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A GENERALIZED INEQUALITY OF OSTROWSKI TYPE FOR MAPPINGS WHOSE SECOND DERIVATIVES BELONG TO $\mathbf{L}_1(a,b)$ AND APPLICATIONS

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Abstract. In this paper, we will improve and generalize for mappings whose second derivatives belong to $\mathrm{L}_1(a,b)$. Some well known inequalities can be derived as special cases of the inequalities obtained here. In addition, perturbed mid-point inequality and perturbed trapezoid inequality are also obtained. The obtained inequalities have immediate applications in numerical integration where new estimates are obtained for the remainder term of the trapezoid and midpoint formulae. Applications to special means are also investigated.

1. Introduction

In the last few decades, the field of mathematical inequalities has proved to be an extensively applicable field. Integral inequalities play an important role in several other branches of mathematics and statistics with reference to its applications. The elementary inequalities are proved to be helpful in the development of many other branches of mathematics. The development of inequalities has been established with the publication of the books by G. H. Hardy, J. E. Littlewood and G. Polya [6] in 1934, E. F. Beckenbach and R. Bellman [7] in 1961 and by D. S. Mitrinovi´c, J. E. Pecari´c and A. M. Fink [8] & [9] in 1991. The publication of later has resulted to bring forward some new integral inequalities involving functions with bounded derivatives that measure bounds on the deviation of functional value from its mean value namely, Ostrowski inequality [5]. This Ostrowski type inequality has powerful applications in numerical integration, probability and optimization theory, stochastic, statistics, information and integral operator theory.

During the last few years, many researchers focused their attention on the study and generalizations (see for example [1], [2], [10] and [11] and reference there in) of the Ostrowski inequality.

Ostrowski [5] proved the classical integral inequality which is stated as:

Theorem 1. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I° *(*I° *is the interior of* I*) and let* $a, b \in I^\circ$ *with* $a < b$*. If* $f' : (a,b) \to \mathbb{R}$ *is bounded on* (a,b) *i.e.* $\|f'\|_\infty = \sup_{t\in[a,b]} |f''(t)| < \infty$*, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\|f'\|_\infty \tag{1.1}$$

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for all $x \in [a,b]$*. The constant* $\frac{1}{4}$ *is sharp in the sense that it can not be replaced by a smaller one.*

For some applications of Ostrowski's inequality to special means and numerical quadrature rules, we refer to the paper [2] by Dragomir et al.

In 1976, Milovanović et al. proved a generalization of Ostrowski's inequality for n-time differentiable mappings (see for example [3], p.46) from which we would like to mention only the case of twice differentiable mappings ([4], p.470).

Dragomir and Wang [1] proved (1.1) for $f' \in L_1[a,b]$, as follows:

Theorem 2. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping in* I° *and* $a, b \in I^\circ$ *with* $a < b$*. If* $f' \in L_1[a,b]$*, then the inequality holds*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{2} + \frac{\left|x - \frac{a+b}{2}\right|}{b-a} \right] \|f'\|_1 \tag{1.2}$$

for all $x \in [a,b]$*.*

They also pointed out some applications of (1.2) in Numerical Integration as well as for special means.

J. Roumeliotis [3], presented product inequalities and weighted quadrature. The weighted inequlity was also obtained in Lebesgue spaces involving first derivative of the function, which is given by

$$\begin{aligned} &\left| \frac{1}{b-a}\int_a^b w(t)f(t)dt - m(a,b)f(x) \right| \\ &\leq \frac{1}{2}\left[m(a,b) + |m(a,x) - m(x,b)|\right]\|f''\|_1 \end{aligned} \tag{1.3}$$

Qayyum [4], also proved the result for $L_1[a,b]$ by using weight function, which is stated as:

Theorem 3. *Let* $f : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *and satisfy the condition* $\theta \leq f' \leq \Phi$, $x \in (a,b)$*. Then we have the inequality*

$$\begin{aligned} &\left| f(x) - \frac{1}{m(a,b)}w(x)(b-a)\left(x - \frac{a+b}{2}\right)f'(x) - \frac{1}{m(a,b)}\int_a^b f(t)w(t)dt \right| \\ &\leq \frac{1}{2m^2(a,b)}w(x)\left(\frac{1}{2}(b-a)^2 + 2\left(x - \frac{a+b}{2}\right)^2\right) \\ &\times \left(\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right)\|f''\|_{w,1} \end{aligned} \tag{1.4}$$

for all $x \in [a,b]$*.*

Barnett et,al, [2] proved out an inequality of Ostrowski type for twice differentiable mappings which is in terms of the $\|.\|_1$ norm of the second derivative f'' and apply it in numerical integration and for some special means.

The following inequality of Ostrowski's type for mappings which are twice differentiable, holds [3].

Theorem 4. *Let* $f:[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$ *and twice differentiable in* (a,b) *and* $f''\in L_1(a,b)$. *Then the inequality obtained*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-\left(x-\frac{a+b}{2}\right)f'(x)\right| \tag{1.5}$$
$$\leq\frac{1}{2(b-a)}\left(\left|x-\frac{a+b}{2}\right|+\frac{1}{2}(b-a)\right)^2\|f''\|_1$$
$$\leq\frac{b-a}{2}\|f''\|_1$$

for all $x\in[a,b]$.

Motivated and inspired by the work of the above mentioned renowned mathematicians, we will establish a new generalized inequality. Some other interesting inequalities are also presented as special cases. In the last, we will present applications for some special means and in numerical integration.

2. Main Results

We now give our main result.

Theorem 5. *Let* $f:[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$ *and twice differentiable on* (a,b) *with second derivative and* $f'':(a,b)\to\mathbb{R}$, *for all* $x\in\left[a+h\frac{b-a}{2},b-h\frac{b-a}{2}\right]$, *it follows that*

$$\left|\begin{array}{c}(1-h)\ f(x)-(1-h)\left(x-\frac{a+b}{2}\right)f'(x)+\frac{h}{2}\left(f(a)+f(b)\right)\\ -\frac{h^2(b-a)}{8}\left(f'(b)-f'(a)\right)-\frac{1}{b-a}\int_a^b f(t)dt\end{array}\right|$$
$$\leq\frac{1}{2(b-a)}\left[\frac{1}{2}(b-a)(1-h)+\left|x-\frac{a+b}{2}\right|\right]^2\|f''\|_1$$
$$\leq\frac{(b-a)}{2}\left(1-\frac{h}{2}\right)^2\|f''\|_1 \tag{2.1}$$

for all $x\in\left[a+h\frac{b-a}{2},b-h\frac{b-a}{2}\right]$ *and* $h\in[0,1]$.

Proof. Let us define the mapping $K:[a,b]^2\to\mathbb{R}$ given in [3] by

$$K(x,t)=\begin{cases}\frac{1}{2}\left[t-\left(a+h\frac{b-a}{2}\right)\right]^2, & \text{if } t\in[a,x]\\ \frac{1}{2}\left[t-\left(b-h\frac{b-a}{2}\right)\right]^2, & \text{if } t\in(x,b]\end{cases}$$

The proof uses the following identity:

$$\int_a^b f(t)dt=(b-a)(1-h)f(x)-(b-a)(1-h)\left(x-\frac{a+b}{2}\right)f'(x) \tag{2.2}$$
$$+h\frac{b-a}{2}\left(f(a)+f(b)\right)-\frac{h^2(b-a)^2}{8}\left(f'(b)-f'(a)\right)+\int_a^b K(x,t)f''(t)\,dt$$

for all $x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right]$.

Using the identity (2.2), we have

$$\left| \begin{array}{l} (1-h)\, f(x) - \frac{1}{(b-a)} \int\limits_a^b f(t)dt + \frac{h}{2(1-h)} \left(f\left(a\right) + f\left(b\right)\right) \\ \\ -\frac{h^2(b-a)}{8} \left(f'\left(b\right) - f'\left(a\right)\right) - (1-h)\left(x - \frac{a+b}{2}\right) f'\left(x\right) \end{array} \right|$$

$$\begin{aligned}
&= \frac{1}{(b-a)} \left| \int\limits_a^b K(x,t) f''\left(t\right) dt \right| \\
&= \frac{1}{(b-a)} \left| \int\limits_a^x \frac{\left[t - \left(a + h\frac{b-a}{2}\right)\right]^2}{2} f''\left(t\right) dt + \int\limits_x^b \frac{\left[t - \left(b - h\frac{b-a}{2}\right)\right]^2}{2} f''\left(t\right) dt \right| \\
&\le \frac{1}{(b-a)} \left[\int\limits_a^x \frac{\left[t - \left(a + h\frac{b-a}{2}\right)\right]^2}{2} \left\| f''\left(t\right)\right\| dt + \int\limits_x^b \frac{\left[t - \left(b - h\frac{b-a}{2}\right)\right]^2}{2} \left\| f''\left(t\right)\right\| dt \right] \\
&\le \frac{1}{(b-a)} \left[\frac{\left[x - \left(a + h\frac{b-a}{2}\right)\right]^2}{2} \int\limits_a^x \left\| f''\left(t\right)\right\| dt + \frac{\left[\left(b - h\frac{b-a}{2}\right) - x\right]^2}{2} \int\limits_x^b \left\| f''\left(t\right)\right\| dt \right] \\
&\le \frac{1}{(b-a)} \max\left\{ \frac{\left[x - \left(a + h\frac{b-a}{2}\right)\right]^2}{2}, \frac{\left[\left(b - h\frac{b-a}{2}\right) - x\right]^2}{2} \right\} \\
&\times \left[\int\limits_a^x \left\| f''\left(t\right)\right\| dt + \int\limits_x^b \left\| f''\left(t\right)\right\| dt \right] \qquad (2.3)
\end{aligned}$$

Now, observe that

$$\max\left\{ \frac{\left[x - \left(a + h\frac{b-a}{2}\right)\right]^2}{2}, \frac{\left[\left(b - h\frac{b-a}{2}\right) - x\right]^2}{2} \right\}$$

$$= \frac{1}{2}\left[\frac{1}{2}\left(b-a\right)\left(1-h\right) + \left|x - \frac{a+b}{2}\right|\right]^2 \qquad (2.4)$$

Using (2.4) in (2.3), we get our required result (2.1). □

Remark 1. *For $h = 0$, in (2.1), we obtain Barnett's result (1.5) proved in [2]. It shows that our result contains Barnett's result (1.5) as a special case.*

Remark 2. *For $h = 1$ in (2.1), we obtain another result.*

$$\begin{aligned}
&\left| \frac{1}{2}\left(f\left(a\right) + f\left(b\right)\right) - \frac{(b-a)}{8}\left(f'\left(b\right) - f'\left(a\right)\right) - \frac{1}{b-a}\int\limits_a^b f(t)dt \right| \\
&\le \frac{1}{2\left(b-a\right)} \left[\left|x - \frac{a+b}{2}\right|\right]^2 \left\| f''\right\|_1 \\
&\le \frac{(b-a)}{8} \left\| f''\right\|_1 \qquad (2.5)
\end{aligned}$$

Hence, for different values of h*, we can obtain variety of results.*

Corollary 1. *If* f *is as in Theorem* 5 *, then we have the following perturbed midpoint inequality*

$$\left|\begin{array}{c}(1-h)\ f\left(\frac{a+b}{2}\right)+\frac{h}{2}\left(f\left(a\right)+f\left(b\right)\right)\\ \\ -\frac{h^2(b-a)}{8}\left(f'\left(b\right)-f'\left(a\right)\right)-\frac{1}{b-a}\int\limits_a^b f(t)dt\end{array}\right|$$

$$\leq \frac{1}{8}\left(b-a\right)\left(1-h\right)^2\left\|f''\right\|_1 \tag{2.6}$$

giving,

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int\limits_a^b f(t)dt\right|\leq\frac{(b-a)}{8}\left\|f''\right\|_1 \tag{2.7}$$

for $h=0$.

Corollary 2. *Let* f *be as in Theorem* 5*, then:*

$$\left|\begin{array}{c}(1-h)\,\frac{f(a)+f(b)}{2}-(1-h)\left(b-a\right)\frac{f'(b)-f'(a)}{4}+\frac{h}{2}\left(f\left(a\right)+f\left(b\right)\right)\\ \\ -\frac{h^2(b-a)}{8}\left(f'\left(b\right)-f'\left(a\right)\right)-\frac{1}{b-a}\int\limits_a^b f(t)dt\end{array}\right|$$

$$\leq\frac{(b-a)}{2}\left(1-\frac{h}{2}\right)^2\left\|f''\right\|_1 \tag{2.8}$$

Proof. Choose in (2.1), $x=a$ and $x=b$ to get:

$$\left|\begin{array}{c}(1-h)\ f(a)+(1-h)\left(\frac{b-a}{2}\right)f'\left(a\right)+\frac{h}{2}\left(f\left(a\right)+f\left(b\right)\right)\\ \\ -\frac{h^2(b-a)}{8}\left(f'\left(b\right)-f'\left(a\right)\right)-\frac{1}{b-a}\int\limits_a^b f(t)dt\end{array}\right|$$

$$\leq\frac{(b-a)}{2}\left(1-\frac{h}{2}\right)^2\left\|f''\right\|_1$$

and

$$\left|\begin{array}{c}(1-h)\ f(b)-(1-h)\left(\frac{b-a}{2}\right)f'\left(b\right)+\frac{h}{2}\left(f\left(a\right)+f\left(b\right)\right)\\ \\ -\frac{h^2(b-a)}{8}\left(f'\left(b\right)-f'\left(a\right)\right)-\frac{1}{b-a}\int\limits_a^b f(t)dt\end{array}\right|$$

$$\leq\frac{(b-a)}{2}\left(1-\frac{h}{2}\right)^2\left\|f''\right\|_1$$

Adding the above two inequalities, using the triangle inequality and dividing by 2, we get the desired inequality (2.8). □

Corollary 3. *Let f be as in Theorem 5, then we have the perturbed trapezoidal inequality:*

$$\left| \frac{f(a)+f(b)}{2} - (b-a)\frac{f'(b)-f'(a)}{4} - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{1}{6}(b-a)^2 \|f''\|_\infty . \tag{2.9}$$

Proof. Put $h=0$, in (2.8). □

Remark 3. *The estimation provided by* (2.8)*, is similar to that of the classical trapezoidal inequality.*

3. Applications in Numerical integration

Let $I_n : a = x_0 < x_1 < x_2 < .... < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$, $\xi_i \in \left[x_i + \delta\frac{h_i}{2}, x_{i+1} - \delta\frac{h_i}{2}\right]$, $(i = 0,1,.....,n-1)$ a sequence of intermediate points and $h_i = x_{i+1} - x_i$, $(i = 0,1,.....,n-1)$. then we have the following quadrature rule:

Theorem 6. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable on (a,b) whose second derivative and $f'' : (a,b) \to \mathbb{R}$ belongs to $L_1(a,b)$, i.e $\|f''\|_1 := \int_a^b \|f''\| dt < \infty$. Then the perturbed Riemann's quadrature formula holds:*

$$\int_a^b f(t)dt = A\left(f, f', I_n, \xi, \delta\right) + R\left(f, f', I_n, \xi, \delta\right) \tag{3.1}$$

where

$$A\left(f, f', I_n, \xi, \delta\right) = (1-\delta)\sum_{i=0}^{n-1} h_i f(\xi_i) - (1-\delta)\sum_{i=0}^{n-1} h_i\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right) f'(\xi_i)$$

$$+ \frac{\delta}{2}\sum_{i=0}^{n-1} h_i\left(f(x_i) + f(x_{i+1})\right) - \frac{\delta^2}{8}\sum_{i=0}^{n-1} h_i^2\left(f'(x_{i+1}) - f'(x_i)\right) \tag{3.2}$$

and the remainder $R\left(f, f', I_n, \xi, \delta\right)$ satisfies the estimation

$$\left|R\left(f, f', I_n, \xi, \delta\right)\right| \leq \frac{1}{2}\sum_{i=0}^{n-1}\left[\frac{h_i(1-\delta)}{2} + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]^2 \|f''\|_1 \leq \left(1 - \frac{\delta}{2}\right)^2 \sum_{i=0}^{n-1}\frac{h_i^2}{2}\|f''\|_1 \tag{3.3}$$

where $\delta \in [0,1]$ and $x_i + \delta\frac{h_i}{2} \leq \xi_i \leq x_{i+1} - \delta\frac{h_i}{2}$.

Proof. Apply Theorem 5 on the interval $[x_i, x_{i+1}]$, $(i = 0, 1, ...n-1)$, gives

$$\left| \begin{array}{c} (1-\delta) h_i f(\xi_i) - (1-\delta) h_i \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right) f'(\xi_i) + \frac{\delta}{2} h_i \left(f(x_i) + f(x_{i+1})\right) \\ -\frac{\delta^2}{8} h_i^2 \left(f'(x_{i+1}) - f'(x_i)\right) - \int\limits_{x_i}^{x_{i+1}} f(t)dt \end{array} \right|$$

$$\leq \frac{1}{2}\left[\frac{h_i(1-\delta)}{2} + \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]^2 \|f''\|_1$$

$$\leq \left(1 - \frac{\delta}{2}\right)^2 \frac{h_i^2}{2} \|f''\|_1$$

for any choice ξ of the intermediate points.

Summing over i from 0 to $n-1$ and using the generalized triangular inequality, we deduce the desired estimation (3.3). □

Corollary 4. *The following perturbed midpoint rule holds:*

$$\int\limits_a^b f(x)dx = M\left(f, f', I_n\right) + R_M\left(f, f', I_n\right),$$

where

$$M\left(f, f', I_n\right) = \sum_{i=0}^{n-1} h_i f\left(\frac{x_i + x_{i+1}}{2}\right) \tag{3.4}$$

and the remainder term $R_M\left(f, f', I_n\right)$ satisfies the estimation:

$$\left|R_M\left(f, f', I_n\right)\right| \leq \|f''\|_1 \sum_{i=0}^{n-1} \frac{h_i^2}{8} \tag{3.5}$$

Corollary 5. *The following perturbed trapezoidal rule holds:*

$$\int\limits_a^b f(x)dx = T\left(f, f', I_n\right) + R_T\left(f, f', I_n\right) \tag{3.6}$$

where

$$T\left(f, f', I_n\right) = \frac{1}{2}\sum_{i=0}^{n-1} h_i \left(f(x_i) + f(x_{i+1})\right) - \frac{1}{8}\sum_{i=0}^{n-1} h_i^2 \left(f'(x_{i+1}) - f'(x_i)\right) \tag{3.7}$$

and the remainder term

$$\left|R_T\left(f, f', I_n\right)\right| \leq \sum_{i=0}^{n-1} \frac{h_i^2}{8} \|f''\|_1 \tag{3.8}$$

Remark 4. *Note that the above mentioned perturbed midpoint formula* (3.4) *and perturbed trapezoid formula* (3.7) *can give better approximations of the integral* $\int\limits_a^b f(x)dx$ *for general classes of mappings.*

4. Application for some special means

Let us recall the following means:
The Arithmetic Mean

$$A = A(a,b) = \frac{a+b}{2}, \qquad a, b \geq 0;$$

The Geometric Mean

$$G = G(a,b) = \sqrt{ab}\ , \qquad a,\ b \geq 0;$$

The Harmonic Mean

$$H = H(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}}\ , \qquad a, b \geq 0;$$

The Logarithmic Mean

$$L = L(a,b) = \begin{cases} a & \text{if } \ a = b, \qquad a, b \geq 0; \\ \frac{b-a}{\ln b - \ln a} & \text{if } \ a \neq b\ ; \end{cases}$$

The Identric Mean

$$I = I(a,b) = \begin{cases} a & \text{if } \quad a = b, \quad a,\ b > 0; \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a \neq b \ \ ; \end{cases}$$

The P-Logarithmic Mean

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & \text{if } a \neq b \\ a & \text{if } \quad a = b, \end{cases}$$

where $p \in \mathbb{R}\backslash\{-1, 0\}$, $a, b > 0$.

The following simple relationships are known in this paper.

$$H \leq G \leq L \leq I \leq A$$

It is also known that L_p is monotonically increasing in $p \in \mathbb{R}$ with $L_0 = I$ and $L_{-1} = L$.

We may now apply inequality (2.1) , to deduce some inequalities for special means by the use of particular mappings as follows:

Remark 5. *Consider mapping* $f : (0, \infty) \to \mathbb{R},\ f(x) = x^r\ , r \in \mathbb{R}\backslash\{-1, 0\}$, *then we have for* $0 < a < b$
then

$$\frac{1}{b-a}\int_a^b f(t)dt = L_r^r\,(a,b)$$

and

$$\|f\,''\|_1 = \left|r\,(r-1)\,(b-a)\,L_{r-1}^{r-1}\,(a,b)\right|.$$

Using the inequality (2.1) *we get:*

$$\left|\begin{array}{c}(1-h)\,x^r-(1-h)\,(x-A)\,rx^{r-1}\\ +\frac{h}{2}\,(a^r+b^r)-\frac{h^2(b-a)r}{8}\left[b^{r-1}-a^{r-1}\right]-L_r^r\,(a,b)\end{array}\right|$$

$$\leq\frac{1}{2}\left[\frac{1}{2}\,(b-a)\,(1-h)+|x-A|\right]^2\left|r\,(r-1)\,L_{r-1}^{r-1}\,(a,b)\right| \quad (4.1)$$

If in (4.1), *we choose* $x=A$, *we get*

$$\left|(1-h)\,A^r+\frac{h}{2}\,(a^r+b^r)-\frac{h^2\,(b-a)\,r}{8}\left[b^{r-1}-a^{r-1}\right]-L_r^r\,(a,b)\right|$$

$$\leq\frac{1}{8}\left[(b-a)\,(1-h)\right]^2\left|r\,(r-1)\,L_{r-1}^{r-1}\,(a,b)\right| \quad (4.2)$$

also choosing $h=0$ *in* (4.2), *we get*

$$\left|A^r-L_r^r\,(a,b)\right|\leq\frac{1}{8}\,(b-a)^2\left|r\,(r-1)\,L_{r-1}^{r-1}\,(a,b)\right| \quad (4.3)$$

Remark 6. *Consider the mapping* $f:(0,\infty)\to\mathbb{R},\ \ f(x)=\frac{1}{x}$.
Then we have for $0<a<b$

$$\frac{1}{b-a}\int_a^b f(t)dt=L^{-1}\,(a,b)$$

and

$$\left\|f\,''\right\|_1=2\,(b-a)\,L_{-3}^{-3}\,(a,b)$$

Using the inequality (2.1) *we get:*

$$\left|(1-h)\ \frac{1}{x}+(1-h)\,(x-A)\,\frac{1}{x^2}+\frac{h}{H}\ -\frac{h^2\,(b-a)}{8}\left(\frac{b^2-a^2}{a^2b^2}\right)-L^{-1}\,(a,b)\right|$$

$$\leq\left[\frac{1}{2}\,(b-a)\,(1-h)+|x-A|\right]^2L_{-3}^{-3}\,(a,b) \quad (4.4)$$

If in (4.4), *we choose* $x=A$, *we get*

$$\left|(1-h)\ \frac{1}{A}+\frac{h}{H}\ -\frac{h^2\,(b-a)}{8}\left(\frac{b^2-a^2}{a^2b^2}\right)-L^{-1}\,(a,b)\right|$$

$$\leq\frac{1}{4}\,(b-a)^2\,(1-h)^2\,L_{-3}^{-3}\,(a,b) \quad (4.5)$$

also choosing $h=0$ *in* (4.5), *we get*

$$\left|\ A^{-1}\ -L^{-1}\,(a,b)\right|\leq\frac{1}{4}\,(b-a)^2\,L_{-3}^{-3}\,(a,b) \quad (4.6)$$

Remark 7. *Let us consider the mapping* $f(x) = \ln x$, $x \in [a,b] \subset (0,\infty)$.
Then we have:

$$\frac{1}{b-a}\int_a^b f(t)dt = \ln I(a,b)$$

and

$$\|f''\|_1 = (b-a) L_{-2}^{-2}(a,b)$$

Using the inequality (2.1) *we get:*

$$\left| \begin{array}{c} (1-h) \quad \ln x - (1-h)(x-A)\frac{1}{x} + \frac{h}{2}(\ln a + \ln b) \\ + \frac{h^2(b-a)^2}{8ab} - \ln I(a,b) \end{array} \right|$$

$$\leq \frac{1}{2}\left[\frac{1}{2}(b-a)(1-h) + |x-A|\right]^2 L_{-2}^{-2}(a,b) \tag{4.7}$$

If in (4.7)*, we choose* $x = A$*, we get*

$$\left|(1-h) \quad \ln A - \frac{h}{2}(\ln a + \ln b) + \frac{h^2(b-a)^2}{8ab} - \ln I(a,b)\right|$$

$$\leq \frac{1}{8}(b-a)^2 (1-h)^2 L_{-2}^{-2}(a,b) \tag{4.8}$$

also choosing $h = 0$ *in* (4.8)*, we get*

$$\left|\frac{A}{I}\right| \leq \exp\frac{(b-a)^2}{8} L_{-2}^{-2}(a,b) \tag{4.9}$$

5. Conclusion:

We established generalized Ostrowski type inequality for bounded differentiable mappings which generalizes the previous inequalities developed and discussed in [1],[2] and [3] . Perturbed midpoint and trapezoid inequalities are obtained. Some closely new results are also given.These generalized inequalities add up to the literature in the sense that they have immediate applications in Numerical Integration and Special Means. These generalized inequalities will also be useful for the researchers working in the field of the approximation theory, applied mathematics, probability theory, stochastic and numerical analysis to solve their problems in engineering and in practical life.

References

[1] S.S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_1-norm and applications to some special means and to some numerical quadrature rules, Tamkang J. of Math., 28 (1997), 239-244.

[2] N.S. Barnett, P. Cerone, S.S. Dragomir, J. Roumeliotis, A. Sofo, A survey on Ostrowski type inequalities for twice differentiable mappings and applications, Inequality Theory and Applications 1 (2001) 24–30.

[3] S.S.Dragomir and Th.M.Rassias(Eds.) Ostrowski type inequalities and Applications in Numerical Integration, Kluwer Academic Publishers,2002.

[4] A. Qayyum, S. S. Dragomir and M. Shoaib, A Weighted Ostrowski Type Inequality for $L_1[a,b]$ and Applications, RGMIA Research Report Collection, (17) (2014).
[5] A. Ostrowski, Uber die Absolutabweichung einer di erentienbaren Funktionen von ihren Integralimittelwert, Comment. Math. Hel. 10(1938), 226-227.
[6] G. H. Hardy, J. E. Littlewood and G. Polya, Inequalities, Cambridge University Press, 1934.
[7] E. F. Beckenbach and R. Bellman, Springer-Verlag, Berlin-Gottinggon-Heidelberg, 1961.
[8] D. S. Mitrinovi ́c , J. E. Pecari ́c , and A. M. Fink. "Classical and New Inequalities in Analysis". Kluwer Academic Publishers, Dordrecht (1991).
[9] D. S. Mitrinovi ́c , J. E. Pecari ́c, and A. M. Fink. "Inequalities Involving Functions and their Integrals and Derivatives". Kluwer Academic Publishers, Dordrecht (1991).
[10] S.S. Dragomir, P. Cerone and J. Roumeliotis, A new generalization of Ostrowski integral inequality for mappings whose derivatives are bounded and applications in numerical integration and for special means, Appl. Math. Lett., 13 (2000), 19–25.
[11] S.S. Dragomir and S. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, Appl. Math. Lett., 11 (1998), 105-109.

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