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A COMPANION OF OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS OF TWO VARIABLES WITH BOUNDED VARIATION

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Abstract. In this paper, a companion of Ostrowski type inequalities for functions of two independent variables with bounded variation is obtained and some sharp inequalities are proved. Application to quadrature rule is also provided.

1. Introduction

Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whoose derivative $f':(a,b)\to\mathbb{R}$ is baunded on (a,b), i.e. $\|f'\|_\infty:=\sup_{t\in(a,b)}|f'(t)|<\infty$. Then we have the inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty, \tag{1.1}$$

for all $x\in[a,b]$[24]. The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality.*

In [16], Dragomir proved following Ostrowski type inequalities related functions of bounded variation:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left|\int_a^b f(t)dt-(b-a)f(x)\right|\leq\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(f)$$

holds for all $x\in[a,b]$. The constant $\frac{1}{2}$ is the best possible.

In [17], Dragomir proved following Ostrowski type inequalities related functions of bounded variation:

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Theorem 2. *Assume that the function $f:[a,b]\to R$ is of bounded variation on $[a,b]$. Then we have the inequalities:*

(1.2)

$$\begin{aligned}&\frac{1}{2}\left|f(x)+f(a+b-x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\\ \leq\ &\frac{1}{b-a}\left[(x-a)\bigvee_a^x(f)+\left(\frac{a+b}{2}-x\right)\bigvee_x^{a+b-x}(f)+(x-a)\bigvee_{a+b-x}^b(f)\right]\end{aligned}$$

$$\leq\begin{cases}\left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\bigvee_a^b(f), \\ \left[2\left(\frac{x-a}{b-a}\right)^{\alpha}+\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{\alpha}\right]^{\frac{1}{\alpha}} \\ \times\left[\left[\bigvee_a^x(f)\right]^{\beta}+\left[\bigvee_x^{a+b-x}(f)\right]^{\beta}+\left[\bigvee_{a+b-x}^b(f)\right]^{\beta}\right]^{\frac{1}{\beta}} & , \text{ if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \left[\frac{x-a+\frac{b-a}{2}}{b-a}\right]\max\left\{\bigvee_a^x(f),\bigvee_x^{a+b-x}(f),\bigvee_{a+b-x}^b(f)\right\}\end{cases}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$ where $\bigvee_c^d(f)$ denotes the total variation of f on $[c,d]$. The constant $\frac{1}{4}$ is the best possible in the first branch of second inequality in (1.2).

2. Preliminaries and Lemmas

In 1910, Fréchet [21] has given the following characterization for the double Riemann-Stieltjes integral. Assume that $f(x,y)$ and $\alpha(x,y)$ are defined over the rectangle $Q=[a,b]\times[c,d]$; let R be the divided into rectangular subdivisions, or cells, by the net of straight lines $x=x_i$, $y=y_i$,

$$a=x_0<x_1<...<x_n=b, \text{ and } c=y_0<y_1<...<y_m=d;$$

let ξ_i,η_j be any numbers satisfying $\xi_i\in[x_{i-1},x_i]$, $\eta_j\in[y_{j-1},y_j]$, $(i=1,2,...,n;$ $j=1,2,...,m)$; and for all i,j let

$$\Delta_{11}\alpha(x_i,y_j)=\alpha(x_{i-1},y_{j-1})-\alpha(x_{i-1},y_j)-\alpha(x_i,y_{j-1})+\alpha(x_i,y_j).$$

Then if the sum

$$S=\sum_{i=1}^{n}\sum_{j=1}^{m}f\left(\xi_i,\eta_j\right)\Delta_{11}\alpha(x_i,y_j)$$

tends to a finite limit as the norm of the subdivisions approaches zero, the integral of f with respect to α is said to exist. We call this limit the restricted integral, and designate it by the symbol

$$\int_a^b\int_c^d f(x,y)d_yd_x\alpha(x,y). \tag{2.1}$$

If in the above formulation S is replaced by the sum

$$S^{*}=\sum_{i=1}^{n}\sum_{j=1}^{m}f\left(\xi_{ij},\eta_{ij}\right)\Delta_{11}\alpha(x_{i},y_{j}),$$

where ξ_{ij},η_{ij} are numbers satisfying $\xi_{ij}\in[x_{i-1},x_{i}]$, $\eta_{ij}\in[y_{j-1},y_{j}]$, we call the limit, when it exist, the unrestricted integral, and designate it by the symbol

$$\int_{a}^{b}\int_{c}^{d}f(x,y)d_{y}d_{x}\alpha(x,y). \tag{2.2}$$

Clearly, the existence of (2.2) implies both the existence of (2.1) and its equality (2.2). On the other hand, Clarkson ([13]) has shown that the existence of (2.1) does not imply the existence of (2.2).

In [12], Clarkson and Adams gave the following definitions of bounded variation for functions of two variables:

2.1. **Definitions.** The function $f(x,y)$ is assumed to be defined in rectangle $R(a\leq x\leq b,\ c\leq y\leq d)$. By the term *net* we shall, unless otherwise specified mean a set of parallels to the axes:

$$\begin{aligned} x &= x_{i}(i=0,1,2,...,m),\ a=x_{0}<x_{1}<...<x_{m}=b;\\ y &= y_{j}(j=0,1,2,...,n),\ c=y_{0}<y_{1}<...<y_{n}=d. \end{aligned}$$

Each of the smaller rectangles into which R is devided by a net will be called a *cell*. We employ the notation

$$\Delta_{11}f(x_{i},y_{j})=f(x_{i+1},y_{j+1})-f(x_{i+1},y_{j})-f(x_{i},y_{j+1})+f(x_{i},y_{j}),$$

$$\Delta f(x_{i},y_{j})=f(x_{i+1},y_{j+1})-f(x_{i},y_{j}).$$

The total variation function, $\phi(\overline{x})$ $[\psi(\overline{y})]$, is defined as the total variation of $f(\overline{x},y)$ $[f(x,\overline{y})]$ considered as a function of y $[x]$ alone in interval (c,d) $[(a,b)]$, or as $+\infty$ if $f(\overline{x},y)$ $[f(x,\overline{y})]$ is of unbounded variation.

Definition 1. *(Vitali-Lebesque-Fréchet-de la Vallée Poussin). The function $f(x,y)$ is said tobe of bounded variation if the sum*

$$\sum_{i=0\ ,\ j=0}^{m-1\ ,\ n-1}\left|\Delta_{11}f(x_{i},y_{j})\right|$$

is bounded for all nets.

Definition 2. *(Fréchet). The function $f(x,y)$ is said tobe of bounded variation if the sum*

$$\sum_{i=0\ ,\ j=0}^{m-1\ ,\ n-1}\epsilon_{i}\overline{\epsilon_{j}}\left|\Delta_{11}f(x_{i},y_{j})\right|$$

is bounded for all nets and all possible choices of $\epsilon_{i}=\pm1$ and $\overline{\epsilon_{j}}=\pm1$.

Definition 3. *(Hardy-Krause). The function $f(x,y)$ is said tobe of bounded variation if it satisfies the condition of Definition 1 and if in addition $f(\overline{x},y)$ is of bounded variation in y (i.e. $\phi(\overline{x})$ is finite) for at least one $\overline{x}$ and $f(x,\overline{y})$ is of bounded variation in y (i.e. $\psi(\overline{y})$ is finite) for at least one $\overline{y}$.*

Definition 4. *(Arzelà). Let (x_i, y_i) $(i = 0, 1, 2, ..., m)$ be any set of points satisfiying the conditions*

$$\begin{aligned} a &= x_0 < x_1 < ... < x_m = b; \\ c &= y_0 < y_1 < ... < y_m = d. \end{aligned}$$

Then $f(x, y)$ is said tobe of bounded variation if the sum

$$\sum_{i=1}^{m} |\Delta f(x_i, y_i)|$$

is bounded for all such sets of points.

Therefore, one can define the consept of total variation of a function of variables, as follows:

Let f be of bounded variation on $Q = [a, b] \times [c, d]$, and let $\sum (P)$ denote the sum $\sum_{i=1}^{n} \sum_{j=1}^{m} |\Delta_{11} f(x_i, y_j)|$ corresponding to the partition P of Q. The number

$$\bigvee_{Q} (f) := \bigvee_{c}^{d} \bigvee_{a}^{b} (f) := \sup \left\{ \sum (P) : P \in P(Q) \right\},$$

is called the total variation of f on Q. Here $P([a, b])$ denotes the family of partitions of $[a, b]$.

In [22], authors proved foolowing Lemmas related double Riemann-Stieltjes integral:

Lemma 1. *(Integrating by parts) If $f \in RS(\alpha)$ on Q, then $\alpha \in RS(f)$ on Q, and we have*

$$\begin{aligned} & \int_{c}^{d} \int_{a}^{b} f(t, s) d_t d_s \alpha(t, s) + \int_{c}^{d} \int_{a}^{b} \alpha(t, s) d_t d_s f(t, s) \\ = & \ f(b, d)\alpha(b, d) - f(b, c)\alpha(b, c) - f(a, d)\alpha(a, d) + f(a, c)\alpha(a, c). \end{aligned} \tag{2.3}$$

Lemma 2. *Assume that $g \in RS(\alpha)$ on Q and α is of bounded variation on Q, then*

$$\left| \int_{c}^{d} \int_{a}^{b} g(x, y) d_x d_y \alpha(x, y) \right| \leq \sup_{(x,y) \in Q} |g(x, y)| \bigvee_{Q} (\alpha). \tag{2.4}$$

In [22], Jawarneh and Noorani proved the following Ostrowski type inequality or functions of two variables with bounded variation:

Theorem 3. *Let $f : Q \to\to \mathbb{R}$ be mapping of bounded variation on Q. Then for all $(x, y) \in Q$, we have inequality*

$$\begin{aligned} \left| (b-a)(d-c) f(x, y) - \int_{c}^{d} \int_{a}^{b} f(t, s) dt ds \right| &\leq \left[\frac{1}{2} (b-a) + \left| x - \frac{a+b}{2} \right| \right] \\ &\quad \times \left[\frac{1}{2} (d-c) + \left| y - \frac{c+d}{2} \right| \right] \bigvee_{Q} (f) \end{aligned} \tag{2.5}$$

where $\bigvee_{Q}(f)$ *denotes the total (double) variation of* f *on* Q.

In [7], Budak and Sarıkaya have proved the following genaralization of the inequality (2.5):

Theorem 4. *Let* $f:Q\rightarrow\mathbb{R}$ *be mapping of bounded variation on* Q. *Then for all* $(x,y)\in Q$, *we have inequality*

$$\begin{aligned}&\left|(b-a)(d-c)\left[(1-\lambda)(1-\eta)f(x,y)+\frac{(1-\lambda)\eta}{2}\left[f(a,y)+f(b,y)\right]\right.\right.\\&\left.\left.+\frac{\lambda(1-\eta)}{2}\left[f(x,c)+f(x,d)\right]+\frac{\lambda\eta}{4}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right]\right]-\int_a^b\int_c^d f(t,s)dsdt\right|\\\leq\ &\max\left\{\lambda\frac{b-a}{2},\left(x-\frac{(2-\lambda)a+\lambda b}{2}\right),\left(\frac{(2-\lambda)b+\lambda a}{2}-x\right)\right\}\\&\times\max\left\{\eta\frac{d-c}{2},\left(y-\frac{(2-\eta)c+\eta d}{2}\right),\left(\frac{(2-\eta)d+\eta c}{2}-y\right)\right\}\bigvee_a^b\bigvee_c^d(f)\end{aligned}\tag{2.6}$$

for any $\lambda,\eta\in[0,1]$ *and* $a+\lambda\frac{b-a}{2}\leq x\leq b-\lambda\frac{b-a}{2}$, $c+\eta\frac{d-c}{2}\leq y\leq d-\eta\frac{d-c}{2}$, *where* $\bigvee_a^b\bigvee_c^b(f)$ *denotes he total variation of* f *on* Q.

A companion of Ostrowski type inequalities (1.2) for functions of single variable with bounded variation were given by Dragomir in [17]. Then, in [3], Alomari gave the generalization of inequalities in [17]. In recent works [1, 2], authors obtained the weighted companion of Ostrowski type inequalities. For other results see [4, 5]. Recently, many of inequalities for functions of single variable with bounded variation have been proved. For more information and recent developments on inequalities for mappings of single variable with bounded variation, please refer to ([8],[10],[11],[14],[15],[18]-[20],[23],[25]-[30]). In the literature, there are a few study for functions of two variables with bounded variation(see [6],[7],[9],[22]).

The aim of this paper is to establish new Ostrowski type inequalities for functions of two independent variables with bounded variation similar to inequalities in (1.2).

3. Main Results

First, we give the following notations to simplify presentetion of some intervals.

$$\begin{aligned}Q_1&=[a,x]\times[c,y]\,,\ Q_2=[a,x]\times[y,c+d-y]\,,\ Q_3=[a,x]\times[c+d-y,d]\,,\\Q_4&=[x,a+b-x]\times[c,y]\,,\ Q_5=[x,a+b-x]\times[y,c+d-y]\,,\\Q_6&=[x,a+b-x]\times[c+d-y,d]\,,\ Q_7=[a+b-x,b]\times[c,y]\,,\\Q_8&=[a+b-x,b]\times[y,c+d-y]\,,\ Q_9=[a+b-x,b]\times[c+d-y,d]\,.\end{aligned}$$

To prove our theorem, we need the following lemma:

Lemma 3. *Let $f : Q = [a,b] \times [c,d] \to \mathbb{R}$ is of bounded variatin on Q. Then, we have the following equality*

$$
\begin{aligned}
&\frac{1}{4}\left[f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)\right] \\
&-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \\
=\;& \frac{1}{(b-a)(d-c)}\Bigg[\int_a^x\int_c^y (t-a)(s-c)\,d_sd_tf(t,s)+\int_a^x\int_c^{c+d-y}(t-a)\left(s-\frac{c+d}{2}\right)d_sd_tf(t,s) \\
&+\int_a^x\int_{c+d-y}^d (t-a)(s-d)\,d_sd_tf(t,s)+\int_x^{a+b-x}\int_c^y\left(t-\frac{a+b}{2}\right)(s-c)\,d_sd_tf(t,s) \\
&+\int_x^{a+b-x}\int_y^{c+d-y}\left(t-\frac{a+b}{2}\right)\left(s-\frac{c+d}{2}\right)d_sd_tf(t,s) \\
&+\int_x^{a+b-x}\int_{c+d-y}^d\left(t-\frac{a+b}{2}\right)(s-d)\,d_sd_tf(t,s)+\int_{a+b-x}^b\int_c^y (t-b)(s-c)\,d_sd_tf(t,s) \\
&+\int_{a+b-x}^b\int_y^{c+d-y}(t-b)\left(s-\frac{c+d}{2}\right)d_sd_tf(t,s)+\int_{a+b-x}^b\int_{c+d-y}^d (t-b)(s-d)\,d_sd_tf(t,s)\Bigg]
\end{aligned}
\tag{3.1}
$$

for any $x \in \left[a, \frac{a+b}{2}\right]$ *and* $y \in \left[c, \frac{c+d}{2}\right]$.

Proof. Using Lemma 1, we have

$$\int_a^x\int_c^y (t-a)(s-c)\,d_sd_tf(t,s) = (x-a)(y-c)f(x,y)-\int_a^x\int_c^y f(t,s)dsdt,$$

and similarly,

$$
\begin{aligned}
&\int_a^x\int_c^{c+d-y}(t-a)\left(s-\frac{c+d}{2}\right)d_sd_tf(t,s) \\
=\;&(x-a)\left(\frac{c+d}{2}-y\right)f(x,c+d-y)+(x-a)\left(\frac{c+d}{2}-y\right)f(x,y)-\int_a^x\int_c^{c+d-y}f(t,s)dsdt,
\end{aligned}
$$

$$\int_a^x\int_{c+d-y}^d (t-a)(s-d)\,d_sd_tf(t,s) = (x-a)(y-c)f(x,c+d-y)-\int_a^x\int_{c+d-y}^d f(t,s)dsdt,$$

$$\int_{x}^{a+b-x}\int_{c}^{y}\left(t-\frac{a+b}{2}\right)(s-c)\,d_sd_tf(t,s)$$
$$=\left(\frac{a+b}{2}-x\right)(y-c)\,f(a+b-x,y)+\left(\frac{a+b}{2}-x\right)(y-c)\,f(x,y)-\int_{x}^{a+b-x}\int_{c}^{y}f(t,s)dsdt,$$

$$\int_{x}^{a+b-x}\int_{y}^{c+d-y}\left(t-\frac{a+b}{2}\right)\left(s-\frac{c+d}{2}\right)d_sd_tf(t,s)$$
$$=\left(\frac{a+b}{2}-x\right)\left(\frac{c+d}{2}-y\right)f(a+b-x,c+d-y)$$
$$+\left(\frac{a+b}{2}-x\right)\left(\frac{c+d}{2}-y\right)f(a+b-x,y)$$
$$+\left(\frac{a+b}{2}-x\right)\left(\frac{c+d}{2}-y\right)f(x,c+d-y)$$
$$+\left(\frac{a+b}{2}-x\right)\left(\frac{c+d}{2}-y\right)f(x,y)-\int_{x}^{a+b-x}\int_{y}^{c+d-y}f(t,s)dsdt,$$

$$\int_{x}^{a+b-x}\int_{c+d-y}^{d}\left(t-\frac{a+b}{2}\right)(s-d)\,d_sd_tf(t,s)$$
$$=\left(\frac{a+b}{2}-x\right)(y-c)\,f(a+b-x,c+d-y)$$
$$+\left(\frac{a+b}{2}-x\right)(y-c)\,f(x,c+d-y)-\int_{x}^{a+b-x}\int_{c+d-y}^{d}f(t,s)dsdt,$$

$$\int_{a+b-x}^{b}\int_{c}^{y}(t-b)(s-c)\,d_sd_tf(t,s)=(x-a)(y-c)\,f(a+b-x,y)-\int_{a+b-x}^{d}\int_{c}^{y}f(t,s)dsdt,$$

$$\int_{a+b-x}^{b}\int_{y}^{c+d-y}(t-b)\left(s-\frac{c+d}{2}\right)d_sd_tf(t,s)$$
$$=(x-a)\left(\frac{c+d}{2}-y\right)f(a+b-x,c+d-y)$$
$$+(x-a)\left(\frac{c+d}{2}-y\right)f(a+b-x,y)-\int_{a+b-x}^{d}\int_{y}^{c+d-y}f(t,s)dsdt,$$

$$\int_{a+b-x}^{b}\int_{c+d-y}^{d}(t-b)(s-d)\,d_s d_t f(t,s)$$
$$= (x-a)(y-c)f(a+b-x,c+d-y)-\int_{a+b-x}^{d}\int_{c+d-y}^{d}f(t,s)dsdt.$$

Summing the above equalities we deduce (3.1). □

Theorem 5. *If the function $f:Q=[a,b]\times[c,d]\to R$ is of bounded variation on Q, then we have*

(3.2)
$$\left|\frac{1}{4}\left[f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)\right]\right.$$
$$\left.-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\leq \frac{1}{(b-a)(d-c)}\left[(x-a)(y-c)\bigvee_{Q_1}(f)+(x-a)\left(\frac{c+d}{2}-y\right)\bigvee_{Q_2}(f)\right.$$
$$+(x-a)(y-c)\bigvee_{Q_3}(f)+\left(\frac{a+b}{2}-x\right)(y-c)\bigvee_{Q_4}(f)$$
$$+\left(\frac{a+b}{2}-x\right)\left(\frac{c+d}{2}-y\right)\bigvee_{Q_5}(f)+\left(\frac{a+b}{2}-x\right)(y-c)\bigvee_{Q_6}(f)$$
$$\left.(x-a)(y-c)\bigvee_{Q_7}(f)+(x-a)\left(\frac{c+d}{2}-y\right)\bigvee_{Q_8}(f)+(x-a)(y-c)\bigvee_{Q_9}(f)\right]$$

$$\leq\begin{cases}\left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\left[\frac{1}{4}+\left|\frac{y-\frac{3c+d}{4}}{d-c}\right|\right]\bigvee_{Q}(f), \\ \left[4\left(\frac{x-a}{b-a}\right)^{\alpha}\left(\frac{y-c}{d-c}\right)^{\alpha}+2\left(\frac{x-a}{b-a}\right)^{\alpha}\left(\frac{\frac{c+d}{2}-y}{d-c}\right)^{\alpha}+2\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{\alpha}\left(\frac{y-c}{d-c}\right)^{\alpha}+\left(\frac{\frac{a+b}{2}-x}{b-a}\right)^{\alpha}\left(\frac{\frac{c+d}{2}-y}{d-c}\right)^{\alpha}\right]^{\frac{1}{\alpha}} \\ \times\left[\left[\bigvee_{Q_1}(f)\right]^{\beta}+\left[\bigvee_{Q_2}(f)\right]^{\beta}+...+\left[\bigvee_{Q_9}(f)\right]^{\beta}\right]^{\frac{1}{\beta}},\ \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \left\{\left(\frac{x-a}{b-a}\right)+\left(\frac{y-c}{d-c}\right)+\left(\frac{\frac{a+b}{2}-x}{b-a}\right)\left(\frac{\frac{c+d}{2}-y}{d-c}\right)\right\}\max\left\{\bigvee_{Q_1}(f),\bigvee_{Q_2}(f),...,\bigvee_{Q_9}(f)\right\}\end{cases}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$ *and* $y \in \left[c, \frac{c+d}{2}\right]$, *where* $\bigvee_{Q}(f)$ *denotes the total (double) variation of* f *on* Q.

Proof. Taking modulus in (3.1) and applying Lemma 2 in (3.3), we have

$$\begin{aligned}
&\left|\frac{1}{4}\left[f(x,y)+f(x,c+d-y)+f(a+b-x,y)+f(a+b-x,c+d-y)\right]\right. \\
&\left.-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right| \\
\leq\ &\frac{1}{(b-a)(d-c)}\left[(x-a)(y-c)\bigvee_{Q_1}(f)+(x-a)\left(\frac{c+d}{2}-y\right)\bigvee_{Q_2}(f)\right. \\
&+(x-a)(y-c)\bigvee_{Q_3}(f)+\left(\frac{a+b}{2}-x\right)(y-c)\bigvee_{Q_4}(f) \\
&+\left(\frac{a+b}{2}-x\right)\left(\frac{c+d}{2}-y\right)\bigvee_{Q_5}(f)+\left(\frac{a+b}{2}-x\right)(y-c)\bigvee_{Q_6}(f) \\
&(x-a)(y-c)\bigvee_{Q_7}(f)+(x-a)\left(\frac{c+d}{2}-y\right)\bigvee_{Q_8}(f) \\
&\left.+(x-a)(y-c)\bigvee_{Q_9}(f)\right]:=M(x,y)
\end{aligned}\tag{3.3}$$

which completes first inequality in (3.2).

$$\begin{aligned}
M(x,y)\ \leq\ &\frac{1}{(b-a)(d-c)}\max_{x,y}\left\{(x-a)(y-c),(x-a)\left(\frac{c+d}{2}-y\right),\right. \\
&\left.\left(\frac{a+b}{2}-x\right)(y-c),\left(\frac{a+b}{2}-x\right)\left(\frac{c+d}{2}-y\right)\right\} \\
&\times\left[\bigvee_{Q_1}(f)+\bigvee_{Q_2}(f)+...+\bigvee_{Q_9}(f)\right] \\
=\ &\frac{1}{(b-a)(d-c)}\max_{x}\left\{(x-a)\max_{y}\left\{(y-c),\left(\frac{c+d}{2}-y\right)\right\},\right. \\
&\left.\left(\frac{a+b}{2}-x\right)\max_{y}\left\{(y-c),\left(\frac{c+d}{2}-y\right)\right\}\right\} \\
&\times\left[\bigvee_{Q_1}(f)+\bigvee_{Q_2}(f)+...+\bigvee_{Q_9}(f)\right].
\end{aligned}$$

Since $\max\limits_{y}$ is independent of x, we have

$$
\begin{aligned}
M(x,y) &\leq \frac{1}{(b-a)(d-c)} \max_{x}\left\{(x-a),\left(\frac{a+b}{2}-x\right)\right\} \\
&\quad \times \max_{y}\left\{(y-c),\left(\frac{c+d}{2}-y\right)\right\}\bigvee_{Q}(f) \\
&= \left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\left[\frac{1}{4}+\left|\frac{y-\frac{3c+d}{4}}{d-c}\right|\right]\bigvee_{Q}(f)
\end{aligned}
$$

and the first branch in the second inequality in (3.2) is proved.

Using Hölder's discrete inequality, we have

$$
\begin{aligned}
M(x,y) &\leq \frac{1}{(b-a)(d-c)}\left[4(x-a)^{\alpha}(y-c)^{\alpha}+2(x-a)^{\alpha}\left(\frac{c+d}{2}-y\right)^{\alpha}\right. \\
&\quad \left. +2\left(\frac{a+b}{2}-x\right)^{\alpha}(y-c)^{\alpha}+\left(\frac{a+b}{2}-x\right)^{\alpha}\left(\frac{c+d}{2}-y\right)^{\alpha}\right]^{\frac{1}{\alpha}} \\
&\quad \times\left[\left[\bigvee_{Q_1}(f)\right]^{\beta}+\left[\bigvee_{Q_2}(f)\right]^{\beta}+...+\left[\bigvee_{Q_9}(f)\right]^{\beta}\right]^{\frac{1}{\beta}}
\end{aligned}
$$

where $\alpha>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$. This comletes the proof of the second branch in the second inequality.

Finally, we have

$$
\begin{aligned}
M(x,y) &\leq \frac{1}{(b-a)(d-c)}\max\left\{\bigvee_{Q_1}(f),\bigvee_{Q_2}(f),...,\bigvee_{Q_9}(f)\right\} \\
&\quad \times\left\{4(x-a)(y-c)+2(x-a)\left(\frac{c+d}{2}-y\right)\right. \\
&\quad \left.+2\left(\frac{a+b}{2}-x\right)(y-c)+\left(\frac{a+b}{2}-x\right)\left(\frac{c+d}{2}-y\right)\right\} \\
&\leq \frac{1}{(b-a)(d-c)}\max\left\{\bigvee_{Q_1}(f),\bigvee_{Q_2}(f),...,\bigvee_{Q_9}(f)\right\} \\
&\quad \left\{(x-a)(d-c)+(y-c)(b-a)+\left(\frac{a+b}{2}-x\right)\left(\frac{c+d}{2}-y\right)\right\}
\end{aligned}
$$

which completes the proof. □

Remark 1. *If we take $x = a$ and $y = c$ in Theorem 5, then we have the "trapezoid inequality"*

(3.4)
$$\left| \frac{f(b,d)+f(a,d)+f(b,c)+f(a,c)}{4} - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt \right| \leq \frac{1}{4} \bigvee_Q (f)$$

which is proved by Jawarneh and Noorani in [22]. *The constant $\frac{1}{4}$ is the best possible. For a simple proof of sharpness of constant see* [7].

Remark 2. *If we take $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in Theorem 5, then we have the "midpoint inequality"*

$$\left| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt \right| \leq \frac{1}{4} \bigvee_Q (f). \tag{3.5}$$

The constant $\frac{1}{4}$ is the best possible in (3.5).

Proof. Assume that (3.5) holds with a constant $B > 0$,i.e.,

$$\left| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt \right| \leq \frac{1}{4} \bigvee_Q (f). \tag{3.6}$$

If we choose $f : Q \to \mathbb{R}$ with

$$f(x,y) = \begin{cases} 1 & \text{if } x = \frac{a+b}{2},\ y = \frac{c+d}{2} \\ \\ 0 & \text{if } (x,y) \in [a,b] \times [c,d] \setminus \left\{ \frac{a+b}{2}, \frac{c+d}{2} \right\} \end{cases}$$

then f is of bounded variation on Q, and

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) = 1, \ \int_a^b \int_c^d f(t,s)dsdt = 0, \text{ and } \bigvee_Q (f) = 4$$

which gives that $1 \leq 4B$, thus $B \geq \frac{1}{4}$, which proves the sharpness of (3.5). □

Corollary 1. *If we take $x = \frac{3a+b}{2}$ and $y = \frac{3c+d}{2}$ in Theorem 5, then we have*

(3.7)
$$\begin{aligned} &\left| \frac{f\left(\frac{3a+b}{2}, \frac{3c+d}{2}\right) + f\left(\frac{3a+b}{2}, \frac{c+3d}{2}\right) + f\left(\frac{a+3b}{2}, \frac{3c+d}{2}\right) + f\left(\frac{a+3b}{2}, \frac{c+3d}{2}\right)}{4} \right. \\ &\left. - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt \right| \\ \leq\ & \frac{1}{16} \bigvee_Q (f). \end{aligned}$$

The constant $\frac{1}{16}$ is the best possible in (3.7).

Proof. Assume that (3.7) holds with a constant $C>0$, i.e.,

(3.8)
$$\begin{aligned}&\left|\frac{f\left(\frac{3a+b}{2},\frac{3c+d}{2}\right)+f\left(\frac{3a+b}{2},\frac{c+3d}{2}\right)+f\left(\frac{a+3b}{2},\frac{3c+d}{2}\right)+f\left(\frac{a+3b}{2},\frac{c+3d}{2}\right)}{4}\right.\\&\left.-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|\\ \leq\ & C\bigvee_Q(f).\end{aligned}$$

If we choose $f:Q\to\mathbb{R}$ with

$$f(x,y)=\begin{cases}1 \text{ if} & (x,y)\in\left\{\left(\frac{3a+b}{2},\frac{3c+d}{2}\right),\left(\frac{3a+b}{2},\frac{c+3d}{2}\right),\left(\frac{a+3b}{2},\frac{3c+d}{2}\right),\left(\frac{a+3b}{2},\frac{c+3d}{2}\right)\right\}\\ 0 \text{ if} & (x,y)\in[a,b]\times[c,d]\setminus\left\{\left(\frac{3a+b}{2},\frac{3c+d}{2}\right),\left(\frac{3a+b}{2},\frac{c+3d}{2}\right),\left(\frac{a+3b}{2},\frac{3c+d}{2}\right),\left(\frac{a+3b}{2},\frac{c+3d}{2}\right)\right\}\end{cases}$$

then f is of bounded variation on Q, and

$$\frac{f\left(\frac{3a+b}{2},\frac{3c+d}{2}\right)+f\left(\frac{3a+b}{2},\frac{c+3d}{2}\right)+f\left(\frac{a+3b}{2},\frac{3c+d}{2}\right)+f\left(\frac{a+3b}{2},\frac{c+3d}{2}\right)}{4}=1,$$

$$\begin{aligned}\int_a^b\int_c^d f(t,s)dsdt &= 0,\\ \text{and } \bigvee_Q(f) &= 16.\end{aligned}$$

Therefore, we get in (3.8), $1\leq 16C$, thus $C\geq\frac{1}{16}$, which implies the constant $\frac{1}{16}$ is the best possible. This completes the proof. □

4. A Composite Quadrature Formula

Let us consider the arbitrary division $I_n: a=x_0<x_1<...<x_n=b$,and $J_m=c=y_0<y_1<...<y_m=d, h_i:=x_{i+1}-x_i$,and $l_j:=y_{j+1}-y_j$

$$\upsilon(h):=\max\{h_i|\ i=0,...,n-1\}$$

$$\upsilon(l):=\max\{l_j|\ j=0,...,m-1\}.$$

Consider the Riemann sum

$$
\begin{aligned}
T(f,I_n,J_m) &= \frac{1}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f\left(\frac{3x_i+x_{i+1}}{2},\frac{3y_j+y_{j+1}}{2}\right)h_i l_j \\
&+\frac{1}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f\left(\frac{3x_i+x_{i+1}}{2},\frac{y_j+3y_{j+1}}{2}\right)h_i l_j \\
&+\frac{1}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f\left(\frac{x_i+3x_{i+1}}{2},\frac{3y_j+y_{j+1}}{2}\right)h_i l_j \\
&+\frac{1}{4}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f\left(\frac{x_i+3x_{i+1}}{2},\frac{y_j+3y_{j+1}}{2}\right)h_i l_j
\end{aligned} \tag{4.1}
$$

The following result holds.

Theorem 6. *Let the function $f:Q=[a,b]\times[c,d]\to R$ is of bounded variation on Q. Then we have*

$$\int_a^b\int_c^d f(t,s)dsdt = T(f,I_n,J_m)+R(f,I_n,J_m)$$

where $T(f,I_n,J_m)$ is the Riemann sum defined in (4.1) and the remainder t $R(f,I_n,J_m)$ satisfies the estimate

$$|R(f,I_n,J_m)| \le \frac{1}{16}\upsilon(h)\upsilon(l)\bigvee_a^b\bigvee_c^d(f). \tag{4.2}$$

The constant $\frac{1}{16}$ is the best possible.

Proof. Applying Corollary 1 on the bidimentional interval $[x_i,x_{i+1}]\times[y_j,y_{j+1}]$, we have

$$
\begin{aligned}
&\left|\frac{1}{4}\left[f\left(\frac{3x_i+x_{i+1}}{2},\frac{3y_j+y_{j+1}}{2}\right)+f\left(\frac{3x_i+x_{i+1}}{2},\frac{y_j+3y_{j+1}}{2}\right)+f\left(\frac{x_i+3x_{i+1}}{2},\frac{3y_j+y_{j+1}}{2}\right)\right.\right. \\
&\left.\left.+f\left(\frac{x_i+3x_{i+1}}{2},\frac{y_j+3y_{j+1}}{2}\right)\right](x_{i+1}-x_i)(y_{j+1}-y_j)-\int_{xi}^{x_{i+1}}\int_{y_j}^{y_{j+1}} f(t,s)dsdt\right| \\
\le\ & \frac{1}{16}(x_{i+1}-x_i)(y_{j+1}-y_j)\bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f)
\end{aligned} \tag{4.3}
$$

for any $i\in\{1,2,...,n-1\}$ and $j\in\{1,2,...,m-1\}$.

Summing the inequality (4.3) over i from 0 to $n-1$ and j from 0 to $m-1$,then we get

$$\begin{aligned} |R(f,I_n,J_m)| &\leq \frac{1}{16}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left(x_{i+1}-x_i\right)\left(y_{j+1}-y_j\right)\bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f) \\ &\leq \frac{1}{16}\upsilon(h)\upsilon(l)\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\bigvee_{x_i}^{x_{i+1}}\bigvee_{y_j}^{y_{j+1}}(f) \\ &= \frac{1}{16}\upsilon(h)\upsilon(l)\bigvee_a^b\bigvee_c^d(f). \end{aligned}$$

This completes the proof. □

Particularly, if we choose the division $I_n : x_i = a + i\frac{b-a}{n}$, $i = 1,2,...,n$ and $J_m : y_j = c + j\frac{d-c}{m}$, $j = 1,2,...,m$, then

(4.4)

$$\begin{aligned} T(f) \quad : \quad &= \frac{(b-a)(d-c)}{4nm}\left[\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f\left(2a+\left(\frac{4i+1}{2n}\right)(b-a), 2c+\left(\frac{4j+1}{2m}\right)(d-c)\right)\right. \\ &+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f\left(2a+\left(\frac{4i+1}{2n}\right)(b-a), 2c+\left(\frac{4j+3}{2m}\right)(d-c)\right) \\ &+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f\left(2a+\left(\frac{4i+3}{2n}\right)(b-a), 2c+\left(\frac{4j+1}{2m}\right)(d-c)\right) \\ &\left.+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1} f\left(2a+\left(\frac{4i+3}{2n}\right)(b-a), 2c+\left(\frac{4j+3}{2m}\right)(d-c)\right)\right]. \end{aligned}$$

Corollary 2. *Under assumption of Theorem 6, we have*

$$\int_a^b\int_c^d f(t,s)dsdt = T(f) + R(f)$$

where where $T(f)$ is the Riemann sum defined in (4.4) and the remainder term $R(f)$ satisfies the estimate

$$|R(f)| \leq \frac{(b-a)(d-c)}{16nm}\bigvee_a^b\bigvee_c^d(f). \tag{4.5}$$

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