j := y_{j+1} - y_j$,

$$\upsilon(h) := \max\left\{ h_i | \ i = 0, ..., n-1 \right\},$$

$$\upsilon(l) := \max\left\{ l_j | \ j = 0, ..., m-1 \right\}.$$

Then the following Theorem holds.

Theorem 4. *Let $f : Q \to \mathbb{R}$ is of bounded variatin on Q and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$, $\tau_j \in [y_j, y_{j+1}]$ $(j = 0, ..., m-1)$. Then we have the quadrature formula:*

$$\begin{aligned}
\int_a^b \int_c^d f(t,s)dsdt &= \frac{4}{9}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f(\xi_i, \tau_j)h_i l_j \\
&+\frac{1}{9}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} \left[f(x_i,\tau_j) + f(x_{i+1},\tau_j) + f(\xi_i, y_j) + f(\xi_i, y_{j+1})\right] h_i l_j \\
&+\frac{1}{36}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} \left[f(x_i,y_j) + f(x_i,y_{j+1}) + f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1})\right] h_i l_j \\
&+R(I_n, J_m, \xi, \tau, f)
\end{aligned}$$

The remainder $R(I_n, J_m, \xi, \eta, f)$ satisfies

$$\begin{aligned}
|R(I_n, J_m, \xi, \eta, f)| &\le \max_{i\in\{0,\dots,n-1\}}\left\{\max\left\{\frac{h_i}{6}, \left(\xi_i - \frac{5x_i + x_{i+1}}{6}\right), \left(\frac{5x_{i+1}+x_i}{6} - \xi_i\right)\right\}\right\} \\
&\times \max_{j\in\{j=0,\dots,m-1\}}\left\{\max\left\{\frac{l_j}{6}, \left(\tau_j - \frac{5y_j + y_{j+1}}{6}\right), \left(\frac{5y_{j+1}+y_j}{6} - \tau_j\right)\right\}\right\} \\
&\times \bigvee_a^b \bigvee_c^d (f).
\end{aligned}$$

Proof. Applying Corollary 1 to bidimentional interval $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$, $\tau_j \in [y_j, y_{j+1}]$ $(j = 0, ..., m-1)$, we have the inequality

$$
\begin{aligned}
&\left|(b-a)(d-c)\left[\frac{4}{9}f(\xi_i, y) + \frac{f(x_i,\tau_j) + f(x_{i+1},\tau_j) + f(\xi_i, y_j) + f(\xi_i, y_{j+1})}{9}\right.\right. \\
&\left.\left. + + \frac{f(x_i,y_j) + f(x_i,y_{j+1}) + f(x_{i+1},y_j) + f(x_{i+1},y_{j+1})}{36}\right] - \int_a^b\int_c^d f(t,s)dsdt\right| \\
\leq\ & \max\left\{\frac{h_i}{6}, \left(\xi_i - \frac{5x_i + x_{i+1}}{6}\right), \left(\frac{5x_{i+1} + x_i}{6} - \xi_i\right)\right\} \\
&\times \max\left\{\frac{l_j}{6}, \left(\tau_j - \frac{5y_j + y_{j+1}}{6}\right), \left(\frac{5y_{j+1} + y_j}{6} - \tau_j\right)\right\} \bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f).
\end{aligned} \tag{4.1}
$$

Summing the inequality (4.1) over i from 0 to $n-1$ and j from 0 to $m-1$, then we get

$$
\begin{aligned}
|R(I_n, J_m, \xi, \tau, f)| \leq\ & \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[\max\left\{\frac{h_i}{6}, \left(\xi_i - \frac{5x_i + x_{i+1}}{6}\right), \left(\frac{5x_{i+1} + x_i}{6} - \xi_i\right)\right\}\right. \\
&\left.\times \max\left\{\frac{l_j}{6}, \left(\tau_j - \frac{5y_j + y_{j+1}}{6}\right), \left(\frac{5y_{j+1} + y_j}{6} - \tau_j\right)\right\} \bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f)\right] \\
\leq\ & \max_{i\in\{0,\ldots,n-1\}}\left\{\max\left\{\frac{h_i}{6}, \left(\xi_i - \frac{5x_i + x_{i+1}}{6}\right), \left(\frac{5x_{i+1} + x_i}{6} - \xi_i\right)\right\}\right\} \\
&\times \max_{j\in\{j=0,\ldots,m-1\}}\left\{\max\left\{\frac{l_j}{6}, \left(\tau_j - \frac{5y_j + y_{j+1}}{6}\right), \left(\frac{5y_{j+1} + y_j}{6} - \tau_j\right)\right\}\right\} \\
&\times \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f) \\
=\ & \max_{i\in\{0,\ldots,n-1\}}\left\{\max\left\{\frac{h_i}{6}, \left(\xi_i - \frac{5x_i + x_{i+1}}{6}\right), \left(\frac{5x_{i+1} + x_i}{6} - \xi_i\right)\right\}\right\} \\
&\times \max_{j\in\{j=0,\ldots,m-1\}}\left\{\max\left\{\frac{l_j}{6}, \left(\tau_j - \frac{5y_j + y_{j+1}}{6}\right), \left(\frac{5y_{j+1} + y_j}{6} - \tau_j\right)\right\}\right\} \\
&\times \bigvee_a^b\bigvee_c^d(f)
\end{aligned}
$$

which is the required result. $\square$

Corollary 4. *Let I_n, J_m and f be as above. If we choose $\xi_i = \frac{x_i+x_{i+1}}{2}$ and $\tau_j = \frac{y_j+y_{j+1}}{2}$ in Theorem 4, then we have the "Simpson's rule"*

$$\begin{aligned}
&\int_a^b \int_c^d f(t,s)dsdt \\
= \quad & \frac{4}{9}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f\left(\frac{x_i+x_{i+1}}{2}, \frac{y_j+y_{j+1}}{2}\right) h_i l_j \\
&+\frac{1}{9}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f\left(\frac{x_i+x_{i+1}}{2}, y_j\right)+f\left(\frac{x_i+x_{i+1}}{2}, y_{j+1}\right)\right] h_i l_j \\
&+\frac{1}{9}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f\left(x_i, \frac{y_j+y_{j+1}}{2}\right)+f\left(x_{i+1}, \frac{y_j+y_{j+1}}{2}\right)\right] h_i l_j \\
&+\frac{1}{36}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left[f(x_i,y_j)+f(x_i,y_{j+1})+f(x_{i+1},y_j)+f(x_{i+1},y_{j+1})\right] h_i l_j \\
&+R_S(I_n, J_m, f).
\end{aligned}$$

The remainder $R_S(I_n, J_m, f)$ satisfies

$$|R_S(I_n, J_m, f)| \leq \frac{1}{9}\upsilon(h)\upsilon(l)\bigvee_a^b\bigvee_c^d(f).$$

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