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PERTURBED COMPANIONS OF OSTROWSKI'S INEQUALITY FOR ABSOLUTELY CONTINUOUS FUNCTIONS (II)

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Abstract. Perturbed companions of Ostrowski's inequality for absolutely continuous functions whose derivatives are Lipschitzian are given. The case of convex functions is also analyzed. Some applications are provided.

1. Introduction

In [16] we established the following companion of Ostrowski inequality [27] for Lebesgue *sup-norm*:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f'\in L_\infty[a,b]$, then we have the inequalities*

$$\left|\frac{1}{2}\left[f(x)+f(a+b-x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.1}$$
$$\leq \frac{1}{b-a}\left[\frac{(x-a)^2}{2}\|f'\|_{[a,x],\infty}+\left(\frac{a+b}{2}-x\right)^2\|f'\|_{[x,a+b-x],\infty}+\frac{(x-a)^2}{2}\|f'\|_{[a+b-x,b],\infty}\right]$$
$$\leq\begin{cases}\left[\frac{1}{8}+2\left(\frac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\|f'\|_{[a,b],\infty}\\[2mm] \left[\frac{1}{2^{\alpha-1}}\left(\frac{x-a}{b-a}\right)^{2\alpha}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^{2\alpha}\right]^{\frac{1}{\alpha}}\\ \quad\times\left[\|f'\|^{\beta}_{[a,x],\infty}+\|f'\|^{\beta}_{[x,a+b-x],\infty}+\|f'\|^{\beta}_{[a+b-x,b],\infty}\right]^{\frac{1}{\beta}}(b-a)\\ \qquad\text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1,\\[2mm] \max\left\{\frac{1}{2}\left(\frac{x-a}{b-a}\right)^2,\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right\}\\ \quad\times\left[\|f'\|_{[a,x],\infty}+\|f'\|_{[x,a+b-x],\infty}+\|f'\|_{[a+b-x,b],\infty}\right](b-a)\end{cases}$$

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for any $x \in \left[a, \frac{a+b}{2}\right]$, *where*

$$\|g\|_{[c,d],\infty} := ess \sup_{t\in[c,d]} |g(s)| .$$

The inequality (1.1), the first inequality in (1.1) and the constant $\frac{1}{8}$ *are sharp.*

If in Theorem 1 we choose $x = a$, then we get

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{4}(b-a)\|f'\|_{[a,b],\infty} \tag{1.2}$$

with $\frac{1}{4}$ as a sharp constant (see for example [21, p. 25]).

If in the same theorem we now choose $x = \frac{a+b}{2}$, then we get

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| &\le \frac{1}{8}(b-a)\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],\infty} + \|f'\|_{\left[\frac{a+b}{2},b\right],\infty}\right] \\ &\le \frac{1}{4}(b-a)\|f'\|_{[a,b],\infty}\end{aligned} \tag{1.3}$$

with the constants $\frac{1}{8}$ and $\frac{1}{4}$ being sharp. This result was obtained in [15] by a different argument.

It is natural to consider the following corollary.

Corollary 1. *With the assumptions in Theorem 1, one has the inequality:*

$$\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{8}(b-a)\|f'\|_{[a,b],\infty} . \tag{1.4}$$

The constant $\frac{1}{8}$ *is best possible in the sense that it cannot be replaced by a smaller quantity.*

In the same paper [16] we established the corresponding inequalities for Lebesgue p-norms with $p \ge 1$ as well as have provided some applications for cumulative distribution functions and some quadrature rules.

For a monograph devoted to Ostrowski type inequalities, see [21].

For research papers on Ostrowski's inequality see [1]-[20], [22]-[24] and [26].

In the recent paper [19] we established the following identity:

Lemma 1. *Assume that* $f : [a, b] \to \mathbb{C}$ *is an absolutely continuous function on* $[a, b]$*. Then we have the equality*

$$\begin{aligned}&\frac{1}{2}[f(x) + f(a+b-x)] + \frac{1}{2}(x-a)^2\frac{\lambda_3(x) - \lambda_1(x)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= \frac{1}{b-a}\int_a^x (t-a)[f'(t) - \lambda_1(x)]\,dt \\ &+ \frac{1}{b-a}\int_x^{a+b-x}\left(t - \frac{a+b}{2}\right)[f'(t) - \lambda_2(x)]\,dt \\ &+ \frac{1}{b-a}\int_{a+b-x}^b (t-b)[f'(t) - \lambda_3(x)]\,dt,\end{aligned} \tag{1.5}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$ *and* $\lambda_j(x)$, $j = 1, 2, 3$ *complex numbers.*

The following particular cases are of interest:

Corollary 2. *With the assumption of Lemma 1 we have the equalities*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt=\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)[f'(t)-\lambda_2]\,dt, \tag{1.6}$$

$$\begin{aligned} &f\left(\frac{a+b}{2}\right)+\frac{1}{8}(b-a)(\lambda_3-\lambda_1)-\frac{1}{b-a}\int_a^b f(t)\,dt \\ &=\frac{1}{b-a}\int_a^{\frac{a+b}{2}}(t-a)[f'(t)-\lambda_1]\,dt+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b(t-b)[f'(t)-\lambda_3]\,dt, \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} &\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]+\frac{1}{32}(b-a)(\lambda_3-\lambda_1)-\frac{1}{b-a}\int_a^b f(t)\,dt \\ &=\frac{1}{b-a}\int_a^{\frac{3a+b}{4}}(t-a)[f'(t)-\lambda_1]\,dt \\ &+\frac{1}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}\left(t-\frac{a+b}{2}\right)[f'(t)-\lambda_2]\,dt \\ &+\frac{1}{b-a}\int_{\frac{a+3b}{4}}^b(t-b)[f'(t)-\lambda_3]\,dt \end{aligned} \tag{1.8}$$

for any $\lambda_1,\lambda_2,\lambda_3\in\mathbb{C}$.

The following particular result with no parameter in the left hand term holds:

Corollary 3. *Assume that* $f:[a,b]\to\mathbb{C}$ *is absolutely continuous on* $[a,b]$. *Then we have the equality*

$$\begin{aligned} &\frac{1}{2}[f(x)+f(a+b-x)]-\frac{1}{b-a}\int_a^b f(t)\,dt \\ &=\frac{1}{b-a}\int_a^x(t-a)[f'(t)-\lambda_1(x)]\,dt \\ &+\frac{1}{b-a}\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)[f'(t)-\lambda_2(x)]\,dt \\ &+\frac{1}{b-a}\int_{a+b-x}^b(t-b)[f'(t)-\lambda_1(x)]\,dt, \end{aligned} \tag{1.9}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$ *and* $\lambda_i(x), i=1,2$ *complex numbers.*

Remark 1. *We get from (1.7) the following particular case:*

$$\begin{aligned} &f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt \\ &=\frac{1}{b-a}\int_a^{\frac{a+b}{2}}(t-a)[f'(t)-\lambda_1]\,dt+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b(t-b)[f'(t)-\lambda_1]\,dt, \end{aligned} \tag{1.10}$$

for any $\lambda_1 \in \mathbb{C}$, *while from (1.8) we get*

$$\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{1.11}$$
$$=\frac{1}{b-a}\int_a^{\frac{3a+b}{4}}(t-a)\left[f'(t)-\lambda_1\right]dt$$
$$+\frac{1}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}}\left(t-\frac{a+b}{2}\right)\left[f'(t)-\lambda_2\right]dt$$
$$+\frac{1}{b-a}\int_{\frac{a+3b}{4}}^{b}(t-b)\left[f'(t)-\lambda_1\right]dt$$

for any $\lambda_1, \lambda_2 \in \mathbb{C}$.

Utilising these identities we obtained in [19] some inequalities for absolutely continuous functions whose derivatives are Lipschitzian are given. The case of convex functions is also analyzed. Applications for cumulative distribution function are provided as well.

2. Inequalities for Lipschitzian Derivatives

We say that the function $g:[a,b]\to\mathbb{C}$ is *Lipschitzian* with the constant $L>0$ if

$$|g(t)-g(s)|\le L\,|t-s|$$

for any $t,s\in[a,b]$.

Theorem 2. *Assume that* $f: I \to \mathbb{C}$ *is an absolutely continuous function on* $[a,b]\subset \mathring{I}$, *the interior of* I. *If the derivative* f' *is Lipschitzian with the constant* $K>0$, *then*

$$\left|\frac{1}{2}\left[f(x)+f(a+b-x)\right]+\frac{1}{2}(x-a)^2\,\frac{f'(b)-f'(a)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.1}$$
$$\le\frac{2K}{3(b-a)}\left[(x-a)^3+\left(\frac{a+b}{2}-x\right)^3\right]$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

Proof. If we take in the equality (1.5) $\lambda_1(x)=f'(a)$, $\lambda_2(x)=f'\left(\frac{a+b}{2}\right)$ and $\lambda_3(x)=f'(b)$ then we have

$$\frac{1}{2}\left[f(x)+f(a+b-x)\right]+\frac{1}{2}(x-a)^2\,\frac{f'(b)-f'(a)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{2.2}$$
$$=\frac{1}{b-a}\int_a^x(t-a)\left[f'(t)-f'(a)\right]dt$$
$$+\frac{1}{b-a}\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)\left[f'(t)-f'\left(\frac{a+b}{2}\right)\right]dt$$
$$+\frac{1}{b-a}\int_{a+b-x}^{b}(t-b)\left[f'(t)-f'(b)\right]dt,$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

Taking the modulus in (2.2) we have

$$\begin{aligned}
&\left|\frac{1}{2}\left[f\left(x\right)+f\left(a+b-x\right)\right]+\frac{1}{2}\left(x-a\right)^{2}\frac{f'\left(b\right)-f'\left(a\right)}{b-a}-\frac{1}{b-a}\int_{a}^{b}f\left(t\right)dt\right|\\
&\leq\frac{1}{b-a}\int_{a}^{x}\left(t-a\right)\left|f'\left(t\right)-f'\left(a\right)\right|dt\\
&+\frac{1}{b-a}\int_{x}^{a+b-x}\left|t-\frac{a+b}{2}\right|\left|f'\left(t\right)-f'\left(\frac{a+b}{2}\right)\right|dt\\
&+\frac{1}{b-a}\int_{a+b-x}^{b}\left(b-t\right)\left|f'\left(b\right)-f'\left(t\right)\right|dt\\
&\leq\frac{K}{b-a}\left[\int_{a}^{x}\left(t-a\right)^{2}dt+\int_{x}^{a+b-x}\left(t-\frac{a+b}{2}\right)^{2}dt+\int_{a+b-x}^{b}\left(b-t\right)^{2}dt\right]\\
&=\frac{2}{3}\frac{K}{b-a}\left[\left(x-a\right)^{3}+\left(\frac{a+b}{2}-x\right)^{3}\right]
\end{aligned}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$ and the inequality (2.1) is proved. □

Corollary 4. *With the assumptions of Theorem 2 we have the inequalities*

$$\left|\frac{1}{2}\left[f\left(a\right)+f\left(b\right)\right]-\frac{1}{b-a}\int_{a}^{b}f\left(t\right)dt\right|\leq\frac{1}{12}K\left(b-a\right)^{2},\tag{2.3}$$

$$\left|f\left(\frac{a+b}{2}\right)+\frac{1}{8}\left(b-a\right)\left[f'\left(b\right)-f'\left(a\right)\right]-\frac{1}{b-a}\int_{a}^{b}f\left(t\right)dt\right|\leq\frac{1}{12}K\left(b-a\right)^{2}\tag{2.4}$$

and

$$\begin{aligned}
&\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]+\frac{1}{32}\left(b-a\right)\left[f'\left(b\right)-f'\left(a\right)\right]\right.\\
&\left.-\frac{1}{b-a}\int_{a}^{b}f\left(t\right)dt\right|\\
&\leq\frac{1}{48}K\left(b-a\right)^{2}
\end{aligned}\tag{2.5}$$

The following dual result also holds:

Theorem 3. *With the assumptions of Theorem 2 we have*

$$\begin{aligned}
&\left|\frac{1}{2}\left[f\left(x\right)+f\left(a+b-x\right)\right]+\frac{1}{2}\left(x-a\right)^{2}\frac{f'\left(a+b-x\right)-f'\left(x\right)}{b-a}\right.\\
&\left.-\frac{1}{b-a}\int_{a}^{b}f\left(t\right)dt\right|\\
&\leq\frac{K}{3\left(b-a\right)}\left[\left(x-a\right)^{3}+2\left(\frac{a+b}{2}-x\right)^{3}\right]
\end{aligned}\tag{2.6}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

Proof. If we take in the equality (1.5) $\lambda_1(x) = f'(x)$, $\lambda_2(x) = f'\left(\frac{a+b}{2}\right)$ and $\lambda_3(x) = f'(a+b-x)$ then we have

$$\begin{aligned}
(2.7)\quad & \frac{1}{2}[f(x)+f(a+b-x)] + \frac{1}{2}(x-a)^2 \frac{f'(a+b-x)-f'(x)}{b-a} \\
& - \frac{1}{b-a}\int_a^b f(t)\,dt \\
& = \frac{1}{b-a}\int_a^x (t-a)[f'(t)-f'(x)]\,dt \\
& + \frac{1}{b-a}\int_x^{a+b-x} \left(t-\frac{a+b}{2}\right)\left[f'(t)-f'\left(\frac{a+b}{2}\right)\right]dt \\
& + \frac{1}{b-a}\int_{a+b-x}^b (t-b)[f'(t)-f'(a+b-x)]\,dt,
\end{aligned}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

Taking the modulus in (2.7) we have

$$\begin{aligned}
(2.8)\quad & \left|\frac{1}{2}[f(x)+f(a+b-x)] + \frac{1}{2}(x-a)^2 \frac{f'(a+b-x)-f'(x)}{b-a}\right. \\
& \left. -\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
& \le \frac{1}{b-a}\int_a^x (t-a)\,|f'(t)-f'(x)|\,dt \\
& + \frac{1}{b-a}\int_x^{a+b-x} \left|t-\frac{a+b}{2}\right|\left|f'(t)-f'\left(\frac{a+b}{2}\right)\right|dt \\
& + \frac{1}{b-a}\int_{a+b-x}^b (b-t)\,|f'(t)-f'(a+b-x)|\,dt \\
& \le \frac{K}{b-a}\left[\int_a^x (t-a)(x-t)\,dt + \int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2 dt\right. \\
& \left. + \int_{a+b-x}^b (b-t)(t-a-b+x)\,dt\right] \\
& := J,
\end{aligned}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

However

$$\int_a^x (t-a)(x-t)\,dt = \frac{1}{6}(x-a)^3,$$

$$\int_{a+b-x}^b (b-t)(t-a-b+x)\,dt = \frac{1}{6}(x-a)^3$$

and

$$\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)^2 dt = \frac{2}{3}\left(\frac{a+b}{2}-x\right)^3$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

Then

$$J=\frac{K}{3(b-a)}\left[(x-a)^{3}+2\left(\frac{a+b}{2}-x\right)^{3}\right]$$

and by (2.8) we get the desired result (2.6). □

Corollary 5. *With the assumptions of Theorem 2 we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\right|\leq\frac{1}{24}K(b-a)^{2} \tag{2.9}$$

and

$$\begin{aligned}&\left|\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\right.\\&\left.+\frac{1}{32}(b-a)\left[f'\left(\frac{a+3b}{4}\right)-f'\left(\frac{3a+b}{4}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\right|\\&\leq\frac{1}{64}K(b-a)^{2}.\end{aligned} \tag{2.10}$$

3. Inequalities for Convex Functions

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f:I\to\mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x,y\in\mathring{I}$ and $x<y$, then $f'_{-}(x)\leq f'_{+}(x)\leq f'_{-}(y)\leq f'_{+}(y)$ which shows that both f'_{-} and f'_{+} are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f:I\to\mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi:I\to[-\infty,\infty]$ such that $\varphi\left(\mathring{I}\right)\subset\mathbb{R}$ and

$$f(x)\geq f(a)+(x-a)\varphi(a)\ \text{ for any } x,a\in I.$$

It is also well known that if f is convex on I, then ∂f is nonempty, f'_{-}, $f'_{+}\in\partial f$ and if $\varphi\in\partial f$, then

$$f'_{-}(x)\leq\varphi(x)\leq f'_{+}(x)\ \text{ for any } x\in\mathring{I}.$$

In particular, φ is a nondecreasing function.

If f is differentiable and convex on $\mathring{I}$, then $\partial f=\{f'\}$.

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be a convex function with the lateral derivatives $f'_{-}(b)$ and $f'_{+}(a)$ finite. Then we have the inequality*

$$\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\leq\frac{1}{2}\left[f(x)+f(a+b-x)\right]+\frac{1}{2}(x-a)^{2}\frac{f'_{-}(b)-f'_{+}(a)}{b-a} \tag{3.1}$$

for any $x\in\left[a,\frac{a+b}{2}\right]$.

Proof. If we take in the equality (1.5) $\lambda_1(x) = f'_+(a)$, $\lambda_2(x) = f'_+\left(\frac{a+b}{2}\right)$ and $\lambda_3(x) = f'_-(b)$ then we have

$$\begin{aligned} &\frac{1}{2}\left[f(x)+f(a+b-x)\right]+\frac{1}{2}(x-a)^2\frac{f'_-(b)-f'_+(a)}{b-a}-\frac{1}{b-a}\int_a^b f(t)\,dt \\ &=\frac{1}{b-a}\int_a^x (t-a)\left[\varphi(t)-f'_+(a)\right]dt \\ &+\frac{1}{b-a}\int_x^{a+b-x}\left(t-\frac{a+b}{2}\right)\left[\varphi(t)-f'_+\left(\frac{a+b}{2}\right)\right]dt \\ &+\frac{1}{b-a}\int_{a+b-x}^b (t-b)\left[\varphi(t)-f'_-(b)\right]dt \end{aligned} \tag{3.2}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$ and $\varphi \in \partial f$, since $f' = \varphi$ almost everywhere on $[a,b]$.

Let $x \in \left(a, \frac{a+b}{2}\right)$. We have

$$(t-a)\left[\varphi(t)-f'_+(a)\right] \geq 0 \text{ for any } t \in [a,x]$$

and

$$(t-b)\left[\varphi(t)-f'_-(b)\right] = (b-t)\left[f'_-(b)-\varphi(t)\right] \geq 0 \text{ for any } t \in [a+b-x,b].$$

Also

$$\left(t-\frac{a+b}{2}\right)\left[\varphi(t)-f'_+\left(\frac{a+b}{2}\right)\right] \geq 0 \text{ for any } t \in [a,a+b-x].$$

Therefore the right hand side of (3.2) is nonnegative and the inequality (3.1) is proved. □

Corollary 6. *With the assumptions in Theorem 4 we have*

$$0 \leq \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \leq \frac{1}{8}(b-a)\left[f'_-(b)-f'_+(a)\right] \tag{3.3}$$

and

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(t)\,dt &\leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\ &+\frac{1}{32}(b-a)\left[f'_-(b)-f'_+(a)\right]. \end{aligned} \tag{3.4}$$

Remark 2. *If the $f:[a,b]\to\mathbb{R}$ is a twice differentiable convex function with the lateral derivatives $f'_-(b)$ and $f'_+(a)$ finite, then we have the inequality*

$$\begin{aligned} 0 &\leq \frac{1}{2}\left[f(x)+f(a+b-x)\right]+\frac{1}{2}(x-a)^2\frac{f'_-(b)-f'_+(a)}{b-a} \\ &-\frac{1}{b-a}\int_a^b f(t)\,dt \\ &\leq \frac{2S}{3(b-a)}\left[(x-a)^3+\left(\frac{a+b}{2}-x\right)^3\right] \end{aligned} \tag{3.5}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$, provided that $0 \leq f''(t) \leq S$ for any $t \in (a,b)$.

In particular we have

$$0 \le f\left(\frac{a+b}{2}\right) + \frac{1}{8}(b-a)\left[f'(b) - f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{1}{12}S(b-a)^2 \tag{3.6}$$

and

$$0 \le \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] + \frac{1}{32}(b-a)\left[f'(b) - f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{1}{48}S(b-a)^2. \tag{3.7}$$

4. Applications for PDF

Now, let X be a random variable taking values in the finite interval $[a,b]$, with the *probability density function* (PDF) $f : [a,b] \to [0,\infty)$ and with the *cumulative distribution function* (CDF) $F(x) = \Pr(X \le x) = \int_a^x f(t)\,dt$. We know that F is monotonic nondecreasing and absolutely continuous on $[a,b]$, $F' = f$ almost everywhere on $[a,b]$ and $F(a) = 0$, $F(b) = \int_a^b f(t)\,dt = 1$.

Proposition 1. *Let X be a random variable taking values in the finite interval $[a,b]$, with PDF $f : [a,b] \to [0,\infty)$ and with CDF $F(x) = \Pr(X \le x) = \int_a^x f(t)\,dt$. If f is Lipschitzian with the constant $L > 0$, then*

$$\left|\frac{1}{2}\left[F(x) + F(a+b-x)\right] + \frac{1}{2}(x-a)^2 \frac{f(b) - f(a)}{b-a} - \frac{b - E(X)}{b-a}\right| \le \frac{2L}{3(b-a)}\left[(x-a)^3 + \left(\frac{a+b}{2} - x\right)^3\right] \tag{4.1}$$

and

$$\left|\frac{1}{2}\left[F(x) + F(a+b-x)\right] + \frac{1}{2}(x-a)^2 \frac{f(a+b-x) - f(x)}{b-a} - \frac{b - E(X)}{b-a}\right| \le \frac{L}{3(b-a)}\left[(x-a)^3 + 2\left(\frac{a+b}{2} - x\right)^3\right] \tag{4.2}$$

for any $x \in \left[a, \frac{a+b}{2}\right]$.

Proof. Follows by Theorem 2 and 4 applied for the CDF $F(x) = \int_a^x f(t)\,dt$ and taking into account that

$$\int_a^b F(t)\,dt = b - E(X).$$

□

In particular, we have

Corollary 7. *With the assumptions in Proposition 1 we have*

$$\left|\frac{1}{2}\left[F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)\right]+\frac{1}{32}(b-a)\left[f(b)-f(a)\right]\right.$$
$$\left.-\frac{b-E(X)}{b-a}\right|$$
$$\leq\frac{1}{48}L(b-a)^{2} \tag{4.3}$$

and

$$\left|\frac{1}{2}\left[F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)\right]\right.$$
$$\left.+\frac{1}{32}(b-a)\left[f\left(\frac{a+3b}{4}\right)-f\left(\frac{3a+b}{4}\right)\right]-\frac{b-E(X)}{b-a}\right|$$
$$\leq\frac{1}{64}L(b-a)^{2}. \tag{4.4}$$

References

[1] G. A. Anastassiou, Univariate Ostrowski inequalities, revisited. *Monatsh. Math.,* **135** (2002), no. 3, 175–189.

[2] P. Cerone and S. S. Dragomir, Midpoint-type rules from an inequalities point of view, Ed. G. A. Anastassiou, *Handbook of Analytic-Computational Methods in Applied Mathematics*, CRC Press, New York. 135-200.

[3] P. Cerone and S. S. Dragomir, New bounds for the three-point rule involving the Riemann-Stieltjes integrals, in *Advances in Statistics Combinatorics and Related Areas*, C. Gulati, et al. (Eds.), World Science Publishing, 2002, 53-62.

[4] P. Cerone, S. S. Dragomir, and C. E. M. Pearce, A generalised trapezoid inequality for functions of bounded variation, *Turkish J. Math.,* **24**(2) (2000), 147-163.

[5] P. Cerone, S. S. Dragomir and J. Roumeliotis, Some Ostrowski type inequalities for n-time differentiable mappings and applications, *Demonstratio Mathematica*, **32**(2) (1999), 697—712.

[6] S. S. Dragomir, Ostrowski's inequality for monotonous mappings and applications, *J. KSIAM,* **3**(1) (1999), 127-135.

[7] S. S. Dragomir, The Ostrowski's integral inequality for Lipschitzian mappings and applications, *Comp. Math. Appl.,* **38** (1999), 33-37.

[8] S. S. Dragomir, The Ostrowski integral inequality for mappings of bounded variation, *Bull. Austral. Math. Soc.,* **60**(1) (1999), 495-508.

[9] S. S. Dragomir, On the midpoint quadrature formula for mappings with bounded variation and applications, *Kragujevac J. Math.,* **22** (2000), 13-18.

[10] S. S. Dragomir, On the Ostrowski's inequality for Riemann-Stieltjes integral, *Korean J. Appl. Math.,* **7** (2000), 477-485.

[11] S. S. Dragomir, On the Ostrowski's integral inequality for mappings with bounded variation and applications, *Math. Ineq. & Appl.,* **4**(1) (2001), 33-40.

[12] S. S. Dragomir, On the Ostrowski inequality for Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ where f is of Hölder type and u is of bounded variation and applications, *J. KSIAM,* **5**(1) (2001), 35-45.

[13] S. S. Dragomir, Ostrowski type inequalities for isotonic linear functionals, *J. Inequal. Pure & Appl. Math.,* **3**(5) (2002), Art. 68.

[14] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products. *J. Inequal. Pure Appl. Math.* **3** (2002), no. 2, Article 31, 8 pp.

[15] S. S. Dragomir, A refinement of Ostrowski's inequality for absolutely continuous functions whose derivatives belong to L_∞ and applications, *Libertas Math.* **22** (2002), 49–63.

[16] S. S. Dragomir, Some companions of Ostrowski's inequality for absolutely continuous functions and applications. Preprint *RGMIA Res. Rep. Coll.* **5** (2002), Suppl. Art. 29. [Online

http://rgmia.org/papers/v5e/COIACFApp.pdf]. *Bull. Korean Math. Soc.* **42** (2005), no. 2, 213–230.

[17] S. S. Dragomir, An Ostrowski like inequality for convex functions and applications, *Revista Math. Complutense*, **16**(2) (2003), 373-382.

[18] S. S. Dragomir, *Operator Inequalities of Ostrowski and Trapezoidal Type.* Springer Briefs in Mathematics. Springer, New York, 2012. x+112 pp. ISBN: 978-1-4614-1778-1

[19] S. S. Dragomir, Perturbed companions of Ostrowski's inequality for absolutely continuous functions (I), Preprint *RGMIA Res. Rep. Coll.* **17** (2014), Art 7. 15 pp. [Onlinehttp://rgmia.org/papers/v17/v17a07.pdf].

[20] S. S. Dragomir, P. Cerone, J. Roumeliotis and S. Wang, A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis, *Bull. Math. Soc. Sci. Math. Romanie,* **42(90)** (4) (1999), 301-314.

[21] S. S. Dragomir and Th. M. Rassias (Eds), *Ostrowski Type Inequalities and Applications in Numerical Integration,* Kluwer Academic Publishers, Dordrecht/Boston/London, 2002.

[22] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_1-norm and applications to some special means and to some numerical quadrature rules, *Tamkang J. of Math.*, **28** (1997), 239-244.

[23] S. S. Dragomir and S. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Appl. Math. Lett.*, **11** (1998), 105-109.

[24] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_p-norm and applications to some special means and to some numerical quadrature rules, *Indian J. of Math.,* **40**(3) (1998), 245-304.

[25] A. Guessab and G. Schmeisser, Sharp integral inequalities of the Hermite-Hadamard type, *J. Approx. Th.,* **115** (2002), 260-288.

[26] A. M. Fink, Bounds on the deviation of a function from its averages, *Czechoslovak Math. J.,* **42**(117) (1992), No. 2, 298-310.

[27] A. Ostrowski, Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, *Comment. Math. Hel,* **10** (1938), 226-227.

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