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INTEGRAL INEQUALITIES OF JENSEN TYPE FOR λ-CONVEX FUNCTIONS

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Abstract. Some integral inequalities of Jensen type for λ-convex functions defined on real intervals are given.

1. Introduction

1.1. h-Convex Functions. We recall here some concepts of convexity that are well known in the literature.

Let I be an interval in $\mathbb{R}$.

Definition 1 ([40]). *We say that $f: I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0,1)$ we have*

$$f(tx + (1-t)y) \leq \frac{1}{t} f(x) + \frac{1}{1-t} f(y). \tag{1.1}$$

Some further properties of this class of functions can be found in [30], [31], [33], [46], [49] and [50]. Among others, its has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

The above concept can be extended for functions $f: C \subseteq X \to [0, \infty)$ where C is a convex subset of the real or complex linear space X and the inequality (1.1) is satisfied for any vectors $x, y \in C$ and $t \in (0,1)$. If the function $f: C \subseteq X \to \mathbb{R}$ is non-negative and convex, then is of Godunova-Levin type.

Definition 2 ([33]). *We say that a function $f: I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0,1]$ we have*

$$f(tx + (1-t)y) \leq f(x) + f(y). \tag{1.2}$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions,* i. e. nonnegative functions satisfying

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\} \tag{1.3}$$

for all $x, y \in I$ and $t \in [0,1]$.

For some results on P-functions see [33] and [47] while for quasi convex functions, the reader can consult [32].

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If $f : C \subseteq X \to [0, \infty)$, where C is a convex subset of the real or complex linear space X, then we say that it is of P-type (or quasi-convex) if the inequality (1.2) (or (1.3)) holds true for $x, y \in C$ and $t \in [0, 1]$.

Definition 3 ([7]). *Let s be a real number, $s \in (0, 1]$. A function $f : [0, \infty) \to [0, \infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if*

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0, \infty)$ and $t \in [0, 1]$.

For some properties of this class of functions see [1], [2], [7], [8], [28], [29], [41], [43] and [52].

The concept of Breckner s-convexity can be similarly extended for functions defined on convex subsets of linear spaces.

It is well known that if $(X, \|\cdot\|)$ is a normed linear space, then the function $f(x) = \|x\|^p$, $p \ge 1$ is convex on X.

Utilising the elementary inequality $(a+b)^s \le a^s + b^s$ that holds for any $a, b \ge 0$ and $s \in (0, 1]$, we have for the function $g(x) = \|x\|^s$ that

$$\begin{aligned} g(tx + (1-t)y) &= \|tx + (1-t)y\|^s \le (t\|x\| + (1-t)\|y\|)^s \\ &\le (t\|x\|)^s + [(1-t)\|y\|]^s \\ &= t^s g(x) + (1-t)^s g(y) \end{aligned}$$

for any $x, y \in X$ and $t \in [0, 1]$, which shows that g is Breckner s-convex on X.

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}$, $(0, 1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 4 ([55]). *Let $h : J \to [0, \infty)$ with h not identical to 0. We say that $f : I \to [0, \infty)$ is an h-convex function if for all $x, y \in I$ we have*

$$f(tx + (1-t)y) \le h(t) f(x) + h(1-t) f(y) \tag{1.4}$$

for all $t \in (0, 1)$.

For some results concerning this class of functions see [55], [6], [44], [53], [51] and [54].

This concept can be extended for functions defined on convex subsets of linear spaces in the same way as above replacing the interval I be the corresponding convex subset C of the linear space X.

We can introduce now another class of functions.

Definition 5. *We say that the function $f : C \subseteq X \to [0, \infty)$ is of s-Godunova-Levin type, with $s \in [0, 1]$, if*

$$f(tx + (1-t)y) \le \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y), \tag{1.5}$$

for all $t \in (0, 1)$ and $x, y \in C$.

We observe that for $s = 0$ we obtain the class of P-functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(C)$ the class of s-Godunova-Levin functions defined on C, then we obviously have

$$P(C) = Q_0(C) \subseteq Q_{s_1}(C) \subseteq Q_{s_2}(C) \subseteq Q_1(C) = Q(C)$$

for $0 \leq s_1 \leq s_2 \leq 1$.

For different inequalities related to these classes of functions, see [1]-[4], [6], [9]-[39], [42]-[44] and [47]-[54].

A function $h : J \to \mathbb{R}$ is said to be *supermultiplicative* if

$$h(ts) \geq h(t) h(s) \text{ for any } t, s \in J. \tag{1.6}$$

If the inequality (1.6) is reversed, then h is said to be *submultiplicative*. If the equality holds in (1.6) then h is said to be a multiplicative function on J.

In [55] it has been noted that if $h : [0, \infty) \to [0, \infty)$ with $h(t) = (x + c)^{p-1}$, then for $c = 0$ the function h is multiplicative. If $c \geq 1$, then for $p \in (0, 1)$ the function h is supermultiplicative and for $p > 1$ the function is submultiplicative.

We observe that, if h, g are nonnegative and supermultiplicative, the same is their product. In particular, if h is supermultiplicative then its product with a power function $\ell_r(t) = t^r$ is also supermultiplicative.

1.2. **λ-Convex Functions.** We start with the following definition (see also [25]):

Definition 6. *Let $\lambda : [0, \infty) \to [0, \infty)$ be a function with the property that $\lambda(t) > 0$ for all $t > 0$. A mapping $f : C \to \mathbb{R}$ defined on convex subset C of a linear space X is called λ-convex on C if*

$$f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) \leq \frac{\lambda(\alpha) f(x) + \lambda(\beta) f(y)}{\lambda(\alpha + \beta)} \tag{1.7}$$

for all $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ and $x, y \in C$.

We observe that if $f : C \to \mathbb{R}$ is λ-convex on C, then f is h-convex on C with $h(t) = \frac{\lambda(t)}{\lambda(1)}$, $t \in [0, 1]$.

If $f : C \to [0, \infty)$ is h-convex function with h supermultiplicative on $[0, \infty)$, then f is λ-convex with $\lambda = h$.

Indeed, if $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ and $x, y \in C$ then

$$\begin{aligned} f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) &\leq h\left(\frac{\alpha}{\alpha + \beta}\right) f(x) + h\left(\frac{\beta}{\alpha + \beta}\right) f(y) \\ &\leq \frac{h(\alpha) f(x) + h(\beta) f(y)}{h(\alpha + \beta)}. \end{aligned}$$

The following proposition contain some properties of λ-convex functions [25].

Proposition 1. *Let $f : C \to \mathbb{R}$ be a λ-convex function on C.*

(i) If $\lambda(0) > 0$, then we have $f(x) \geq 0$ for all $x \in C$;

(ii) If there exists $x_0 \in C$ so that $f(x_0) > 0$, then

$$\lambda(\alpha + \beta) \leq \lambda(\alpha) + \lambda(\beta)$$

for all $\alpha, \beta > 0$, i.e. the mapping λ is subadditive on $(0, \infty)$.

(iii) If there exists $x_0, y_0 \in C$ with $f(x_0) > 0$ and $f(y_0) < 0$, then

$$\lambda(\alpha + \beta) = \lambda(\alpha) + \lambda(\beta)$$

for all $\alpha, \beta > 0$, i.e. the mapping λ is additive on $(0, \infty)$.

We have the following result providing many examples of subadditive functions $\lambda : [0, \infty) \to [0, \infty)$.

Theorem 1 ([25])**.** *Let $h(z)=\sum_{n=0}^{\infty}a_nz^n$ a power series with nonnegative coefficients $a_n\geq 0$ for all $n\in\mathbb{N}$ and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$. If $r\in(0,R)$ then the function $\lambda_r:[0,\infty)\to[0,\infty)$ given by*

$$\lambda_r(t):=\ln\left[\frac{h(r)}{h(r\exp(-t))}\right] \tag{1.8}$$

is nonnegative, increasing and subadditive on $[0,\infty)$.

We have the following fundamental examples of power series with positive coefficients

$$h(z)=\sum_{n=0}^{\infty}z^n=\frac{1}{1-z},\ z\in D(0,1) \tag{1.9}$$
$$h(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^n=\exp(z)\qquad z\in\mathbb{C},$$
$$h(z)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n}=\cosh z,\ z\in\mathbb{C};$$
$$h(z)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1}=\sinh z,\ z\in\mathbb{C};$$
$$h(z)=\sum_{n=1}^{\infty}\frac{1}{n}z^n=\ln\frac{1}{1-z},\ z\in D(0,1).$$

Other important examples of functions as power series representations with positive coefficients are:

$$h(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}=\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right),\qquad z\in D(0,1); \tag{1.10}$$
$$h(z)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}z^{2n+1}=\sin^{-1}(z),\qquad z\in D(0,1);$$
$$h(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}=\tanh^{-1}(z),\qquad z\in D(0,1);$$
$$h(z)={}_2F_1(\alpha,\beta,\gamma,z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}z^n,\alpha,\beta,\gamma>0,$$
$$z\in D(0,1);$$

where Γ is *Gamma function.*

Remark 1. *Now, if we take $h(z)=\frac{1}{1-z}$, $z\in D(0,1)$, then*

$$\lambda_r(t)=\ln\left[\frac{1-r\exp(-t)}{1-r}\right] \tag{1.11}$$

is nonnegative, increasing and subadditive on $[0,\infty)$ for any $r\in(0,1)$.

If we take $h(z)=\exp(z)$, $z\in\mathbb{C}$ then

$$\lambda_r(t)=r[1-\exp(-t)] \tag{1.12}$$

is nonnegative, increasing and subadditive on $[0,\infty)$ for any $r>0$.

Corollary 1 ([25])**.** *Let $h(z)=\sum_{n=0}^{\infty}a_n z^n$ a power series with nonnegative coefficients $a_n\geq 0$ for all $n\in\mathbb{N}$ and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$ and $r\in(0,R)$. For a mapping $f:C\to\mathbb{R}$ defined on convex subset C of a linear space X, the following statements are equivalent:*

(i) The function f is λ_r-convex with $\lambda_r:[0,\infty)\to[0,\infty)$,

$$\lambda_r(t):=\ln\left[\frac{h(r)}{h(r\exp(-t))}\right];$$

(ii) We have the inequality

$$\left[\frac{h(r)}{h(r\exp(-\alpha-\beta))}\right]^{f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)}\leq\left[\frac{h(r)}{h(r\exp(-\alpha))}\right]^{f(x)}\left[\frac{h(r)}{h(r\exp(-\beta))}\right]^{f(y)}\tag{1.13}$$

for any $\alpha,\beta\geq 0$ with $\alpha+\beta>0$ and $x,y\in C$.

(iii) We have the inequality

$$\frac{[h(r\exp(-\alpha))]^{f(x)}[h(r\exp(-\beta))]^{f(y)}}{[h(r\exp(-\alpha-\beta))]^{f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)}}\leq[h(r)]^{f(x)+f(y)-f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)}\tag{1.14}$$

for any $\alpha,\beta\geq 0$ with $\alpha+\beta>0$ and $x,y\in C$.

Remark 2. *We observe that, in the case when*

$$\lambda_r(t)=r[1-\exp(-t)],\ t\geq 0$$

then the function f is λ_r-convex on convex subset C of a linear space X iff

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)\leq\frac{[1-\exp(-\alpha)]f(x)+[1-\exp(-\beta)]f(y)}{1-\exp(-\alpha-\beta)}\tag{1.15}$$

for any $\alpha,\beta\geq 0$ with $\alpha+\beta>0$ and $x,y\in C$.

We observe that this definition is independent of $r>0$.

The inequality (1.15) is equivalent with

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)\leq\frac{\exp(\beta)[\exp(\alpha)-1]f(x)+\exp(\alpha)[\exp(\beta)-1]f(y)}{\exp(\alpha+\beta)-1}\tag{1.16}$$

for any $\alpha,\beta\geq 0$ with $\alpha+\beta>0$ and $x,y\in C$.

2. Unweighted Jensen Integral Inequalities

The following discrete inequality of Jensen's type has been obtained in [26]:

Theorem 2. *Let $\lambda:[0,\infty)\to[0,\infty)$ be a function with the property that $\lambda(t)>0$ for all $t>0$ and a mapping $f:C\to\mathbb{R}$ defined on convex subset C of a linear space X. The following statements are equivalent:*

(i) f is λ-convex on C;

(ii) For all $x_i\in C$ and $p_i\geq 0$ with $i\in\{1,...,n\}$, $n\geq 2$ so that $P_n>0$, we have the inequality

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i x_i\right)\leq\frac{1}{\lambda(P_n)}\sum_{i=1}^{n}\lambda(p_i)f(x_i).\tag{2.1}$$

The proof can be done by induction over $n\geq 2$.

Corollary 2. *Let $f : C \to \mathbb{R}$ be a λ-convex function on C and $\alpha_i \in [0,1]$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} \alpha_i = 1$. Then for any $x_i \in C$ with $i \in \{1, ..., n\}$ we have the inequality*

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \frac{1}{\lambda(1)} \sum_{i=1}^{n} \lambda(\alpha_i) f(x_i). \tag{2.2}$$

In particular, we have

$$f\left(\frac{x_1 + ... + x_n}{n}\right) \leq c(n) \frac{f(x_1) + ... + f(x_n)}{n}, \tag{2.3}$$

where

$$c(n) := \frac{n\lambda\left(\frac{1}{n}\right)}{\lambda(1)}.$$

We have the following version of Jensen's inequality:

Corollary 3. *Let $f : C \to \mathbb{R}$ be a λ-convex function on C and $x_i \in C$ and $p_i \geq 0$ with $i \in \{1, ..., n\}$, $n \geq 2$ so that $P_n > 0$. Then we have the inequality*

$$f\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{\lambda(1)} \sum_{i=1}^{n} \lambda\left(\frac{p_i}{P_n}\right) f(x_i). \tag{2.4}$$

The proof follows by (2.2) for $\alpha_i = \frac{p_i}{P_n}$, $i \in \{1, ..., n\}$.

Corollary 4. *Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ a power series with nonnegative coefficients $a_n \geq 0$ for all $n \in \mathbb{N}$ and convergent on the open disk $D(0, R)$ with $R > 0$ or $R = \infty$. For a mapping $f : C \to \mathbb{R}$ defined on convex subset C of a linear space X, the following statements are equivalent:*

(i) The function f is λ_r-convex with $\lambda_r : [0, \infty) \to [0, \infty)$

$$\lambda_r(t) := \ln\left[\frac{h(r)}{h(r\exp(-t))}\right]$$

on C;

(ii) We have the inequality

$$\left[\frac{h(r)}{h(r\exp(-P_n))}\right]^{f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)} \leq \prod_{i=1}^{n} \left[\frac{h(r)}{h(r\exp(-p_i))}\right]^{f(x_i)} \tag{2.5}$$

for any $x_i \in C$ and $p_i \geq 0$ with $i \in \{1, ..., n\}$, $n \geq 2$ so that $P_n > 0$.

We have the following Jensen inequality for the Riemann integral:

Theorem 3. *Let $u : [a, b] \to [m, M]$ be a Riemann integrable function on $[a, b]$. Let $\lambda : [0, \infty) \to [0, \infty)$ be a function with the property that $\lambda(t) > 0$ for all $t > 0$ and the function $f : [m, M] \to [0, \infty)$ is λ-convex and Riemann integrable on the interval $[m, M]$. If the following limit exists*

$$\lim_{t \to 0+} \frac{\lambda(t)}{t} = k \in (0, \infty) \tag{2.6}$$

then

$$f\left(\frac{1}{b-a} \int_a^b u(t)\, dt\right) \leq \frac{k}{\lambda(b-a)} \int_a^b f(u(t))\, dt. \tag{2.7}$$

Proof. Consider the sequence of divisions

$$d_n:\ x_i^{(n)}=a+\frac{i}{n}(b-a),\ i\in\{0,...,n\}$$

and the intermediate points

$$\xi_i^{(n)}=a+\frac{i}{n}(b-a),\ i\in\{0,...,n\}.$$

We observe that the norm of the division $\Delta_n:=\max_{i\in\{0,\dots,n-1\}}\left(x_{i+1}^{(n)}-x_i^{(n)}\right)=\frac{b-a}{n}\to 0$ as $n\to\infty$ and since u is Riemann integrable on $[a,b]$, then

$$\begin{aligned}\int_a^b u(t)\,dt &= \lim_{n\to\infty}\sum_{i=0}^{n-1}u\left(\xi_i^{(n)}\right)\left[x_{i+1}^{(n)}-x_i^{(n)}\right]\\ &= \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=0}^{n-1}u\left(a+\frac{i}{n}(b-a)\right).\end{aligned}$$

Also, since $f:[m,M]\to[0,\infty)$ is Riemann integrable, then $f\circ u$ is Riemann integrable and

$$\int_a^b f(u(t))\,dt=\lim_{n\to\infty}\frac{b-a}{n}\sum_{i=0}^{n-1}f\left[u\left(a+\frac{i}{n}(b-a)\right)\right].$$

Utilising the inequality (2.1) for $p_i:=\frac{b-a}{n}$ and $x_i:=u\left(a+\frac{i}{n}(b-a)\right)$ we have

$$\begin{aligned}&f\left(\frac{1}{b-a}\frac{b-a}{n}\sum_{i=0}^{n-1}u\left(a+\frac{i}{n}(b-a)\right)\right)\\ &\le\frac{1}{\lambda(b-a)}\sum_{i=0}^{n-1}\lambda\left(\frac{b-a}{n}\right)f\left(u\left(a+\frac{i}{n}(b-a)\right)\right)\\ &=\frac{1}{\lambda(b-a)}\lambda\left(\frac{b-a}{n}\right)\sum_{i=0}^{n-1}f\left(u\left(a+\frac{i}{n}(b-a)\right)\right)\\ &=\frac{n}{\lambda(b-a)(b-a)}\lambda\left(\frac{b-a}{n}\right)\frac{b-a}{n}\sum_{i=0}^{n-1}f\left(u\left(a+\frac{i}{n}(b-a)\right)\right)\end{aligned}\tag{2.8}$$

for any $n\ge 1$.

Observe that

$$\lim_{n\to\infty}\frac{\lambda\left(\frac{b-a}{n}\right)}{\frac{b-a}{n}}=\lim_{t\to 0+}\frac{\lambda(t)}{t}=k\in(0,\infty),$$

and by taking the limit over $n\to\infty$ in the inequality (2.8), we deduce the desired result (2.7). □

Corollary 5. *Let $u:[a,b]\to[m,M]$ be a Riemann integrable function on $[a,b]$ and $h(z)=\sum_{n=0}^{\infty}a_nz^n$ a power series with nonnegative coefficients $a_n\ge 0$ for all $n\in\mathbb{N}$ and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$ and $r\in(0,R)$. Let $\lambda_r:[0,\infty)\to[0,\infty)$ be given by*

$$\lambda_r(t):=\ln\left[\frac{h(r)}{h(r\exp(-t))}\right]$$

and the function $f:[m,M]\to[0,\infty)$ is λ_r-convex and Riemann integrable on the interval $[m,M]$. Then

$$f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right)\le\frac{rh'(r)}{h(r)\ln\left[\frac{h(r)}{h(r\exp(-(b-a)))}\right]}\int_a^b f(u(t))\,dt. \tag{2.9}$$

Proof. We know that λ_r is differentiable on $(0,\infty)$ and

$$\lambda_r'(t):=\frac{r\exp(-t)\,h'(r\exp(-t))}{h(r\exp(-t))}$$

for $t\in(0,\infty)$, where

$$h'(z)=\sum_{n=1}^{\infty}na_nz^{n-1}.$$

Since $\lambda_r(0)=0$, therefore

$$k=\lim_{s\to0+}\frac{\lambda(s)}{s}=\lambda_+'(0)=\frac{rh'(r)}{h(r)}>0\text{ for }r\in(0,R).$$

Utilising (2.7) we get the desired result (2.9). □

The following Hermite-Hadamard type inequality holds:

Corollary 6. *With the assumptions of Theorem 3 for f and λ and if $[a,b]=[m,M]$, then we have the Hermite-Hadamard type inequality*

$$f\left(\frac{a+b}{2}\right)\le\frac{k}{\lambda(b-a)}\int_a^b f(t)\,dt. \tag{2.10}$$

Remark 3. *Assume that the function $f:[m,M]\to[0,\infty)$ is λ-convex and Riemann integrable on the interval $[m,M]$ with*

$$\lambda(t)=1-\exp(-t),\ t\ge0.$$

If $u:[a,b]\to[m,M]$ is a Riemann integrable function on $[a,b]$, then

$$f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right)\le\frac{1}{1-\exp(-(b-a))}\int_a^b f(u(t))\,dt.$$

In particular, for $[a,b]=[m,M]$ and $u(t)=t$ we have the Hermite-Hadamard type inequality

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{1-\exp(-(b-a))}\int_a^b f(t)\,dt.$$

The proof follows from (2.7) observing that

$$k=\lim_{t\to0+}\frac{\lambda(t)}{t}=\lambda_+'(0)=1.$$

Utilising a similar argument and the inequality (2.4) we can state the following result as well:

Theorem 4. *Let $u:[a,b]\to[m,M]$ be a Riemann integrable function on $[a,b]$. Let $\lambda:[0,\infty)\to[0,\infty)$ be a function with the property that $\lambda(t)>0$ for all $t>0$*

and the function $f:[m,M]\to[0,\infty)$ *is* λ*-convex and Riemann integrable on the interval* $[m,M]$. *If the limit (2.6) exists, then*

$$f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right)\le\frac{k}{\lambda(1)(b-a)}\int_a^b f(u(t))\,dt. \tag{2.11}$$

Examples of such inequalities are incorporated below:

Corollary 7. *Let* $u:[a,b]\to[m,M]$ *be a Riemann integrable function on* $[a,b]$ *and* $h(z)=\sum_{n=0}^{\infty}a_nz^n$ *a power series with nonnegative coefficients* $a_n\ge0$ *for all* $n\in\mathbb{N}$ *and convergent on the open disk* $D(0,R)$ *with* $R>0$ *or* $R=\infty$ *and* $r\in(0,R)$. *Let* $\lambda_r:[0,\infty)\to[0,\infty)$ *be given by*

$$\lambda_r(t):=\ln\left[\frac{h(r)}{h(r\exp(-t))}\right]$$

and the function $f:[m,M]\to[0,\infty)$ *is* λ_r*-convex and Riemann integrable on the interval* $[m,M]$. *Then*

$$f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right)\le\frac{rh'(r)}{h(r)\ln\left[\frac{h(r)}{h(re^{-1})}\right](b-a)}\int_a^b f(u(t))\,dt. \tag{2.12}$$

We also have the Hermite-Hadamard type inequality:

Corollary 8. *With the assumptions of Theorem 4 for* f *and* λ *and if* $[a,b]=[m,M]$, *then we have the Hermite-Hadamard type inequality*

$$f\left(\frac{a+b}{2}\right)\le\frac{k}{\lambda(1)(b-a)}\int_a^b f(t)\,dt. \tag{2.13}$$

Remark 4. *Assume that the function* $f:[m,M]\to[0,\infty)$ *is* λ*-convex and Riemann integrable on the interval* $[m,M]$ *with*

$$\lambda(t)=1-\exp(-t),\ t\ge0.$$

If $u:[a,b]\to[m,M]$ *is a Riemann integrable function on* $[a,b]$, *then*

$$f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right)\le\frac{e}{e-1}\cdot\frac{1}{b-a}\int_a^b f(u(t))\,dt.$$

In particular, for $[a,b]=[m,M]$ *and* $u(t)=t$ *we have the Hermite-Hadamard type inequality*

$$f\left(\frac{a+b}{2}\right)\le\frac{e}{e-1}\cdot\frac{1}{b-a}\int_a^b f(t)\,dt.$$

3. Weighted Jensen Integral Inequalities

We can prove now a weighted version of Jensen inequality.

Theorem 5. *Let* $u,w:[a,b]\to[m,M]$ *be Riemann integrable functions on* $[a,b]$ *and* $w(t)\ge0$ *for any* $t\in[a,b]$ *with* $\int_a^b w(t)\,dt>0$. *Let* $\lambda:[0,\infty)\to[0,\infty)$ *be a function with the property that* $\lambda(t)>0$ *for all* $t>0$ *and the function* $f:[m,M]\to[0,\infty)$ *is* λ*-convex and Riemann integrable on the interval* $[m,M]$. *If the following limit exists, is finite and*

$$\lim_{t\to\infty}\frac{t}{\lambda(t)}=\ell>0, \tag{3.1}$$

then

$$f\left(\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\,u(t)\,dt\right)\leq \ell\frac{1}{\int_a^b w(t)\,dt}\int_a^b \lambda(w(t))\,f(u(t))\,dt. \tag{3.2}$$

Proof. Consider the sequence of divisions

$$d_n:\ x_i^{(n)}=a+\frac{i}{n}(b-a),\ i\in\{0,...,n\}$$

and the intermediate points

$$\xi_i^{(n)}=a+\frac{i}{n}(b-a),\ i\in\{0,...,n\}.$$

We observe that the norm of the division $\Delta_n:=\max_{i\in\{0,\dots,n-1\}}\left(x_{i+1}^{(n)}-x_i^{(n)}\right)=\frac{b-a}{n}\to 0$ as $n\to\infty$.

If we write the inequality (2.1) for the sequences

$$p_i=w\left(a+\frac{i}{n}(b-a)\right)\text{ and }x_i=u\left(a+\frac{i}{n}(b-a)\right),\ i\in\{0,...,n\}$$

we get

$$\begin{aligned}(3.3)\quad & f\left(\frac{1}{\sum_{i=0}^{n-1}w\left(a+\frac{i}{n}(b-a)\right)}\sum_{i=0}^{n-1}w\left(a+\frac{i}{n}(b-a)\right)u\left(a+\frac{i}{n}(b-a)\right)\right)\\ &\leq\frac{1}{\lambda\left(\sum_{i=0}^{n-1}w\left(a+\frac{i}{n}(b-a)\right)\right)}\\ &\times\sum_{i=0}^{n-1}\lambda\left(w\left(a+\frac{i}{n}(b-a)\right)\right)f\left(u\left(a+\frac{i}{n}(b-a)\right)\right),\end{aligned}$$

for $n\geq 1$.

Observe that

$$\begin{aligned}& f\left(\frac{1}{\sum_{i=0}^{n-1}w\left(a+\frac{i}{n}(b-a)\right)}\sum_{i=0}^{n-1}w\left(a+\frac{i}{n}(b-a)\right)u\left(a+\frac{i}{n}(b-a)\right)\right)\\ &=f\left(\frac{1}{\frac{b-a}{n}\sum_{i=0}^{n-1}w\left(a+\frac{i}{n}(b-a)\right)}\right.\\ &\left.\times\frac{b-a}{n}\sum_{i=0}^{n-1}w\left(a+\frac{i}{n}(b-a)\right)u\left(a+\frac{i}{n}(b-a)\right)\right)\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\lambda\left(\sum_{i=0}^{n-1} w\left(a+\frac{i}{n}(b-a)\right)\right)} \\
&\times \sum_{i=0}^{n-1} \lambda\left(w\left(a+\frac{i}{n}(b-a)\right)\right) f\left(u\left(a+\frac{i}{n}(b-a)\right)\right) \\
&= \frac{\sum_{i=0}^{n-1} w\left(a+\frac{i}{n}(b-a)\right)}{\lambda\left(\sum_{i=0}^{n-1} w\left(a+\frac{i}{n}(b-a)\right)\right)} \times \frac{1}{\frac{b-a}{n}\sum_{i=0}^{n-1} w\left(a+\frac{i}{n}(b-a)\right)} \\
&\times \frac{b-a}{n}\sum_{i=0}^{n-1} \lambda\left(w\left(a+\frac{i}{n}(b-a)\right)\right) f\left(u\left(a+\frac{i}{n}(b-a)\right)\right).
\end{aligned}$$

Then from (3.3) we get

$$\begin{aligned}
&f\left(\frac{1}{\frac{b-a}{n}\sum_{i=0}^{n-1} w\left(a+\frac{i}{n}(b-a)\right)}\right. \\
&\left.\times \frac{b-a}{n}\sum_{i=0}^{n-1} w\left(a+\frac{i}{n}(b-a)\right) u\left(a+\frac{i}{n}(b-a)\right)\right) \\
&\leq \frac{\sum_{i=0}^{n-1} w\left(a+\frac{i}{n}(b-a)\right)}{\lambda\left(\sum_{i=0}^{n-1} w\left(a+\frac{i}{n}(b-a)\right)\right)} \times \frac{1}{\frac{b-a}{n}\sum_{i=0}^{n-1} w\left(a+\frac{i}{n}(b-a)\right)} \\
&\times \frac{b-a}{n}\sum_{i=0}^{n-1} \lambda\left(w\left(a+\frac{i}{n}(b-a)\right)\right) f\left(u\left(a+\frac{i}{n}(b-a)\right)\right)
\end{aligned} \tag{3.4}$$

for all $n \geq 1$.

Since

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{i=0}^{n-1} w\left(a+\frac{i}{n}(b-a)\right) \\
&= \lim_{n\to\infty} \frac{b-a}{n}\sum_{i=0}^{n-1} w\left(a+\frac{i}{n}(b-a)\right) \times \lim \frac{n}{b-a} \\
&= \int_a^b w(t)\,dt \times \infty = \infty,
\end{aligned}$$

then

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} w\left(a+\frac{i}{n}(b-a)\right)}{\lambda\left(\sum_{i=0}^{n-1} w\left(a+\frac{i}{n}(b-a)\right)\right)} = \lim_{n\to\infty} \frac{t}{\lambda(t)} = \ell$$

and by letting $n \to \infty$ in (3.4) we get the desired result (3.2). □

The following unweighted version of Jensen inequality holds:

Corollary 9. *Let $u : [a, b] \to [m, M]$ be a Riemann integrable function on $[a, b]$. Let $\lambda : [0, \infty) \to [0, \infty)$ be a function with the property that $\lambda(t) > 0$ for all $t > 0$ and the function $f : [m, M] \to [0, \infty)$ is λ-convex and Riemann integrable on the* *interval* $[m,M]$. *If the limit (3.1) exists, then*

$$f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right) \le \ell\lambda(1)\,\frac{1}{b-a}\int_a^b f(u(t))\,dt. \tag{3.5}$$

Moreover, if $[a,b]=[m,M]$, *then by taking* $u(t)=t$, $t\in[a,b]$, *we have the Hermite-Hadamard inequality*

$$f\left(\frac{a+b}{2}\right) \le \ell\lambda(1)\,\frac{1}{b-a}\int_a^b f(t)\,dt. \tag{3.6}$$

Remark 5. *In order to give examples of subadditive functions* $\lambda:[0,\infty)\to[0,\infty)$ *with the property that* $\lambda(t)>0$ *for all* $t>0$ *and for which the following limit exists, is finite and*

$$\lim_{t\to\infty}\frac{t}{\lambda(t)}=\ell>0, \tag{3.7}$$

we consider the power series $h(z)=\sum_{n=1}^{\infty}a_n z^n$ *with nonnegative coefficients* $a_n\ge 0$ *for all* $n\ge 1$, $a_1>0$ *and convergent on the open disk* $D(0,R)$ *with* $R>0$ *or* $R=\infty$.

Such examples are the functions

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z},\ \ z\in D(0,1)$$

$$\sum_{n=1}^{\infty}\frac{1}{n!}z^n = \exp(z)-1,\ \ z\in\mathbb{C},$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1} = \sinh z,\ \ z\in\mathbb{C};$$

$$\sum_{n=1}^{\infty}\frac{1}{n}z^n = \ln\frac{1}{1-z},\ \ z\in D(0,1)$$

and others.

Let $\lambda_r:[0,\infty)\to[0,\infty)$ *be given by*

$$\lambda_r(t):=\ln\left[\frac{h(r)}{h(r\exp(-t))}\right].$$

We know that λ_r *is differentiable on* $(0,\infty)$ *and*

$$\lambda_r'(t):=\frac{r\exp(-t)\,h'(r\exp(-t))}{h(r\exp(-t))}$$

for $t\in(0,\infty)$, *where*

$$h'(z)=\sum_{n=1}^{\infty} n a_n z^{n-1}.$$

By l'Hospital's rule we have

$$\lim_{t\to\infty}\frac{t}{\lambda_r(t)}=\lim_{t\to\infty}\frac{1}{\lambda_r'(t)}.$$

Since for the power series $h(z) = a_1 z + a_2 z^2 + a_3 z^3 + \dots$ *we have* $h'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \dots$ *then*

$$\lambda_r'(t) = \frac{r\exp(-t)\left(a_1 + 2a_2 r\exp(-t) + 3a_3 (r\exp(-t))^2 + \dots\right)}{r\exp(-t)\left(a_1 + a_2 r\exp(-t) + a_3 (r\exp(-t))^2 + \dots\right)}$$

$$= \frac{a_1 + 2a_2 r\exp(-t) + 3a_3 (r\exp(-t))^2 + \dots}{a_1 + a_2 r\exp(-t) + a_3 (r\exp(-t))^2 + \dots},\ t \in (0,\infty).$$

Therefore $\lim_{t\to\infty} \lambda_r'(t) = 1$ *and*

$$\lim_{t\to\infty} \frac{t}{\lambda_r(t)} = 1.$$

Corollary 10. *Let* $u, w : [a,b] \to [m,M]$ *be Riemann integrable functions on* $[a,b]$ *and* $w(t) \geq 0$ *for any* $t \in [a,b]$ *with* $\int_a^b w(t)\,dt > 0$. *Consider the power series* $h(z) = \sum_{n=1}^{\infty} a_n z^n$ *with nonnegative coefficients* $a_n \geq 0$ *for all* $n \geq 1$, $a_1 > 0$ *and convergent on the open disk* $D(0,R)$ *with* $R > 0$ *or* $R = \infty$. *Let* $r \in (0,R)$ *and assume that the function* $f : [m,M] \to [0,\infty)$ *is* λ_r*-convex and Riemann integrable on the interval* $[m,M]$ *with*

$$\lambda_r(t) := \ln\left[\frac{h(r)}{h(r\exp(-t))}\right].$$

Then we have the inequality

$$f\left(\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\,u(t)\,dt\right) \leq \frac{1}{\int_a^b w(t)\,dt}\int_a^b \ln\left[\frac{h(r)}{h(r\exp(-w(t)))}\right] f(u(t))\,dt. \tag{3.8}$$

The proof follows by Theorem 5 observing that $\ell = 1$.

Remark 6. *With the assumptions of Corollary 10 for* u, h *and* f *we have*

$$f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right) \leq \ln\left[\frac{h(r)}{h(re^{-1})}\right]\frac{1}{b-a}\int_a^b f(u(t))\,dt. \tag{3.9}$$

In particular, for $[a,b] = [m,M]$ *we have the Hermite-Hadamard inequality*

$$f\left(\frac{a+b}{2}\right) \leq \ln\left[\frac{h(r)}{h(re^{-1})}\right]\frac{1}{b-a}\int_a^b f(t)\,dt. \tag{3.10}$$

4. Interval Dependency

Let $u : [a,b] \to [m,M]$ be a Riemann integrable function on $[a,b]$. Let $\lambda : [0,\infty) \to [0,\infty)$ be a function with the property that $\lambda(t) > 0$ for all $t > 0$ and the function $f : [m,M] \to [0,\infty)$ is λ-convex and Riemann integrable on the interval $[m,M]$. Assume also that the following limit exists

$$\lim_{t\to 0+} \frac{\lambda(t)}{t} = k \in (0,\infty).$$

By Theorem 3 we have that

$$\Delta(f,u,\lambda;[a,b]) := \int_a^b f(u(t))\,dt - \frac{1}{k}\lambda(b-a) f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right) \geq 0. \tag{4.1}$$

Theorem 6. *With the above assumptions for u, λ, f and k we have:*
(i) For any $c \in (a,b)$ we have

$$\Delta(f,u,\lambda;[a,b]) \geq \Delta(f,u,\lambda;[a,c]) + \Delta(f,u,\lambda;[c,b]) \geq 0, \tag{4.2}$$

i.e. $\Delta(f,u,\lambda;\cdot)$ is a superadditive function of intervals.
(ii) For any $c,d \in (a,b)$ with $c<d$ we have

$$\Delta(f,u,\lambda;[a,b]) \geq \Delta(f,u,\lambda;[c,d]) \geq 0, \tag{4.3}$$

i.e. $\Delta(f,u,\lambda;\cdot)$ is a monotonic nondecreasing function of intervals.

Proof. (i) By the λ-convexity of f we have for $c \in (a,b)$ that

$$\begin{aligned}
& f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right) \\
&= f\left(\frac{1}{b-a}\left(\int_a^c u(t)\,dt + \int_c^b u(t)\,dt\right)\right) \\
&= f\left(\frac{c-a}{b-a}\left(\frac{1}{c-a}\int_a^c u(t)\,dt\right) + \frac{b-c}{b-a}\left(\frac{1}{b-c}\int_c^b u(t)\,dt\right)\right) \\
&\leq \frac{\lambda(c-a) f\left(\frac{1}{c-a}\int_a^c u(t)\,dt\right) + \lambda(b-c) f\left(\frac{1}{b-c}\int_c^b u(t)\,dt\right)}{\lambda(b-a)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \Delta(f,u,\lambda;[a,b]) \\
&= \int_a^c f(u(t))\,dt + \int_c^b f(u(t))\,dt - \frac{1}{k}\lambda(b-a) f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right) \\
&\geq \int_a^c f(u(t))\,dt + \int_c^b f(u(t))\,dt - \frac{1}{k}\lambda(b-a) \\
&\times \left[\frac{\lambda(c-a) f\left(\frac{1}{c-a}\int_a^c u(t)\,dt\right) + \lambda(b-c) f\left(\frac{1}{b-c}\int_c^b u(t)\,dt\right)}{\lambda(b-a)}\right] \\
&= \int_a^c f(u(t))\,dt - \frac{1}{k}\lambda(c-a) f\left(\frac{1}{c-a}\int_a^c u(t)\,dt\right) \\
&+ \int_c^b f(u(t))\,dt - \frac{1}{k}\lambda(b-c) f\left(\frac{1}{b-c}\int_c^b u(t)\,dt\right) \\
&= \Delta(f,u,\lambda;[a,c]) + \Delta(f,u,\lambda;[c,b])
\end{aligned}$$

and the inequality (4.2) is proved.

(ii) For any $c,d \in (a,b)$ with $c<d$ we have on applying the property (4.2) that

$$\begin{aligned}\Delta(f,u,\lambda;[a,b]) &\geq \Delta(f,u,\lambda;[a,c]) + \Delta(f,u,\lambda;[c,b]) \\ &\geq \Delta(f,u,\lambda;[a,c]) + \Delta(f,u,\lambda;[c,d]) + \Delta(f,u,\lambda;[d,b]) \\ &\geq \Delta(f,u,\lambda;[c,d])\end{aligned}$$

and the proof is concluded. □

Remark 7. *If $[a,b]=[m,M]$ and $u(t)=t$, $t\in[a,b]$ then the functional*

$$\delta(f,\lambda;[a,b]) := \int_a^b f(t)\,dt - \frac{1}{k}\lambda(b-a)\,f\left(\frac{a+b}{2}\right) \geq 0$$

is a superadditive and monotonic nondecreasing function of intervals.

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