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ON HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR n-TIMES DIFFERENTIABLE s-LOGARITHMICALLY CONVEX FUNCTIONS WITH APPLICATIONS TO SPECIAL MEANS

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Abstract. In this paper, we establish Hermite-Hadamard type inequalities for functions whose nth derivatives are s-logarithmically convex functions. From our results, several results for classical trapezoidal and classical midpoint inequalities are obtained in terms second derivatives that are s-logarithmically convex functions as special cases. Finally, applications to special means of the obtained results are given.

1. Introduction

The classical convexity is defined as follows.

Definition 1. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if*

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{1.1}$$

for all x, $y \in I$ and $\lambda \in [0,1]$. The inequality (1.1) holds in reverse direction if f is a concave function.

The following double inequality holds

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.2}$$

for convex function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ and is know as the Hermite-Hadamard inequality. The inequality (1.2) holds in reverse direction if f is a concave function.

The inequality (1.2) has been subject of extensive research and has been refined and generalized by a number of mathematicians for over one hundred years see for instance [1]-[9], [12]-[16], [19]-[23], [25]-[28] and the references therein.

Many mathematicians are trying to generalize the classical convexity in a number of ways and one of them is so called logarithmically convexity defined as follows.

Definition 2. [27] *If a function $f : I \subseteq \mathbb{R} \to (0,\infty)$ satisfies*

$$f(\lambda x + (1-\lambda) y) \leq [f(x)]^{\lambda}[f(y)]^{1-\lambda}, \tag{1.3}$$

for all x, $y \in I$, $\lambda \in [0,1]$, the function f is called logarithmically convex on I. If the inequality (1.3) reverses, the function f is called logarithmically concave on I.

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The notion of logarithmically convex functions was generalized by Xi el al. in [27].

Definition 3. [27] *For some $s \in (0,1]$, a positive function $f : I \subseteq \mathbb{R} \to (0,\infty)$ is said to be s-logarthmically convex on I if and only if*

$$f(\lambda x + (1-\lambda) y) \leq [f(x)]^{s\lambda} [f(y)]^{s(1-\lambda)}$$

holds for all x, $y \in I$ and $\lambda \in [0,1]$.

It is obvious that when $s = 1$ in Definition 3, the s-logarthmically convex function becomes the usual logarithmically convex.

Xi et al. [27], obtained the following Hermite-Hadamard type inequalites for s-logarthmically convex functions.

Theorem 1. [27] *Let $f : I \subseteq [0,\infty) \to (0,\infty)$ be a differentiable function on I°, a, $b \in I^\circ$ with $a < b$ and $f' \in L([a,b])$. If $|f(x)|^q$ for $q \geq 1$ is s-logarithmically convex on $[a,b]$ for some given $s \in (0,1]$, then*

$$\left| f(a) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \leq \frac{(b-a)}{4}\left(\frac{1}{2}\right)^{1-1/q} \left\{ 3^{(q-1)/q} [L_1(\mu,q)]^{1/q} + [L_2(\mu,q,b)]^{1/q} \right\}, \quad (1.4)$$

where

$$L_1(\mu,q) \leq \begin{cases} \left|f'(a)f'(b)\right|^{sq/2} F_1(\mu_1), & 0 < \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right| \leq 1, \\ \left|f'(a)f'(b)\right|^{q/(2s)} F_1(\mu_2), & 1 \leq \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right|, \\ \left|f'(a)f'(b)\right|^{sq/2} F_1(\mu_3), & 0 < \left|f^{(n)}(a)\right| \leq 1 < \left|f^{(n)}(b)\right|, \\ \left|f'(a)f'(b)\right|^{q/(2s)} F_1(\mu_4), & 0 < \left|f^{(n)}(b)\right| \leq 1 < \left|f^{(n)}(a)\right|, \end{cases}$$

$$L_2(\mu,q,u) \leq \begin{cases} \left|f'(u)\right|^{sq/2} F_1(\mu_1), & 0 < \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right| \leq 1, \\ \left|f'(u)\right|^{q/(2s)} F_1(\mu_2), & 1 \leq \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right|, \\ \left|f'(u)\right|^{sq/2} F_1(\mu_3), & 0 < \left|f^{(n)}(a)\right| \leq 1 < \left|f^{(n)}(b)\right|, \\ \left|f'(u)\right|^{q/(2s)} F_1(\mu_4), & 0 < \left|f^{(n)}(b)\right| \leq 1 < \left|f^{(n)}(a)\right|, \end{cases}$$

$$F_1(\nu) = \begin{cases} \frac{1}{\ln \nu}\left(2\nu - 1 - \frac{\nu-1}{\ln \nu}\right) & \nu \neq 1, \\ \frac{3}{2} & \nu = 1, \end{cases}$$

$$F_2(\nu)=\begin{cases}\frac{1}{\ln\nu}\left(\nu-\frac{\nu-1}{\ln\nu}\right) & \nu\neq 1,\\ \frac{1}{2} & \nu=1,\end{cases}$$

and

$$\mu_1=\left|\frac{f^{(n)}(a)}{f^{(n)}(b)}\right|^{sq/2},\mu_2=\left|\frac{f^{(n)}(a)}{f^{(n)}(b)}\right|^{q/(2s)},\mu_3=\frac{\left|f^{(n)}(a)\right|^{sq/2}}{\left|f^{(n)}(b)\right|^{q/(2s)}},\mu_4=\frac{\left|f^{(n)}(a)\right|^{q/(2s)}}{\left|f^{(n)}(b)\right|^{qs/2}}.$$

Theorem 2. [27] *Under the conditions of Theorem 1, we have*

$$\left|f(b)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{4}\left(\frac{1}{2}\right)^{1-1/q}\left\{\left[L_2(\mu,q,a)\right]^{1/q}+3^{(q-1)/q}\left[L_1\left(\mu^{-1},q\right)\right]^{1/q}\right\}, \quad (1.5)$$

where $L_1(\mu,q)$, $L_2(\mu,q,u)$, $F_1(\nu)$, $F_2(\nu)$ *and* μ_i *for* $i=1,\ 2,\ 3,\ 4$ *are defined as in Theorem 1.*

Theorem 3. [27] *Under the conditions of Theorem 1, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{4}\left(\frac{1}{2}\right)^{1-1/q}\left\{\left[L_2(\mu,q,b)\right]^{1/q}+\left[L_1\left(\mu^{-1},q,a\right)\right]^{1/q}\right\}, \quad (1.6)$$

where $L_1(\mu,q)$, $L_2(\mu,q,u)$, $F_1(\nu)$, $F_2(\nu)$ *and* μ_i *for* $i=1,\ 2,\ 3,\ 4$ *are defined as in Theorem 1.*

Applications to special means of positive numbers of the above results are also given in [27].

Motivated by the above definitions and the results, the main purpose of the present paper to establish new Hermite-Hadamard type inequalities for functions whose nth derivatives in absolute value are s-logarthmically convex. These results not only generalize the results from [27] but many other interesting results can be obtained for functions whose second derivatives in absolute value are s-logarthmically convex which may be better than those from [27].

2. Main Results

First we quote some useful lemmas to prove our mains results.

Lemma 1. [12] *Suppose* $f: I\subseteq\mathbb{R}\to\mathbb{R}$ *is a function such that* $f^{(n)}$ *exists on* I° *for* $n\in\mathbb{N}$, $n\geq 1$. *If* $f^{(n)}$ *is integrable on* $[a,b]$, *for* $a,b\in I$ *with* $a>b$, *the equality holds*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a) = \frac{(b-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)\,f^{(n)}(ta+(1-t)\,b)dt, \quad (2.1)$$

where the sum above takes 0 *when* $n=1$ *and* $n=2$.

Lemma 2. [17] *Suppose $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on I° for $n \in \mathbb{N}$, $n \geq 1$. If $f^{(n)}$ is integrable on $[a,b]$, for $a, b \in I$ with $a > b$, the equality holds*

$$\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{(-1)(b-a)^n}{n!}\int_0^1 K_n(t) f^{(n)}(ta + (1-t)b)dt, \quad (2.2)$$

where

$$K_n(t) := \begin{cases} t^n, & t \in \left[0, \frac{1}{2}\right] \\ (t-1)^n, & t \in \left(\frac{1}{2}, 1\right] \end{cases}.$$

The following useful result will also help us establishing our results.

Lemma 3. [17] *If $\mu > 0$ and $\mu \neq 1$, then*

$$\int_0^1 t^n \mu^t dt = \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n!\mu \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!(\ln \mu)^{k+1}}. \quad (2.3)$$

Lemma 4. [17] *If $\mu > 0$ and $\mu \neq 1$, then*

$$\int_0^{\frac{1}{2}} t^n \mu^t dt = \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n!\mu^{1/2} \sum_{k=0}^{n} \frac{(-1)^k}{2^{n-k}(n-k)!(\ln \mu)^{k+1}}. \quad (2.4)$$

Proof. It follows from Lemma 3 after making use of the substitution $t = \frac{u}{2}$. □

Lemma 5. [17] *If $\mu > 0$ and $\mu \neq 1$, then*

$$\int_{\frac{1}{2}}^1 (1-t)^n \mu^t dt = \frac{n!\mu}{(\ln \mu)^{n+1}} - n!\mu^{1/2} \sum_{k=0}^{n} \frac{1}{2^{n-k}(n-k)!(\ln \mu)^{k+1}}. \quad (2.5)$$

Proof. It follows from Lemma 4 after making the substitution $1 - t = u$. □

Lemma 6. [24] *For $\alpha > 0$ and $\mu > 0$, we have*

$$I(\alpha, \mu) := \int_0^1 t^{\alpha-1}\mu^t dt = \mu \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(\ln \mu)^{k-1}}{(\alpha)_k} < \infty,$$

where

$$(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+k-1).$$

Moreover, it holds

$$\left| I(\alpha, \mu) - \mu \sum_{k=1}^{m} (-1)^{k-1} \frac{(\ln \mu)^{k-1}}{(\alpha)_k} \right| \leq \frac{|\ln \mu|}{\alpha\sqrt{2\pi(m-1)}} \left(\frac{|\ln \mu| e}{m-1}\right)^{m-1}.$$

We are now ready to set off our first result.

Theorem 4. *Let $I \subseteq [0, \infty)$ be an open real interval and let $f : I \to (0, \infty)$ be a function such that $f^{(n)}$ exists on I, a, $b \in I$ with $a < b$ and $f^{(n)}$ is integrable on*

$[a,b]$ for $n\in\mathbb{N}$, $n\geq 2$. If $\left|f^{(n)}\right|^{q}$ is s-logarthmically convex on $[a,b]$ for $q\in[1,\infty)$, $s\in(0,1]$, we have the inequalities

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx -\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right| \leq \frac{(b-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-1/q}$$
$$\times\begin{cases} \left|f^{(n)}(b)\right|^{s}\left[F_1(\mu,n)\right]^{1/q}, & 0<\left|f^{(n)}(a)\right|,\left|f^{(n)}(b)\right|\leq 1\\ \left|f^{(n)}(a)\right|^{(1-s)}\left|f^{(n)}(b)\right|\left[F_1(\mu,n)\right]^{1/q}, & 1\leq\left|f^{(n)}(a)\right|,\left|f^{(n)}(b)\right|\\ \left|f^{(n)}(b)\right|\left[F_1(\mu,n)\right]^{1/q}, & 0<\left|f^{(n)}(a)\right|\leq 1<\left|f^{(n)}(b)\right|\\ \left|f^{(n)}(b)\right|^{s}\left|f^{(n)}(a)\right|^{(1-s)}\left[F_1(\mu,n)\right]^{1/q}, & 0<\left|f^{(n)}(b)\right|\leq 1<\left|f^{(n)}(a)\right|, \end{cases} \tag{2.6}$$

where $\mu=\left|\frac{f^{(n)}(a)}{f^{(n)}(b)}\right|^{sq}$ and

$$F_1(\mu,n)=\begin{cases} \frac{(-1)^n n![\ln\mu+2]}{(\ln\mu)^{n+1}}-\frac{2\mu}{\ln\mu}-n!\mu\sum_{k=1}^{n}\frac{(-1)^k[\ln\mu+2]}{(n-k)!(\ln\mu)^{k+1}} & \mu\neq 1\\ \frac{n-1}{n+1} & \mu=1 \end{cases}.$$

Proof. Suppose $n\geq 2$. By s-logarthmically convexity of $\left|f^{(n)}\right|^{q}$ on $[a,b]$, Lemma 1 and Hölder inequality, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$
$$\leq\frac{(b-a)^n}{2n!}\left(\int_0^1 t^{n-1}(n-2t)\,dt\right)^{1-1/q}$$
$$\times\left(\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(a)\right|^{qt^s}\left|f^{(n)}(b)\right|^{q(1-t)^s}dt\right)^{1/q}. \tag{2.7}$$

Let $0<\xi\leq 1\leq\eta$, $0\leq\lambda\leq 1$ and $0<s\leq 1$. Then

$$\xi^{\lambda^s}\leq\xi^{s\lambda}\text{ and }\eta^{\lambda^s}\leq\eta^{s\lambda+1-s}. \tag{2.8}$$

For $0<\left|f^{(n)}(a)\right|,\left|f^{(n)}(b)\right|\leq 1$, from (2.8) and Lemma 3, we have

$$\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(a)\right|^{qt^s}\left|f^{(n)}(b)\right|^{q(1-t)^s}dt$$
$$\leq\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(a)\right|^{qst}\left|f^{(n)}(b)\right|^{sq(1-t)}dt$$
$$=\left|f^{(n)}(b)\right|^{sq}\int_0^1 t^{n-1}(n-2t)\mu^t dt=\left|f^{(n)}(b)\right|^{sq}F_1(\mu,n), \tag{2.9}$$

where $\mu = \left|\frac{f^{(n)}(a)}{f^{(n)}(b)}\right|^{sq}$.
For $1 < \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right|$, from (2.8) and by using Lemma 3, we have

$$\begin{aligned}
&\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(a)\right|^{qt^s}\left|f^{(n)}(b)\right|^{q(1-t)^s}dt \\
&\leq \left|f^{(n)}(a)\right|^{q(1-s)}\left|f^{(n)}(b)\right|^{q}\int_0^1 t^{n-1}(n-2t)\mu^t dt \\
&= \left|f^{(n)}(a)\right|^{q(1-s)}\left|f^{(n)}(b)\right|^{q}F_1(\mu,n). \quad (2.10)
\end{aligned}$$

For $0 < \left|f^{(n)}(a)\right| \leq 1 < \left|f^{(n)}(b)\right|$, from (2.8) and by Lemma 3, we obtain

$$\begin{aligned}
&\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(a)\right|^{qt^s}\left|f^{(n)}(b)\right|^{q(1-t)^s}dt \\
&\leq \left|f^{(n)}(b)\right|^{q}\int_0^1 t^{n-1}(n-2t)\mu^t dt = \left|f^{(n)}(b)\right|^{q}F_1(\mu,n). \quad (2.11)
\end{aligned}$$

Lastly for $0 < \left|f^{(n)}(b)\right| \leq 1 < \left|f^{(n)}(a)\right|$ from (2.8) and Lemma 3, we get that

$$\begin{aligned}
&\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(a)\right|^{qt^s}\left|f^{(n)}(b)\right|^{q(1-t)^s}dt \\
&\leq \left|f^{(n)}(a)\right|^{q}\left|f^{(n)}(b)\right|^{q(1-s)}\int_0^1 t^{n-1}(n-2t)\mu^t dt \\
&= \left|f^{(n)}(b)\right|^{sq}\left|f^{(n)}(a)\right|^{q(1-s)}F_1(\mu,n). \quad (2.12)
\end{aligned}$$

Combining (2.9), (2.10), (2.11) and (2.12), we get the required result. This completes the proof of the theorem. □

Corollary 1. *Suppose the assumptions of Theorem 4 are satisfied and if $q = 1$, we have the inequalities*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \right. \\
&\left. - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right| \leq \frac{(b-a)^n}{2n!} \\
&\times \begin{cases}
\left|f^{(n)}(b)\right|^{s}F_1(\mu,n), & 0 < \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right| \leq 1 \\
\left|f^{(n)}(a)\right|^{(1-s)}\left|f^{(n)}(b)\right|F_1(\mu,n), & 1 \leq \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right| \\
\left|f^{(n)}(b)\right|F_1(\mu,n), & 0 < \left|f^{(n)}(a)\right| \leq 1 < \left|f^{(n)}(b)\right| \\
\left|f^{(n)}(b)\right|^{s}\left|f^{(n)}(a)\right|^{(1-s)}F_1(\mu,n), & 0 < \left|f^{(n)}(b)\right| \leq 1 < \left|f^{(n)}(a)\right|,
\end{cases} \quad (2.13)
\end{aligned}$$

where $\mu = \left| \frac{f^{(n)}(a)}{f^{(n)}(b)} \right|^s$ *and*

$$F_1(\mu, n) = \begin{cases} \frac{(-1)^n n! [\ln \mu + 2]}{(\ln \mu)^{n+1}} - \frac{2\mu}{\ln \mu} - n! \mu \sum_{k=1}^{n} \frac{(-1)^k [\ln \mu + 2]}{(n-k)! (\ln \mu)^{k+1}} & \mu \neq 1 \\ \frac{n-1}{n+1} & \mu = 1 \end{cases}.$$

Corollary 2. *Under the assumptions of Theorem 4, if $n = 2$, we have the inequalities*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \leq \frac{(b-a)^2}{4} \left(\frac{1}{3} \right)^{1-1/q}$$
$$\times \begin{cases} \left| f''(b) \right|^s [F_1(\mu, 2)]^{1/q}, & 0 < \left| f''(a) \right|, \left| f''(b) \right| \leq 1 \\ \left| f''(a) \right|^{(1-s)} \left| f''(b) \right| [F_1(\mu, 2)]^{1/q}, & 1 \leq \left| f''(a) \right|, \left| f''(b) \right| \\ \left| f''(b) \right| [F_1(\mu, 2)]^{1/q}, & 0 < \left| f''(a) \right| \leq 1 < \left| f''(b) \right| \\ \left| f''(b) \right|^s \left| f''(a) \right|^{(1-s)} [F_1(\mu, 2)]^{1/q}, & 0 < \left| f''(b) \right| \leq 1 < \left| f''(a) \right|, \end{cases} \tag{2.14}$$

where $\mu = \left| \frac{f''(a)}{f''(b)} \right|^{sq}$ *and*

$$F_1(\mu, 2) = \begin{cases} \frac{2(\ln \mu + \mu) \ln \mu + 4(\ln \mu - \mu)}{(\ln \mu)^3} & \mu \neq 1 \\ \frac{1}{3} & \mu = 1. \end{cases}$$

Remark 1. *For $s = 1$, one can get very interesting inequalites from (2.6), (2.13) and (2.14) for log-convex functions.*

Theorem 5. *Let $I \subseteq [0, \infty)$ be an open real interval and let $f : I \to (0, \infty)$ be a function such that $f^{(n)}$ exists on I, a, $b \in I$ with $a < b$ and $f^{(n)}$ is integrable on $[a, b]$ for $n \in \mathbb{N}$, $n \geq 2$. If $\left| f^{(n)} \right|^q$ is s-logarthmically convex on $[a, b]$ for $q \in (1, \infty)$,*

$s \in (0,1]$, we have the inequalities

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$
$$\leq \frac{(b-a)^n\left[n^{(2q-1)/(q-1)}-(n-2)^{(2q-1)/(q-1)}\right]^{1-1/q}}{2^{2-1/q}n!}\left(\frac{q-1}{2q-1}\right)^{1-1/q}$$
$$\times\begin{cases}\left|f^{(n)}(b)\right|^s\left[F_2(\mu,n)\right]^{1/q}, & 0<\left|f^{(n)}(a)\right|,\left|f^{(n)}(b)\right|\leq 1\\ \left|f^{(n)}(a)\right|^{(1-s)}\left|f^{(n)}(b)\right|\left[F_2(\mu,n)\right]^{1/q}, & 1\leq\left|f^{(n)}(a)\right|,\left|f^{(n)}(b)\right|\\ \left|f^{(n)}(b)\right|\left[F_2(\mu,n)\right]^{1/q}, & 0<\left|f^{(n)}(a)\right|\leq 1<\left|f^{(n)}(b)\right|\\ \left|f^{(n)}(b)\right|^s\left|f^{(n)}(a)\right|^{(1-s)}\left[F_2(\mu,n)\right]^{1/q}, & 0<\left|f^{(n)}(b)\right|\leq 1<\left|f^{(n)}(a)\right|,\end{cases} \quad (2.15)$$

where $\mu=\left|\frac{f^{(n)}(a)}{f^{(n)}(b)}\right|^{sq}$,

$$F_2(\mu,n)=\begin{cases}\mu\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(\ln\mu)^{k-1}}{(nq-q+1)_k}<\infty & \mu\neq 1\\ \frac{1}{nq-q+1} & \mu=1\end{cases}$$

and $(nq-q+1)_k=(nq-q+1)(nq-q+2)\cdots(nq-q+k)$.

Proof. Since $\left|f^{(n)}\right|^q$ is s-logarthmically convex on $[a,b]$ for $q\in(1,\infty)$, $s\in(0,1]$, hence from Lemma 1 and the Hölder inequality, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$
$$\leq\frac{(b-a)^n}{2n!}\left(\int_0^1(n-2t)^{q/(q-1)}\,dt\right)^{1-1/q}\left(\int_0^1 t^{q(n-1)}\left|f^{(n)}(ta+(1-t)b\right|^q dt\right)^{1/q}$$
$$\leq \frac{(b-a)^n\left[n^{(2q-1)/(q-1)}-(n-2)^{(2q-1)/(q-1)}\right]^{1-1/q}}{2^{2-1/q}n!}\left(\frac{q-1}{2q-1}\right)^{1-1/q}$$
$$\times\left(\int_0^1 t^{q(n-1)}\left|f^{(n)}(a)\right|^{qt^s}\left|f^{(n)}(b)\right|^{q(1-t)^s}dt\right)^{1/q}. \quad (2.16)$$

From (2.8), Lemma 6 and by using similar arguments as in proving Theorem 4, we have the inequality (2.15). This completes the proof of the theorem. □

Corollary 3. *Suppose the assumptions of Theorem 5 are satisfied and $n=2$. Then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)^2}{2}\left(\frac{q-1}{2q-1}\right)^{1-1/q}$$
$$\times\begin{cases} \left|f''(b)\right|^s\left[F_2(\mu,2)\right]^{1/q}, & 0<\left|f''(a)\right|,\left|f''(b)\right|\le 1 \\ \left|f''(a)\right|^{(1-s)}\left|f''(b)\right|\left[F_2(\mu,2)\right]^{1/q}, & 1\le\left|f''(a)\right|,\left|f''(b)\right| \\ \left|f''(b)\right|\left[F_2(\mu,2)\right]^{1/q}, & 0<\left|f''(a)\right|\le 1<\left|f''(b)\right| \\ \left|f''(b)\right|^s\left|f''(a)\right|^{(1-s)}\left[F_2(\mu,2)\right]^{1/q}, & 0<\left|f''(b)\right|\le 1<\left|f''(a)\right|, \end{cases} \tag{2.17}$$

where $\mu=\left|\frac{f''(a)}{f''(b)}\right|^{sq}$,

$$F_2(\mu,2)=\begin{cases} \mu\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(\ln\mu)^{k-1}}{(q+1)_k}<\infty & \mu\ne 1 \\ \frac{1}{q+1} & \mu=1 \end{cases}$$

and $(q+1)_k=(q+1)(q+2)\cdots(q+k)$.

Corollary 4. *Suppose the assumptions of Theorem 5 are satisfied and $n=2$, $s=1$. Then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\le\frac{(b-a)^2}{2}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left|f''(b)\right|\left[F_2(\mu,2)\right]^{1/q}, \tag{2.18}$$

where

$$\mu=\left|\frac{f''(a)}{f''(b)}\right|^q,\ F_2(\mu,2)=\begin{cases} \mu\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(\ln\mu)^{k-1}}{(q+1)_k}<\infty & \mu\ne 1 \\ \frac{1}{q+1} & \mu=1 \end{cases}$$

and $(q+1)_k=(q+1)(q+2)\cdots(q+k)$.

Now we give some results related to left-side of Hermite-Hadamard's inequality for n-times differentiable s-logarthmically convex functions.

Theorem 6. *Let $I\subseteq[0,\infty)$ be an open real interval and let $f:I\to(0,\infty)$ be a function such that $f^{(n)}$ exists on I, $a,\ b\in I$ with $a<b$ and $f^{(n)}$ is integrable on $[a,b]$ for $n\in\mathbb{N}$, $n\ge 1$. If $\left|f^{(n)}\right|^q$ is s-logarthmically convex on $[a,b]$ for $q\in[1,\infty)$,*

$s \in (0,1]$, *we have the inequalities*

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \begin{cases} \frac{(b-a)^n \left|f^{(n)}(b)\right|^s \left\{[F_3(\mu,n)]^{1/q} + [F_4(\mu,n)]^{1/q}\right\}}{n! 2^{(n+1)(q-1)/q}(n+1)^{1-1/q}}, & 0 < \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right| \leq 1 \\ \frac{(b-a)^n \left|f^{(n)}(a)\right|^{(1-s)} \left|f^{(n)}(b)\right| \left\{[F_3(\mu,n)]^{1/q} + [F_4(\mu,n)]^{1/q}\right\}}{n! 2^{(n+1)(q-1)/q}(n+1)^{1-1/q}}, & 1 \leq \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right| \\ \frac{(b-a)^n \left|f^{(n)}(b)\right| \left\{[F_3(\mu,n)]^{1/q} + [F_4(\mu,n)]^{1/q}\right\}}{n! 2^{(n+1)(q-1)/q}(n+1)^{1-1/q}}, & 0 < \left|f^{(n)}(a)\right| \leq 1 < \left|f^{(n)}(b)\right| \\ \frac{(b-a)^n \left|f^{(n)}(b)\right|^s \left|f^{(n)}(a)\right|^{(1-s)} \left\{[F_3(\mu,n)]^{1/q} + [F_4(\mu,n)]^{1/q}\right\}}{n! 2^{(n+1)(q-1)/q}(n+1)^{1-1/q}}, & 0 < \left|f^{(n)}(b)\right| \leq 1 < \left|f^{(n)}(a)\right| \end{cases}, \tag{2.19}$$

where $\mu = \left|\frac{f^{(n)}(a)}{f^{(n)}(b)}\right|^{sq}$,

$$F_3(\mu,n) = \begin{cases} \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n! \mu^{1/2} \sum_{k=0}^{n} \frac{(-1)^k}{2^{n-k}(n-k)!(\ln \mu)^{k+1}}, & \mu \neq 1 \\ \frac{1}{2^{n+1}}, & \mu = 1 \end{cases}$$

and

$$F_4(\mu,n) = \begin{cases} \frac{n! \mu}{(\ln \mu)^{n+1}} - n! \mu^{1/2} \sum_{k=0}^{n} \frac{1}{2^{n-k}(n-k)!(\ln \mu)^{k+1}}, & \mu \neq 1 \\ \frac{1}{2^{n+1}}, & \mu = 1. \end{cases}.$$

Proof. Suppose $n \geq 1$. By using Lemma 2, the s-logarthmically convexity of $\left|f^{(n)}\right|$ and the Hölder inequality, we have

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)^n}{n!}\left[\left(\int_{\frac{1}{2}}^{1} (1-t)^n\,dt\right)^{1-1/q} \left(\int_{\frac{1}{2}}^{1} (1-t)^n \left|f^{(n)}(a)\right|^{qt^s} \left|f^{(n)}(b)\right|^{q(1-t)^s} dt\right)^{1/q}\right.$$
$$\left. + \left(\int_0^{\frac{1}{2}} t^n dt\right)^{1-1/q} \left(\int_0^{\frac{1}{2}} t^n \left|f^{(n)}(a)\right|^{qt^s} \left|f^{(n)}(b)\right|^{q(1-t)^s} dt\right)^{1/q}\right]. \tag{2.20}$$

From (2.8), Lemma 4, Lemma 5 and the same reasoning as in proving Theorem 4, we have the required inequality (2.19). This completes the proof of the theorem. □

Corollary 5. *Suppose the assumptions of Theorem 6 are fulfilled and if $q=1$, we have*

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\begin{cases}\frac{(b-a)^n\left|f^{(n)}(b)\right|^s\{F_3(\mu,n)+F_4(\mu,n)\}}{n!}, & 0<\left|f^{(n)}(a)\right|,\left|f^{(n)}(b)\right|\leq 1\\ \frac{(b-a)^n\left|f^{(n)}(a)\right|^{(1-s)}\left|f^{(n)}(b)\right|\{F_3(\mu,n)+F_4(\mu,n)\}}{n!}, & 1\leq\left|f^{(n)}(a)\right|,\left|f^{(n)}(b)\right|\\ \frac{(b-a)^n\left|f^{(n)}(b)\right|\{F_3(\mu,n)+F_4(\mu,n)\}}{n!}, & 0<\left|f^{(n)}(a)\right|\leq 1<\left|f^{(n)}(b)\right|\\ \frac{(b-a)^n\left|f^{(n)}(b)\right|^s\left|f^{(n)}(a)\right|^{(1-s)}\{F_3(\mu,n)+F_4(\mu,n)\}}{n!}, & 0<\left|f^{(n)}(b)\right|\leq 1<\left|f^{(n)}(a)\right|\end{cases}, \tag{2.21}$$

where $\mu=\left|\frac{f^{(n)}(a)}{f^{(n)}(b)}\right|^s$ *and* $F_3(\mu,n)$, $F_4(\mu,n)$ *are defined as in Theorem 6.*

Corollary 6. *Suppose the assumptions of Theorem 6 are fulfilled and if $s=1$, we have*

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq\frac{(b-a)^n\left|f^{(n)}(b)\right|\left\{[F_3(\mu,n)]^{1-1/q}+[F_4(\mu,n)]^{1-1/q}\right\}}{n!2^{(n+1)(q-1)/q}(n+1)^{1-1/q}}, \tag{2.22}$$

where $\mu=\left|\frac{f^{(n)}(a)}{f^{(n)}(b)}\right|^q$ *and* $F_3(\mu,n)$, $F_4(\mu,n)$ *are defined as in Theorem 6.*

Corollary 7. *Suppose the assumptions of Theorem 6 are fulfilled and if $n=1$, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{(b-a)}{2^{3(1-1/q)}}$$
$$\times\begin{cases}\left|f'(b)\right|^s\left\{[F_3(\mu,1)]^{1/q}+[F_4(\mu,1)]^{1/q}\right\}, & 0<\left|f'(a)\right|,\left|f'(b)\right|\leq 1\\ \left|f'(a)\right|^{(1-s)}\left|f'(b)\right|\left\{[F_3(\mu,1)]^{1/q}+[F_4(\mu,1)]^{1/q}\right\}, & 1\leq\left|f'(a)\right|,\left|f'(b)\right|\\ \left|f'(b)\right|\left\{[F_3(\mu,1)]^{1/q}+[F_4(\mu,1)]^{1/q}\right\}, & 0<\left|f'(a)\right|\leq 1<\left|f'(b)\right|\\ \left|f'(b)\right|^s\left|f'(a)\right|^{(1-s)}\left\{[F_3(\mu,1)]^{1/q}+[F_4(\mu,1)]^{1/q}\right\}, & 0<\left|f'(b)\right|\leq 1<\left|f'(a)\right|,\end{cases}, \tag{2.23}$$

where $\mu = \left| \frac{f'(a)}{f'(b)} \right|^{sq}$,

$$F_3(\mu, 1) = \begin{cases} \frac{2+\mu^{1/2}(\ln\mu - 2)}{2(\ln\mu)^2}, & \mu \neq 1 \\ \frac{1}{4}, & \mu = 1 \end{cases}$$

and

$$F_4(\mu, 1) = \begin{cases} \frac{2\mu-\mu^{1/2}(\ln\mu - 2)}{2(\ln\mu)^2}, & \mu \neq 1 \\ \frac{1}{4}, & \mu = 1. \end{cases}.$$

Theorem 7. *Let $I \subseteq [0,\infty)$ be an open real interval and let $f : I \to (0,\infty)$ be a function such that $f^{(n)}$ exists on I, a, $b \in I$ with $a < b$ and $f^{(n)}$ is integrable on $[a,b]$ for $n \in \mathbb{N}$, $n \geq 1$. If $\left|f^{(n)}\right|^q$ is s-logarthmically convex on $[a,b]$ for $q \in (1,\infty)$, $s \in (0,1]$, we have the inequalities*

$$\left| \sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx \right|$$
$$\leq \begin{cases} \frac{(b-a)^n \left|f^{(n)}(b)\right|^s \left\{[F_5(\mu)]^{1/q} + [F_6(\mu)]^{1/q}\right\}}{2^{n+1/p}(np+1)^{1/p} n!}, & 0 < \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right| \leq 1 \\ \frac{(b-a)^n \left|f^{(n)}(a)\right|^{(1-s)} \left|f^{(n)}(b)\right| \left\{[F_5(\mu)]^{1/q} + [F_6(\mu)]^{1/q}\right\}}{2^{n+1/p}(np+1)^{1/p} n!}, & 1 \leq \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right| \\ \frac{(b-a)^n \left|f^{(n)}(b)\right| \left\{[F_5(\mu)]^{1/q} + [F_6(\mu)]^{1/q}\right\}}{2^{n+1/p}(np+1)^{1/p} n!}, & 0 < \left|f^{(n)}(a)\right| \leq 1 < \left|f^{(n)}(b)\right| \\ \frac{(b-a)^n \left|f^{(n)}(b)\right|^s \left|f^{(n)}(a)\right|^{(1-s)} \left\{[F_5(\mu)]^{1/q} + [F_6(\mu)]^{1/q}\right\}}{2^{n+1/p}(np+1)^{1/p} n!}, & 0 < \left|f^{(n)}(b)\right| \leq 1 < \left|f^{(n)}(a)\right|, \end{cases}, \tag{2.24}$$

where $\mu = \left| \frac{f^{(n)}(a)}{f^{(n)}(b)} \right|^{sq}$,

$$F_5(\mu) = \begin{cases} \frac{\mu^{1/2}-1}{\ln\mu}, & \mu \neq 1 \\ \frac{1}{2}, & \mu = 1 \end{cases}, F_6(\mu) = \begin{cases} \frac{\mu-\mu^{1/2}}{\ln\mu}, & \mu \neq 1 \\ \frac{1}{2}, & \mu = 1 \end{cases}$$

and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2, the Hölder integral inequality and s-logarthmically convexity of $\left|f^{(n)}\right|^{q}$ on $[a,b]$, we have

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^{k}+1\right](b-a)^{k}}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx\right|$$
$$\leq\frac{(b-a)^{n}}{n!}\left[\left(\int_{0}^{\frac{1}{2}}t^{np}dt\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{2}}\left|f^{(n)}(a)\right|^{qt^{s}}\left|f^{(n)}(b)\right|^{q(1-t)^{s}}dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^{1}(1-t)^{np}dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left|f^{(n)}(a)\right|^{qt^{s}}\left|f^{(n)}(b)\right|^{q(1-t)^{s}}dt\right)^{\frac{1}{q}}\right]. \quad (2.25)$$

Using (2.8) and similar arguments as in proving Theorem 4, we get (2.24). This completes the proof of the theorem. □

Corollary 8. *Under the assumptions of Theorem 7, if $n=1$, we have the inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx\right|$$
$$\leq\begin{cases}\frac{(b-a)\left|f'(b)\right|^{s}\left\{[F_5(\mu)]^{1/q}+[F_6(\mu)]^{1/q}\right\}}{2^{1+1/p}(p+1)^{1/p}}, & 0<\left|f'(a)\right|,\left|f'(b)\right|\leq 1\\ \frac{(b-a)\left|f'(b)\right|\left|f'(a)\right|^{(1-s)}\left\{[F_5(\mu)]^{1/q}+[F_6(\mu)]^{1/q}\right\}}{2^{1+1/p}(p+1)^{1/p}}, & 1\leq\left|f'(a)\right|,\left|f'(b)\right|\\ \frac{(b-a)\left|f'(b)\right|\left\{[F_5(\mu)]^{1/q}+[F_6(\mu)]^{1/q}\right\}}{2^{1+1/p}(p+1)^{1/p}}, & 0<\left|f'(a)\right|\leq 1<\left|f'(b)\right|\\ \frac{(b-a)\left|f'(b)\right|^{s}\left|f'(a)\right|^{(1-s)}\left\{[F_5(\mu)]^{1/q}+[F_6(\mu)]^{1/q}\right\}}{2^{1+1/p}(p+1)^{1/p}}, & 0<\left|f'(b)\right|\leq 1<\left|f'(a)\right|,\end{cases}, \quad (2.26)$$

where $\mu=\left|\frac{f'(a)}{f'(b)}\right|^{sq}$

$$F_5(\mu)=\begin{cases}\frac{\mu^{1/2}-1}{\ln\mu}, & \mu\neq 1\\ \frac{1}{2}, & \mu=1\end{cases},\ F_6(\mu)=\begin{cases}\frac{\mu-\mu^{1/2}}{\ln\mu}, & \mu\neq 1\\ \frac{1}{2}, & \mu=1\end{cases}$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 9. *Under the assumptions of Theorem 7, if $s=1$, we have the inequality*

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^{k}+1\right](\eta(b,a))^{k}}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx\right|$$
$$\leq\frac{(b-a)^{n}\left|f^{(n)}(b)\right|\left\{[F_5(\mu)]^{1/q}+[F_6(\mu)]^{1/q}\right\}}{2^{n+1/p}(np+1)^{1/p}n!}, \quad (2.27)$$

where

$$F_5(\mu)=\begin{cases}\frac{\mu^{1/2}-1}{\ln\mu}, & \mu\neq 1\\ \frac{1}{2}, & \mu=1\end{cases},\ F_6(\mu)=\begin{cases}\frac{\mu-\mu^{1/2}}{\ln\mu}, & \mu\neq 1\\ \frac{1}{2}, & \mu=1\end{cases}$$

$\mu=\left|\frac{f^{(n)}(a)}{f^{(n)}(b)}\right|^q$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

3. Applications to Special Means

For positive numbers $a>0$, $b>0$, define

$$A(a,b)=\frac{a+b}{2},\ G(a,b)=\sqrt{ab},\ H(a,b)=\frac{2ab}{a+b},$$

$$I(a,b)=\begin{cases}\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, & a\neq b,\\ a & a=b,\end{cases}$$

and

$$L_p(a,b)=\begin{cases}\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & p\neq 0,-1 \text{ and } a\neq b,\\ \frac{b-a}{\ln b-\ln a}, & p=-1 \text{ and } a\neq b,\\ I(a,b), & p=0 \text{ and } a\neq b,\\ a, & a=b.\end{cases}$$

It is well known that A, G, H, $L=L_{-1}$, $I=L_0$ and L_p are called the arithmetic, geometric, harmonic, logarithmic, exponential and generalized logarithmic means of positive numbers a and b.

In what follows we will use the above means and the established results of the previous section to obtain some interesting inequalities involving means.

Theorem 8. *Let* $0<a<b\leq 1$, $r<0$, $r\neq -1,-2$, $s\in(0,1]$ *and* $q\geq 1$.

(1) *If* $r\neq -3$, *then*

$$\begin{aligned}&\left|A\left(a^{r+2},b^{r+2}\right)-\left[L_{r+2}(a,b)\right]^{r+2}\right|\\ &\leq\frac{(b-a)^2}{4}\left(\frac{1}{3}\right)^{1-1/q}\left|(r+2)(r+1)\right|\\ &\times\left[\frac{2a^{qr}b^{qr}}{rsq(\ln b-\ln a)}\right]^{1/q}\left[\frac{2\left[L\left(a^{rqs},b^{rqs}\right)\right]^2}{b^{rqs}\left(b^{rqs}-a^{rqs}\right)^2}\right.\\ &\left.+\frac{\left[a^{rqs}+2b^{rqs}\right]L\left(a^{rqs},b^{rqs}\right)}{a^{rqs}b^{rqs}\left(b^{rqs}-a^{rqs}\right)}-\frac{1}{a^{rqs}}\right]^{1/q}.\end{aligned}$$

(2) *If* $r=-3$, *then*

$$\left|\frac{1}{H(a,b)}-\frac{1}{I(a,b)}\right|\le\frac{(b-a)^2}{2}\left(\frac{1}{3}\right)^{1-1/q}$$
$$\times\left[\frac{2}{3qsa^{3q}b^{3q}(\ln a-\ln b)}\right]^{1/q}\left[\frac{2\left[L\left(a^{-3qs},b^{-3qs}\right)\right]^2}{b^{-3qs}\left(b^{-3qs}-a^{-3qs}\right)^2}\right.$$
$$\left.+\frac{\left[a^{-3qs}+2b^{-3qs}\right]L\left(a^{-3qs},b^{-3qs}\right)}{a^{-3qs}b^{-3qs}\left(b^{-3qs}-a^{-3qs}\right)}-a^{3qs}\right]^{1/q}.$$

Proof. Let $f(x)=\frac{x^{r+2}}{(r+2)(r+1)}$ for $0<x\le 1$. Then $\left|f''(x)\right|=x^r$ and

$$\ln\left|f''(\lambda x+(1-\lambda)y)\right|^q\le\lambda^s\ln\left|f''(x)\right|^q+(1-\lambda)^s\left|f''(y)\right|^q$$

for x, $y\in(0,1]$, $\lambda\in[0,1]$, $s\in(0,1]$ and $q\ge 1$. This shows that $f(x)=\frac{x^{r+2}}{(r+2)(r+1)}$ is s-logarthmically convex function on $(0,1]$. Since$\left|f''(a)\right|>\left|f''(b)\right|=b^r\ge 1$, hence

$$\mu=\left|\frac{f''(a)}{f''(b)}\right|^{qs}=\left(\frac{a}{b}\right)^{rqs}$$

and

$$\left|f''(b)\right|^q\left|f''(a)\right|^{q(1-s)}F_1(\mu,2)$$
$$=\left[\frac{2a^{qr}b^{qr}}{rsq(\ln b-\ln a)}\right]\left[\frac{2\left[L\left(a^{rqs},b^{rqs}\right)\right]^2}{b^{rqs}\left(b^{rqs}-a^{rqs}\right)^2}\right.$$
$$\left.+\frac{\left[a^{rqs}+2b^{rqs}\right]L\left(a^{rqs},b^{rqs}\right)}{a^{rqs}b^{rqs}\left(b^{rqs}-a^{rqs}\right)}-\frac{1}{a^{rqs}}\right].$$

Substituting the above quantities in Corollary 2, we get the required inequality. □

Remark 2. *The other results given above may also give very interesting inequalities containing means and the details are left to the interested reader.*

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