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A GENERALIZED INEQUALITY OF OSTROWSKI TYPE FOR TWICE DIFFERENTIABLE BOUNDED MAPPINGS AND APPLICATIONS

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Abstract. In this paper, we will improve and generalize Ostrowski type inequality for twice differentiable mappings in terms of the upper and lower bounds of the second derivative. Some well known inequalities can be derived as special cases of the inequalities obtained here. In addition, perturbed midpoint inequality and perturbed trapezoid inequality are also obtained. The obtained inequalities have immediate applications in numerical integrations where new estimates are obtained for the remainder term of the trapezoid and midpoint formulae. Applications to special means are also investigated.

1. Introduction

Inequalities have proved to be an exalted and applicable tool for the development of many branches of Mathematics. It's importance has increased noticeably during the past few decades and it is now treated as an independent branch of Mathematics. This field is active and experiencing a tremendous boost with the passage of time in theory as well as in applications. One element that particularly signifies its importance is its applications in various fields. Uptill now, a vast number of research papers and books have been dedicated to inequalities and their numerous applications.

Ostrowski's inequalities play an important role in several other branches of mathematics and statistics with reference to its applications. In recent years a number of authors ([9],[10],[11] and [12]) have written about generalizations of Ostrowski's inequality. In 1998, Dragomir et al.[7] presented a new proof to the classical Ostrowski's inequality and for the first time applied it to the estimation of error bounds for some special means and for some numerical quadrature rules. It is with the same viewpoint, the two monographs [8] were written in 2002 and 2004 to present some selected results on Ostrowski type inequalities and their applications. The current paper will obtain bounds for quadrature rules consisting of, at most, three points for twice differentiable functions. These results will be obtained with the help of kernels.

Ostrowski [1] proved the following classical integral inequality.

Theorem 1. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I° *(the interior of* I*) and let* $a, b \in I^\circ$ *with* $a < b$*. If* $f' : (a,b) \to \mathbb{R}$ *is bounded on* (a,b) *i.e.*

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$\|f'\|_\infty = \sup_{t\in[a,b]} |f''(t)| < \infty$, *then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\|f'\|_\infty \tag{1.1}$$

for all $x\in[a,b]$*. The constant* $\frac{1}{4}$ *is sharp in the sense that it can not be replaced by a smaller one.*

Some applications of Ostrowski's inequality to special means and numerical quadrature rules, are given in [2] by Dragomir et al.

In 1976, Milovanović et al. proved a generalization of Ostrowski's inequality for n-time differentiable mappings from which we would like to mention only the case of twice differentiable mappings ([4], p.470).

Theorem 2. *Let* $f:[a,b]\to\mathbb{R}$ *be a twice differentiable mapping such that* $f''(t)$ *is bounded on* (a,b) *i.e* $\|f''\|_\infty = \sup_{t\in(a,b)}|f''(t)| < \infty$*, then the inequality:*

$$\begin{aligned} &\left| \frac{1}{2}\left[f(x) + \frac{(x-a)f(a)+(b-x)f(b)}{b-a} \right] - \frac{1}{b-a}\int_a^b f(t)dt \right| \\ \le\ & \frac{\|f'\|_\infty}{4}(b-a)^2 \left[\frac{1}{12} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} \right] \end{aligned} \tag{1.2}$$

for all $x\in[a,b]$*.*

In [5] Barnett et al. pointed out an inequality of Ostrowski's type which was similar, in a sense, to the Milovanović -Pecarić result and applied it for special means and in numerical integration. Some of the results of the Barnett 's paper has been reported by Cerone et al [6].

Theorem 3. *Let* $f:[a,b]\to\mathbb{R}$ *be a twice differentiable on* (a,b) *and* $f'':(a,b)\to\mathbb{R}$ *is bounded i.e* $\|f''\|_\infty = \sup_{t\in(a,b)}|f''(t)| < \infty$*, then the inequality:*

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \left(x-\frac{a+b}{2}\right) f'(x) \right| \\ \le\ & \left(\frac{1}{24}(b-a)^2 + \frac{1}{2}\left(x-\frac{a+b}{2}\right)^2 \right)\|f''\|_\infty \le \frac{(b-a)^2}{6}\|f''\|_\infty \end{aligned} \tag{1.3}$$

for all $x\in[a,b]$ *.*

Motivated and inspired by the work of the above mentioned renowned mathematicians, we will establish a new generalized inequality. Some other interesting inequalities are also presented as special cases. In the end, we will give applications for some special means and in numerical integration.

2. Main Results

We now give our main result.

Theorem 4. *Let* $f:[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$ *and twice differentiable on* (a,b), *then*

$$\begin{aligned}
&\left| \begin{array}{c} (1-h)\ f(x)-(1-h)\left(x-\frac{a+b}{2}\right)f'(x)+\frac{h}{2}\left(f(a)+f(b)\right) \\ -\frac{h^2(b-a)}{8}\left(f'(b)-f'(a)\right)-\frac{1}{b-a}\int\limits_a^b f(t)dt \end{array}\right| \\
\leq\ & \left[(1-h)\left(\frac{(b-a)^2(1-h)^2}{24}+\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2\right)+\frac{h^3(b-a)^2}{24}\right]\|f''\|_\infty \\
\leq\ & \left[3(1-h)^2+1\right]\frac{(b-a)^2}{24}\|f''\|_\infty \qquad (2.1)
\end{aligned}$$

for all $x\in\left[a+h\frac{b-a}{2},b-h\frac{b-a}{2}\right]$ *and* $h\in[0,1]$.

Proof. Let us define the mapping $K:[a,b]^2\to\mathbb{R}$ [8] by

$$K(x,t)=\begin{cases}\frac{1}{2}\left[t-\left(a+h\frac{b-a}{2}\right)\right]^2, \text{ if } t\in[a,x]\\ \frac{1}{2}\left[t-\left(b-h\frac{b-a}{2}\right)\right]^2, \text{ if } t\in(x,b]\end{cases}$$

Let

$$\int\limits_a^b K(x,t)f''(t)\,dt=\int\limits_a^x\frac{1}{2}\left[t-\left(a+h\frac{b-a}{2}\right)\right]^2f''(t)\,dt+\int\limits_x^b\frac{1}{2}\left[t-\left(b-h\frac{b-a}{2}\right)\right]^2f''(t)\,dt.$$

After some manipulations, we obtained the following identity.

$$\begin{aligned}
\int\limits_a^b f(t)dt =\ & (b-a)(1-h)f(x)-(b-a)(1-h)\left(x-\frac{a+b}{2}\right)f'(x) \qquad (2.2)\\
&+h\frac{b-a}{2}\left(f(a)+f(b)\right)-\frac{h^2(b-a)^2}{8}\left(f'(b)-f'(a)\right)+\int\limits_a^b K(x,t)f''(t)\,dt
\end{aligned}$$

for all $x\in\left[a+h\frac{b-a}{2},b-h\frac{b-a}{2}\right]$.

This is a particular form of the identity given in ([8], p.67, Theorem 28).

Using the identity (2.2), we have

$$\begin{vmatrix} (1-h)\,f(x)-\frac{1}{(b-a)}\int\limits_a^b f(t)dt+\frac{h}{2}\left(f\left(a\right)+f\left(b\right)\right) \\ -\frac{h^2(b-a)}{8}\left(f'\left(b\right)-f'\left(a\right)\right)-(1-h)\left(x-\frac{a+b}{2}\right)f'\left(x\right) \end{vmatrix}$$

$$\begin{aligned}
&\leq \frac{1}{(b-a)}\left|\int\limits_a^b K(x,t)f''\left(t\right)dt\right| \\
&\leq \|f''\|_\infty \frac{1}{(b-a)}\int\limits_a^b |K(x,t)|\,dt \qquad (2.3) \\
&= \|f''\|_\infty \frac{1}{(b-a)}\left[\int\limits_a^x \frac{\left[t-\left(a+h\frac{b-a}{2}\right)\right]^2}{2}dt+\int\limits_x^b \frac{\left[t-\left(b-h\frac{b-a}{2}\right)\right]^2}{2}dt\right]
\end{aligned}$$

Now, observe that

$$\begin{aligned}
&\int\limits_a^x \frac{\left[t-\left(a+h\frac{b-a}{2}\right)\right]^2}{2}dt+\int\limits_x^b \frac{\left[t-\left(b-h\frac{b-a}{2}\right)\right]^2}{2}dt \\
&= (b-a)\left(1-h\right)\left[\frac{\left(b-a\right)^2\left(1-h\right)^2}{24}+\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2\right]+\frac{h^3\left(b-a\right)^3}{24} \qquad (2.4)
\end{aligned}$$

Using (2.4) in (2.3), we get our required result given in (2.1). □

Remark 1. *For $h=0$, in (2.1), we obtain Barnett's result (1.3) proved in [1]. It shows that our result contains Barnett's result (1.3) as a special case.*

Remark 2. *For $h=1$ in (2.1), we obtain another useful inequaliy.*

$$\begin{aligned}
&\left|\frac{(f(a)+f(b))}{2}-\frac{(b-a)}{8}\left(f'\left(b\right)-f'\left(a\right)\right)-\frac{1}{b-a}\int\limits_a^b f(t)dt\right| \\
&\leq \frac{\left(b-a\right)^2}{24}\|f''\|_\infty
\end{aligned}$$

Hence, for different values of h, we can obtain variety of results.

Corollary 1. *If f is as in Theorem 4, then we have the following perturbed midpoint inequality*

$$\begin{vmatrix} (1-h)\,f\left(\frac{a+b}{2}\right)+\frac{h}{2}\left(f\left(a\right)+f\left(b\right)\right)-\frac{h^2(b-a)}{8}\left(f'\left(b\right)-f'\left(a\right)\right) \\ -\frac{1}{b-a}\int\limits_a^b f(t)dt \\ \leq \frac{(b-a)^2}{24}\left[3\left(h-\frac{1}{2}\right)^2+\frac{1}{4}\right]\|f''\|_\infty \end{vmatrix} \qquad (2.5)$$

giving,

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int\limits_a^b f(t)dt\right|\leq \frac{\left(b-a\right)^2}{24}\|f''\|_\infty \qquad (2.6)$$

for $h = 0$, this recaptures the classical midpoint inequality.

Remark 3. *The estimation provided by* (2.5) *is better than the estimation provided by the classical midpoint inequality.*

Corollary 2. *Let f be as in Theorem* 4*, then :*

$$\left| \begin{array}{c} (1-h)\frac{f(a)+f(b)}{2} - (1-h)(b-a)\frac{f'(b)-f'(a)}{4} + \frac{h}{2}(f(a)+f(b)) \\ -\frac{h^2(b-a)}{8}(f'(b)-f'(a)) - \frac{1}{b-a}\int_a^b f(t)dt \end{array} \right|$$

$$\leq \left[3(1-h)^2+1\right]\frac{(b-a)^2}{24}\|f''\|_\infty \tag{2.7}$$

Proof. Put $x = a$ and $x = b$ in (2.1), summing up the obtained inequalities, using the triangle inequality and dividing by 2, we get the required inequality. □

Corollary 3. *Let f be as in Theorem* 4*, then we have the perturbed trapezoidal inequality:*

$$\left| \frac{f(a)+f(b)}{2} - (b-a)\frac{f'(b)-f'(a)}{4} - \frac{1}{b-a}\int_a^b f(t)dt \right|$$

$$\leq \frac{1}{6}(b-a)^2\|f''\|_\infty . \tag{2.8}$$

Proof. Put $h = 0$, in (2.7). □

Remark 4. *The estimation provided by* (2.8)*, is similar to that of the classical trapezoidal inequality.*

3. Applications in Numerical integration

Let $I_n : a = x_0 < x_1 < x_2 < .... < x_{n-1} < x_n = b$ be a division of the interval $[a, b]$, $\xi_i \in \left[x_i + \delta\frac{h_i}{2}, x_{i+1} - \delta\frac{h_i}{2}\right]$, $(i = 0, 1, ..., n-1)$ a sequence of intermediate points and $h_i = x_{i+1} - x_i$, $(i = 0, 1, ..., n-1)$. then we have the following quadrature rule:

Theorem 5. *Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable on (a, b) and $f'' : (a, b) \to \mathbb{R}$ is bounded, i.e $\|f''\|_\infty < \infty$. Then we have the following:*

$$\int_a^b f(t)dt = A\left(f, f', I_n, \xi, \delta\right) + R\left(f, f', I_n, \xi, \delta\right) \tag{3.1}$$

where

$$\begin{aligned} A\left(f, f', I_n, \xi, \delta\right) &= (1-\delta)\sum_{i=0}^{n-1} h_i f(\xi_i) - (1-\delta)\sum_{i=0}^{n-1} h_i\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right) f'(\xi_i) \\ &+\frac{\delta}{2}\sum_{i=0}^{n-1} h_i\left(f(x_i) + f(x_{i+1})\right) - \frac{\delta^2}{8}\sum_{i=0}^{n-1} h_i^2\left(f'(x_{i+1}) - f'(x_i)\right) \end{aligned} \tag{3.2}$$

and the remainder $R\left(f, f', I_n, \xi, \delta\right)$ *satisfies the estimation*

$$\begin{aligned}
&\left|R\left(f, f', I_n, \xi, \delta\right)\right| \\
\leq\ & \sum_{i=0}^{n-1} h_i \left(\left[\frac{h_i^2 (1-\delta)^3}{24} + \frac{1}{2}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2 \right] + \frac{\delta^3 h_i^2}{24} \right) \|f''\|_\infty \\
\leq\ & \left[3(1-\delta)^2 + 1\right] \sum_{i=0}^{n-1} \frac{h_i^3}{24} \|f''\|_\infty \qquad (3.3)
\end{aligned}$$

where $\delta \in [0,1]$ *and* $x_i + \delta \frac{h_i}{2} \leq \xi_i \leq x_{i+1} - \delta \frac{h_i}{2}$.

Proof. Apply Theorem 4 on the interval $[x_i, x_{i+1}]$, $(i = 0, 1, ...n-1)$, to obtain

$$\begin{aligned}
&\left| \begin{array}{c} (1-\delta) h_i f(\xi_i) - (1-\delta) h_i \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right) f'(\xi_i) + \frac{\delta}{2} h_i \left(f(x_i) + f(x_{i+1})\right) \\ -\frac{\delta^2}{8} h_i^2 \left(f'(x_{i+1}) - f'(x_i)\right) - \int_{x_i}^{x_{i+1}} f(t)dt \end{array} \right| \\
\leq\ & \left[(1-\delta) h_i \left(\frac{h_i^2 (1-\delta)^2}{24} + \frac{1}{2}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2 \right) + \frac{\delta^3 h_i^2}{24} \right] \|f''\|_\infty \\
\leq\ & \left[3(1-\delta)^2 + 1\right] \frac{h_i^3}{24} \|f''\|_\infty
\end{aligned}$$

for any choice ξ of the intermediate points.

Summing over i from 0 to $n-1$ and using the generalized triangular inequality, we deduce the desired estimation (3.3). □

Corollary 4. *The following perturbed midpoint rule holds:*

$$\int_a^b f(x)dx = M\left(f, f', I_n\right) + R_M\left(f, f', I_n\right),$$

where

$$M\left(f, f', I_n\right) = \sum_{i=0}^{n-1} h_i f\left(\frac{x_i + x_{i+1}}{2}\right) \qquad (3.4)$$

and the remainder term $R_M\left(f, f', I_n\right)$ *satisfies the estimation:*

$$\left|R_M\left(f, f', I_n\right)\right| \leq \|f''\|_\infty \sum_{i=0}^{n-1} \frac{h_i^3}{6} \qquad (3.5)$$

Corollary 5. *The following perturbed trapezoidal rule holds:*

$$\int_a^b f(x)dx = T\left(f, f', I_n\right) + R_T\left(f, f', I_n\right) \qquad (3.6)$$

where

$$T\left(f, f', I_n\right) = \sum_{i=0}^{n-1} h_i \frac{\left(f(x_i) + f(x_{i+1})\right)}{2} - \frac{1}{8} \sum_{i=0}^{n-1} h_i^2 \left(f'(x_{i+1}) - f'(x_i)\right) \qquad (3.7)$$

and the remainder term

$$\left|R_T\left(f,\acute{f},I_n\right)\right| \leq \|f\,''\|_\infty \sum_{i=0}^{n-1} h_i^3 \left[\frac{\left((x_{i+1})-(x_i)\right)^2}{24}\right] \tag{3.8}$$

Remark 5. *Note that the above mentioned perturbed midpoint formula* (3.4) *and perturbed trapezoid formula* (3.7) *can give better approximations of the integral* $\int_a^b f(x)dx$ *for general classes of mappings.*

4. **Application for some special means**

Let us recall the following means:

The Arithmetic Mean

$$A = A(a,b) = \frac{a+b}{2}, \qquad a,b \geq 0.$$

The Geometric Mean

$$G = G(a,b) = \sqrt{ab}\ , \qquad a,\ b \geq 0.$$

The Harmonic Mean

$$H = H(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}}\ , \qquad a,b \geq 0;$$

The Logarithmic Mean

$$L = L(a,b) = \begin{cases} a & \text{if } a=b, \quad a,b \geq 0; \\ \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b\ ; \end{cases}$$

The Identric Mean

$$I = I(a,b) = \begin{cases} a & \text{if } \quad a=b, \quad a,\ b>0; \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a \neq b\ ; \end{cases}$$

The P-Logarithmic Mean

$$L_p \quad = \quad L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & \text{if } a \neq b \\ a & \text{if } \quad a=b, \end{cases}$$

where $p \in \mathbb{R}\backslash\{-1,0\}$, $a,b>0$.

The following simple relationships are known in this paper.

$$H \leq G \leq L \leq I \leq A$$

It is also known that L_p is monotonically increasing in $p \in \mathbb{R}$ with $L_0 = I$ and $L_{-1} = L$.

We may now apply inequality (2.1) , to deduce some inequalities for special means by the use of particular mappings as follows:

Remark 6. *Consider the mapping* $f:(0,\infty)\to\mathbb{R}$, $f(x)=x^r$, $r\in\mathbb{R}\setminus\{-1,0\}$, *for* $0<a<b$,

then

$$\frac{1}{b-a}\int_a^b f(t)dt = L_r^r(a,b),$$

and

$$\|f''\|_\infty = |r(r-1)\delta_r(a,b)|,$$

where

$$\delta_r(a,b)=\begin{cases} b^{r-1} & \text{if } r\in(1,\infty) \\ a^{r-1} & \text{if } r\in(-\infty,1)\setminus\{0\}\end{cases}.$$

Then the inequality (2.1) *gives*

$$\left|\begin{array}{c}(1-h)x^r-(1-h)(x-A)rx^{r-1} \\ +\frac{h}{2}(a^r+b^r)-\frac{h^2(b-a)r}{8}\left[b^{r-1}-a^{r-1}\right]-L_r^r(a,b)\end{array}\right| \tag{4.1}$$

$$\leq \left((1-h)\left[\frac{(b-a)^2(1-h)^2}{24}+\frac{1}{2}(x-A)^2\right]+\frac{h^3(b-a)^2}{24}\right)|r(r-1)\delta_r(a,b)|.$$

Choosing $x=A$ *in* (4.1), *we get*

$$\left|(1-h)A^r+\frac{h}{2}(a^r+b^r)-\frac{h^2(b-a)r}{8}\left[b^{r-1}-a^{r-1}\right]-L_r^r(a,b)\right|$$
$$\leq \frac{(b-a)^2}{24}\left[3\left(h-\frac{1}{2}\right)^2+\frac{1}{4}\right]|r(r-1)\delta_r(a,b)|. \tag{4.2}$$

Choosing $h=0$ *in* (4.2), *we get*

$$|A^r-L_r^r(a,b)|\leq\frac{(b-a)^2}{24}|r(r-1)\delta_r(a,b)|. \tag{4.3}$$

Remark 7. *Consider the mapping* $f(x)=\frac{1}{x}$, $x\in[a,b]\subset(0,\infty)$, *then*

$$\frac{1}{b-a}\int_a^b f(t)dt = L_{-1}^{-1}(a,b),$$

and

$$\|f''\|_\infty=\frac{2}{a^3}.$$

Then the inequality (2.1) *gives*

$$\left|(1-h)\ \frac{1}{x}+(1-h)(x-A)\frac{1}{x^2}+\frac{h}{H}\ -\frac{h^2(b-a)}{8}\left(\frac{b^2-a^2}{a^2b^2}\right)-L_{-1}^{-1}(a,b)\right|$$
$$\leq \left((1-h)\left[\frac{(b-a)^2(1-h)^2}{24}+\frac{1}{2}(x-A)^2\right]+\frac{h^3(b-a)^2}{24}\right)\frac{2}{a^3}. \tag{4.4}$$

Choosing $x = A$ *in* (4.4), *we get*

$$\left|(1-h)\ \frac{1}{A}+\frac{h}{H}\ -\frac{h^2(b-a)}{8}\left(\frac{b^2-a^2}{a^2b^2}\right)-L_{-1}^{-1}(a,b)\right|$$

$$\leq\ \frac{(b-a)^2}{12}\left[3\left(h-\frac{1}{2}\right)^2+\frac{1}{4}\right]\frac{1}{a^3}. \tag{4.5}$$

Choosing $h = 0$ *in* (4.5), *we get*

$$\left|\ A^{-1}\ -L_{-1}^{-1}(a,b)\right| \leq \frac{(b-a)^2}{12a^3}. \tag{4.6}$$

Remark 8. *Consider the mapping* $f(x) = \ln x$, $x \in [a,b] \subset (0,\infty)$,

$$\frac{1}{b-a}\int_a^b f(t)dt = \ln I(a,b),$$

and

$$\|f\ ''\|_\infty = \frac{1}{a^2}.$$

Then inequality (2.1) *gives*

$$\left|\begin{array}{c}(1-h)\ \ \ln x-(1-h)(x-A)\frac{1}{x}+\frac{h}{2}(\ln a+\ln b)\\ \\ +\frac{h^2(b-a)^2}{8ab}-\ln I(a,b)\end{array}\right|$$

$$\leq\ \left((1-h)\left[\frac{(b-a)^2(1-h)^2}{24}+\frac{1}{2}(x-A)^2\right]+\frac{h^3(b-a)^2}{24}\right)\frac{1}{a^2}. \tag{4.7}$$

Choosing $x = A$ *in* (4.7) , *we get*

$$\left|(1-h)\ \ \ln A-\frac{h}{2}(\ln a+\ln b)+\frac{h^2(b-a)^2}{8ab}-\ln I(a,b)\right|$$

$$\leq\ \frac{(b-a)^2}{8}\left[\left(h-\frac{1}{2}\right)^2+\frac{1}{12}\right]\frac{1}{a^2}. \tag{4.8}$$

Choosing $h = 0$ *in* (4.8), *we get*

$$\left|\frac{A}{I}\right| \leq \exp\frac{(b-a)^2}{24a^2} \tag{4.9}$$

5. Conclusion:

We established generalized Ostrowski type inequality for bounded differentiable mappings. We have shown that the inequalities obtaind in [4],[5],[6] and [7] are special cases of our inequalties. Perturbed midpoint and trapezoid inequalities are also obtained. Improvements to classical trapezoidal and midpoint inequalities are provided.These generalized inequalities add up to the literature in the sense that they have immediate applications in Numerical Integration and Special Means. These generalized inequalities will also be useful for the researchers working in the field of approximation theory, applied mathematics, probability theory, stochastic and numerical analysis.

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