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LEBESGUE INTEGRAL INEQUALITIES OF JENSEN TYPE FOR λ-CONVEX FUNCTIONS

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Abstract. Some Lebesgue integral inequalities of Jensen type for λ-convex functions defined on real intervals are given.

1. Introduction

1.1. **h-Convex Functions.** We recall here some concepts of convexity that are well known in the literature.

Let I be an interval in $\mathbb{R}$.

Definition 1 ([42]). *We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0,1)$ we have*

$$f(tx + (1-t)y) \leq \frac{1}{t} f(x) + \frac{1}{1-t} f(y). \tag{1.1}$$

Some further properties of this class of functions can be found in [32], [33], [35], [48], [51] and [52]. Among others, its has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

The above concept can be extended for functions $f : C \subseteq X \to [0, \infty)$ where C is a convex subset of the real or complex linear space X and the inequality (1.1) is satisfied for any vectors $x, y \in C$ and $t \in (0,1)$. If the function $f : C \subseteq X \to \mathbb{R}$ is non-negative and convex, then is of Godunova-Levin type.

Definition 2 ([35]). *We say that a function $f : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0,1]$ we have*

$$f(tx + (1-t)y) \leq f(x) + f(y). \tag{1.2}$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions,* i. e. nonnegative functions satisfying

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\} \tag{1.3}$$

for all $x, y \in I$ and $t \in [0,1]$.

For some results on P-functions see [35] and [49] while for quasi convex functions, the reader can consult [34].

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If $f : C \subseteq X \to [0, \infty)$, where C is a convex subset of the real or complex linear space X, then we say that it is of P-type (or quasi-convex) if the inequality (1.2) (or (1.3)) holds true for $x, y \in C$ and $t \in [0, 1]$.

Definition 3 ([7])**.** *Let s be a real number, $s \in (0, 1]$. A function $f : [0, \infty) \to [0, \infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if*

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0, \infty)$ and $t \in [0, 1]$.

For some properties of this class of functions see [1], [2], [7], [8], [30], [31], [43], [45] and [54].

The concept of Breckner s-convexity can be similarly extended for functions defined on convex subsets of linear spaces.

It is well known that if $(X, \|\cdot\|)$ is a normed linear space, then the function $f(x) = \|x\|^p$, $p \ge 1$ is convex on X.

Utilising the elementary inequality $(a + b)^s \le a^s + b^s$ that holds for any $a, b \ge 0$ and $s \in (0, 1]$, we have for the function $g(x) = \|x\|^s$ that

$$\begin{aligned} g(tx + (1-t)y) &= \|tx + (1-t)y\|^s \le (t\|x\| + (1-t)\|y\|)^s \\ &\le (t\|x\|)^s + [(1-t)\|y\|]^s \\ &= t^s g(x) + (1-t)^s g(y) \end{aligned}$$

for any $x, y \in X$ and $t \in [0, 1]$, which shows that g is Breckner s-convex on X.

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}$, $(0, 1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I, respectively.

Definition 4 ([58])**.** *Let $h : J \to [0, \infty)$ with h not identical to 0. We say that $f : I \to [0, \infty)$ is an h-convex function if for all $x, y \in I$ we have*

$$f(tx + (1-t)y) \le h(t) f(x) + h(1-t) f(y) \tag{1.4}$$

for all $t \in (0, 1)$.

For some results concerning this class of functions see [58], [6], [46], [55], [53] and [57].

This concept can be extended for functions defined on convex subsets of linear spaces in the same way as above replacing the interval I be the corresponding convex subset C of the linear space X.

We can introduce now another class of functions.

Definition 5. *We say that the function $f : C \subseteq X \to [0, \infty)$ is of s-Godunova-Levin type, with $s \in [0, 1]$, if*

$$f(tx + (1-t)y) \le \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y), \tag{1.5}$$

for all $t \in (0, 1)$ and $x, y \in C$.

We observe that for $s = 0$ we obtain the class of P-functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(C)$ the class of s-Godunova-Levin functions defined on C, then we obviously have

$$P(C) = Q_0(C) \subseteq Q_{s_1}(C) \subseteq Q_{s_2}(C) \subseteq Q_1(C) = Q(C)$$

for $0 \leq s_1 \leq s_2 \leq 1$.

For different inequalities related to these classes of functions, see [1]-[4], [6], [9]-[41], [44]-[46] and [49]-[57].

A function $h : J \to \mathbb{R}$ is said to be *supermultiplicative* if

$$h(ts) \geq h(t) h(s) \text{ for any } t, s \in J. \tag{1.6}$$

If the inequality (1.6) is reversed, then h is said to be *submultiplicative*. If the equality holds in (1.6) then h is said to be a multiplicative function on J.

In [58] it has been noted that if $h : [0, \infty) \to [0, \infty)$ with $h(t) = (x + c)^{p-1}$, then for $c = 0$ the function h is multiplicative. If $c \geq 1$, then for $p \in (0, 1)$ the function h is supermultiplicative and for $p > 1$ the function is submultiplicative.

We observe that, if h, g are nonnegative and supermultiplicative, the same is their product. In particular, if h is supermultiplicative then its product with a power function $\ell_r(t) = t^r$ is also supermultiplicative.

1.2. **λ-Convex Functions.** We start with the following definition (see also [26]):

Definition 6. *Let $\lambda : [0, \infty) \to [0, \infty)$ be a function with the property that $\lambda(t) > 0$ for all $t > 0$. A mapping $f : C \to \mathbb{R}$ defined on convex subset C of a linear space X is called λ-convex on C if*

$$f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) \leq \frac{\lambda(\alpha) f(x) + \lambda(\beta) f(y)}{\lambda(\alpha + \beta)} \tag{1.7}$$

for all $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ and $x, y \in C$.

We observe that if $f : C \to \mathbb{R}$ is λ-convex on C, then f is h-convex on C with $h(t) = \frac{\lambda(t)}{\lambda(1)}$, $t \in [0, 1]$.

If $f : C \to [0, \infty)$ is h-convex function with h supermultiplicative on $[0, \infty)$, then f is λ-convex with $\lambda = h$.

Indeed, if $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ and $x, y \in C$ then

$$\begin{aligned} f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) &\leq h\left(\frac{\alpha}{\alpha + \beta}\right) f(x) + h\left(\frac{\beta}{\alpha + \beta}\right) f(y) \\ &\leq \frac{h(\alpha) f(x) + h(\beta) f(y)}{h(\alpha + \beta)}. \end{aligned}$$

The following proposition contain some properties of λ-convex functions [26].

Proposition 1. *Let $f : C \to \mathbb{R}$ be a λ-convex function on C.*

(i) If $\lambda(0) > 0$, then we have $f(x) \geq 0$ for all $x \in C$;

(ii) If there exists $x_0 \in C$ so that $f(x_0) > 0$, then

$$\lambda(\alpha + \beta) \leq \lambda(\alpha) + \lambda(\beta)$$

for all $\alpha, \beta > 0$, i.e. the mapping λ is subadditive on $(0, \infty)$.

(iii) If there exists $x_0, y_0 \in C$ with $f(x_0) > 0$ and $f(y_0) < 0$, then

$$\lambda(\alpha + \beta) = \lambda(\alpha) + \lambda(\beta)$$

for all $\alpha, \beta > 0$, i.e. the mapping λ is additive on $(0, \infty)$.

We have the following result providing many examples of subadditive functions $\lambda : [0, \infty) \to [0, \infty)$.

Theorem 1 ([26])**.** *Let $h(z)=\sum_{n=0}^{\infty}a_nz^n$ a power series with nonnegative coefficients $a_n\geq 0$ for all $n\in\mathbb{N}$ and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$. If $r\in(0,R)$ then the function $\lambda_r:[0,\infty)\to[0,\infty)$ given by*

$$\lambda_r(t):=\ln\left[\frac{h(r)}{h(r\exp(-t))}\right] \tag{1.8}$$

is nonnegative, increasing and subadditive on $[0,\infty)$.

We have the following fundamental examples of power series with positive coefficients

$$h(z)=\sum_{n=0}^{\infty}z^n=\frac{1}{1-z},\ z\in D(0,1) \tag{1.9}$$
$$h(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^n=\exp(z)\qquad z\in\mathbb{C},$$
$$h(z)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n}=\cosh z,\ z\in\mathbb{C};$$
$$h(z)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1}=\sinh z,\ z\in\mathbb{C};$$
$$h(z)=\sum_{n=1}^{\infty}\frac{1}{n}z^n=\ln\frac{1}{1-z},\ z\in D(0,1).$$

Other important examples of functions as power series representations with positive coefficients are:

$$h(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}=\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right),\qquad z\in D(0,1); \tag{1.10}$$
$$h(z)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}z^{2n+1}=\sin^{-1}(z),\qquad z\in D(0,1);$$
$$h(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}=\tanh^{-1}(z),\qquad z\in D(0,1);$$
$$h(z)={}_2F_1(\alpha,\beta,\gamma,z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}z^n,\alpha,\beta,\gamma>0,$$
$$z\in D(0,1);$$

where Γ is *Gamma function.*

Remark 1. *Now, if we take $h(z)=\frac{1}{1-z}$, $z\in D(0,1)$, then*

$$\lambda_r(t)=\ln\left[\frac{1-r\exp(-t)}{1-r}\right] \tag{1.11}$$

is nonnegative, increasing and subadditive on $[0,\infty)$ for any $r\in(0,1)$.

If we take $h(z)=\exp(z)$, $z\in\mathbb{C}$ then

$$\lambda_r(t)=r[1-\exp(-t)] \tag{1.12}$$

is nonnegative, increasing and subadditive on $[0,\infty)$ for any $r>0$.

Corollary 1 ([26]). *Let $h(z)=\sum_{n=0}^{\infty}a_nz^n$ a power series with nonnegative coefficients $a_n\geq 0$ for all $n\in\mathbb{N}$ and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$ and $r\in(0,R)$. For a mapping $f:C\to\mathbb{R}$ defined on convex subset C of a linear space X, the following statements are equivalent:*

(i) The function f is λ_r-convex with $\lambda_r:[0,\infty)\to[0,\infty)$,

$$\lambda_r(t):=\ln\left[\frac{h(r)}{h(r\exp(-t))}\right];$$

(ii) We have the inequality

$$\left[\frac{h(r)}{h(r\exp(-\alpha-\beta))}\right]^{f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)}\leq\left[\frac{h(r)}{h(r\exp(-\alpha))}\right]^{f(x)}\left[\frac{h(r)}{h(r\exp(-\beta))}\right]^{f(y)} \tag{1.13}$$

for any $\alpha,\beta\geq 0$ with $\alpha+\beta>0$ and $x,y\in C$.

(iii) We have the inequality

$$\frac{[h(r\exp(-\alpha))]^{f(x)}[h(r\exp(-\beta))]^{f(y)}}{[h(r\exp(-\alpha-\beta))]^{f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)}}\leq[h(r)]^{f(x)+f(y)-f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)} \tag{1.14}$$

for any $\alpha,\beta\geq 0$ with $\alpha+\beta>0$ and $x,y\in C$.

Remark 2. *We observe that, in the case when*

$$\lambda_r(t)=r[1-\exp(-t)],\ t\geq 0$$

then the function f is λ_r-convex on convex subset C of a linear space X iff

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)\leq\frac{[1-\exp(-\alpha)]f(x)+[1-\exp(-\beta)]f(y)}{1-\exp(-\alpha-\beta)} \tag{1.15}$$

for any $\alpha,\beta\geq 0$ with $\alpha+\beta>0$ and $x,y\in C$.

We observe that this definition is independent of $r>0$.

The inequality (1.15) is equivalent with

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)\leq\frac{\exp(\beta)[\exp(\alpha)-1]f(x)+\exp(\alpha)[\exp(\beta)-1]f(y)}{\exp(\alpha+\beta)-1} \tag{1.16}$$

for any $\alpha,\beta\geq 0$ with $\alpha+\beta>0$ and $x,y\in C$.

We have the following Jensen inequality for the Riemann integral [28]:

Theorem 2. *Let $u:[a,b]\to[m,M]$ be a Riemann integrable function on $[a,b]$. Let $\lambda:[0,\infty)\to[0,\infty)$ be a function with the property that $\lambda(t)>0$ for all $t>0$ and the function $f:[m,M]\to[0,\infty)$ is λ-convex and Riemann integrable on the interval $[m,M]$. If the following limit exists*

$$\lim_{t\to 0+}\frac{\lambda(t)}{t}=k\in(0,\infty) \tag{1.17}$$

then

$$f\left(\frac{1}{b-a}\int_a^b u(t)\,dt\right)\leq\frac{k}{\lambda(b-a)}\int_a^b f(u(t))\,dt. \tag{1.18}$$

The following weighted version of Jensen inequality for the Riemann integral [28] also holds.

Theorem 3. *Let $u, w : [a,b] \to [m,M]$ be Riemann integrable functions on $[a,b]$ and $w(t) \geq 0$ for any $t \in [a,b]$ with $\int_a^b w(t)\,dt > 0$. Let $\lambda : [0,\infty) \to [0,\infty)$ be a function with the property that $\lambda(t) > 0$ for all $t > 0$ and the function $f : [m,M] \to [0,\infty)$ is λ-convex and Riemann integrable on the interval $[m,M]$. If the following limit exists, is finite and*

$$\lim_{t\to\infty} \frac{t}{\lambda(t)} = \ell > 0, \tag{1.19}$$

then

$$f\left(\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)\,u(t)\,dt\right) \leq \ell \frac{1}{\int_a^b w(t)\,dt}\int_a^b \lambda(w(t))\,f(u(t))\,dt. \tag{1.20}$$

Motivated by the above results we establish in this paper some Jensen type inequalities for the general Lebesgue integral.

2. Some Results for Differentiable Functions

If we assume that the function $f : I \to [0,\infty)$ is differentiable on the interior of I, denoted by $\mathring{I}$, then we have the following *"gradient inequality"* that will play an essential role in the following.

Lemma 1. *Let $\lambda : (0,\infty) \to (0,\infty)$ be a function and such that the right limit*

$$\lim_{t\to 0+} \frac{\lambda(t)}{t} = k \in (0,\infty) \tag{2.1}$$

exists and is finite, and the left derivative in 1 denoted $\lambda'_-(1)$ exists and is finite.

If the function $f : I \to [0,\infty)$ is differentiable on $\mathring{I}$ and λ-convex, then

$$kf(x) - \lambda'_-(1)\,f(y) \geq \lambda(1)\,f'(y)(x-y) \tag{2.2}$$

for any $x, y \in \mathring{I}$ with $x \neq y$.

Proof. Since f is λ-convex on I, then

$$\frac{\lambda(t)\,f(x) + \lambda(1-t)\,f(y)}{\lambda(1)} \geq f(tx + (1-t)\,y)$$

for any $t \in (0,1)$ and for any $x, y \in \mathring{I}$, which is equivalent to

$$\lambda(t)\,f(x) + [\lambda(1-t) - \lambda(1)]\,f(y) \geq \lambda(1)\,[f(tx + (1-t)\,y) - f(y)]$$

and by dividing by $t > 0$ we get

$$\frac{\lambda(t)}{t} f(x) + \left[\frac{\lambda(1-t) - \lambda(1)}{t}\right] f(y) \geq \lambda(1)\,\frac{f(tx + (1-t)\,y) - f(y)}{t} \tag{2.3}$$

for any $t \in (0,1)$.

Now, since f is differentiable on $y \in \mathring{I}$, then we have

$$\begin{aligned} \lim_{t\to 0+} \frac{f(tx + (1-t)\,y) - f(y)}{t} &= \lim_{t\to 0+} \frac{f(y + t(x-y)) - f(y)}{t} \\ &= (x-y) \lim_{t\to 0+} \frac{f(y + t(x-y)) - f(y)}{t(x-y)} \\ &= (x-y)\,f'(y) \end{aligned} \tag{2.4}$$

for any $x \in \mathring{I}$ with $x \neq y$.

Also we have

$$\lim_{t\to 0+}\frac{\lambda(1-t)-\lambda(1)}{t}=\lim_{s\to 1-}\frac{\lambda(s)-\lambda(1)}{1-s}=-\lim_{s\to 1-}\frac{\lambda(s)-\lambda(1)}{s-1}=-\lambda'_{-}(1) \tag{2.5}$$

Taking the limit over $t\to 0+$ in (2.3) and utilizing (2.4) and (2.5) we get the desired result (2.2). □

Remark 3. *If we assume that*

$$k\geq \lambda'_{-}(1), \tag{2.6}$$

then the inequality (2.2) also holds for $x=y$.

Remark 4. *If* $\lambda:[0,\infty)\to[0,\infty)$ *with* $\lambda(0)=0$ *then the condition (2.1) is equivalent with the fact that the right derivative*

$$\lambda_{+}(0)=\lim_{t\to 0+}\frac{\lambda(t)}{t}$$

exist is finite and $\lambda_{+}(0)=k$.

In this situation the inequality (2.2) becomes for $\lambda_{+}(0)>0$

$$\lambda_{+}(0)f(x)-\lambda'_{-}(1)f(y)\geq\lambda(1)f'(y)(x-y) \tag{2.7}$$

for any $x,y\in\mathring{I}$ *with* $x\neq y$.

If the function λ *is subadditive on* $[0,\infty)$ *and has finite lateral derivatives with* $\lambda_{+}(0)>0$, *then*

$$\lambda(t)+\lambda(1-t)\geq\lambda(1),\ t\in(0,1),$$

i.e.

$$\frac{\lambda(t)}{t}\geq\frac{\lambda(1)-\lambda(1-t)}{t},\ t\in(0,1). \tag{2.8}$$

Taking the limit over $t\to 0+$ *in (2.8) we get*

$$\lambda_{+}(0)\geq\lambda'_{-}(1),$$

therefore the inequality (2.7) also holds for $x=y$.

We have the following result.

Corollary 2. *Let* $\lambda:[0,\infty)\to[0,\infty)$ *be a subadditive function with* $\lambda(0)=0$ *and having the lateral derivative* $\lambda_{+}(0),\ \lambda'_{-}(1)\in(0,\infty)$.

If the function $f:I\to[0,\infty)$ *is differentiable on* $\mathring{I}$ *and* λ*-convex, then*

$$\lambda_{+}(0)f(x)-\lambda'_{-}(1)f(y)\geq\lambda(1)f'(y)(x-y) \tag{2.9}$$

for any $x,y\in\mathring{I}$.

As examples of such functions we have:

Proposition 2. *Let* $h(z)=\sum_{n=0}^{\infty}a_nz^n$ *a power series with nonnegative coefficients* $a_n\geq 0$ *for all* $n\in\mathbb{N}$ *and convergent on the open disk* $D(0,R)$ *with* $R>0$ *or* $R=\infty$ *and* $r\in(0,R)$. *If the function* $f:I\to[0,\infty)$ *is differentiable on* $\mathring{I}$ *and* λ_r*-convex with* $\lambda_r:[0,\infty)\to[0,\infty)$,

$$\lambda_r(t):=\ln\left[\frac{h(r)}{h(r\exp(-t))}\right],$$

then

$$\frac{rh'(r)}{h(r)}f(x)-\frac{re^{-1}h'\left(re^{-1}\right)}{h\left(re^{-1}\right)}f(y)\geq\ln\left[\frac{h(r)}{h\left(re^{-1}\right)}\right]f'(y)(x-y), \tag{2.10}$$

for any $x,y\in\mathring{I}$.

Proof. We know that λ_r is differentiable on $(0,\infty)$ and

$$\lambda_r'(t):=\frac{r\exp(-t)\,h'(r\exp(-t))}{h(r\exp(-t))}$$

for $t\in(0,\infty)$, where

$$h'(z)=\sum_{n=1}^{\infty}na_nz^{n-1}.$$

Since $\lambda_r(0)=0$, then

$$k=\lim_{s\to0+}\frac{\lambda(s)}{s}=\lambda_+'(0)=\frac{rh'(r)}{h(r)}>0\text{ for }r\in(0,R).$$

Also

$$\lambda_r'(1)=\frac{re^{-1}h'\left(re^{-1}\right)}{h\left(re^{-1}\right)}$$

and

$$\lambda_r(1)=\ln\left[\frac{h(r)}{h\left(re^{-1}\right)}\right].$$

Applying Corollary 2 we deduce the desired result (2.10). □

Corollary 3. *If the function* $f:I\to[0,\infty)$ *is differentiable on* $\mathring{I}$ *and* λ*-convex with* $\lambda:[0,\infty)\to[0,\infty)$, $\lambda(t)=1-\exp(-t)$, *then we have*

$$ef(x)-f(y)\geq(e-1)f'(y)(x-y) \tag{2.11}$$

for any $x,y\in\mathring{I}$.

It follows by Proposition 2 observing that $\lambda'(t)=\exp(-t)$, $t>0$.

3. Jensen Type Inequalities

Let $(\Omega,\mathcal{A},\mu)$ be a measurable space consisting of a set Ω, a $\sigma-$algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. For a $\mu-$measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\geq0$ for $\mu-$ a.e.(almost every) $x\in\Omega$, consider the Lebesgue space

$$L_w(\Omega,\mu):=\{f:\Omega\to\mathbb{R},\ f\text{ is }\mu\text{-measurable and }\int_\Omega w(x)|f(x)|\,d\mu(x)<\infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega wd\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$.

Theorem 4. *Let* $\lambda:[0,\infty)\to[0,\infty)$ *be a subadditive function with* $\lambda(0)=0$ *and having the lateral derivative* $\lambda_+(0)$, $\lambda_-'(1)\in(0,\infty)$. *If the function* $f:I\to[0,\infty)$ *is differentiable on* $\mathring{I}$ *and* λ*-convex, then for any* $u:\Omega\to[m,M]\subset\mathring{I}$ *so that* $f\circ u$, $u\in L_w(\Omega,\mu)$, *where* $w\geq0$ μ*-a.e. (*μ*-almost everywhere) on* Ω *with* $\int_\Omega wd\mu=1$ *we have*

$$\int_\Omega w\cdot(f\circ u)\,d\mu\geq\frac{\lambda_-'(1)}{\lambda_+(0)}f\left(\int_\Omega wud\mu\right). \tag{3.1}$$

Proof. Observe that, since $u:\Omega\to[m,M]$ and $u\in L_w(\Omega,\mu)$, then $\int_\Omega wud\mu\in[m,M]$. Applying Corollary 2 we have

$$\lambda_+(0)\,f(u(t))-\lambda'_-(1)\,f\left(\int_\Omega wud\mu\right)\geq\lambda(1)\,f'\left(\int_\Omega wud\mu\right)\left(u(t)-\int_\Omega wud\mu\right)\tag{3.2}$$

for any $t\in\Omega$.

If we multiply (3.2) by $w(t)\geq0$ for μ-almost every $t\in\Omega$ we get

$$\lambda_+(0)\,w(t)\,f(u(t))-\lambda'_-(1)\,f\left(\int_\Omega wud\mu\right)w(t)\geq\lambda(1)\,f'\left(\int_\Omega wud\mu\right)\left(w(t)\,u(t)-\left(\int_\Omega wud\mu\right)w(t)\right)\tag{3.3}$$

for μ-almost every $t\in\Omega$.

Integrating (3.3) over t on Ω we get

$$\begin{aligned}&\lambda_+(0)\int_\Omega w(t)\,f(u(t))\,d\mu(t)-\lambda'_-(1)\,f\left(\int_\Omega wud\mu\right)\int_\Omega w(t)\,d\mu(t)\\&\geq\lambda(1)\,f'\left(\int_\Omega wud\mu\right)\\&\times\left(\int_\Omega w(t)\,u(t)\,d\mu(t)-\left(\int_\Omega wud\mu\right)\int_\Omega w(t)\,d\mu(t)\right)\end{aligned}\tag{3.4}$$

and since $\int_\Omega w(t)\,d\mu(t)=1$, we deduce the desired result (3.1). □

The following inequality of Hermite-Hadamard type holds:

Corollary 4. *Let $\lambda:[0,\infty)\to[0,\infty)$ be a subadditive function with $\lambda(0)=0$ and having the lateral derivative $\lambda_+(0)$, $\lambda'_-(1)\in(0,\infty)$. If the function $f:I\to[0,\infty)$ is differentiable on $\mathring{I}$ and λ-convex, then for any $[a,b]\subset\mathring{I}$ we have*

$$\frac{1}{b-a}\int_a^b f(t)\,dt\geq\frac{\lambda'_-(1)}{\lambda_+(0)}f\left(\frac{a+b}{2}\right).\tag{3.5}$$

Follows from Theorem 4 by taking $\Omega=[a,b]$, $u:[a,b]\to[a,b]$, $u(t)=t$, $w(t)=\frac{1}{b-a}$ and $d\mu=dt$ is the Lebesgue measure on the interval $[a,b]$.

If we consider the discrete measure, then we have:

Corollary 5. *Let $\lambda:[0,\infty)\to[0,\infty)$ be a subadditive function with $\lambda(0)=0$ and having the lateral derivative $\lambda_+(0)$, $\lambda'_-(1)\in(0,\infty)$. If the function $f:I\to[0,\infty)$ is differentiable on $\mathring{I}$ and λ-convex, then for any $x_i\in\mathring{I}$ and $p_i\geq0$, $i\in\{1,...,n\}$ with $\sum_{i=1}^n p_i=1$ we have*

$$\sum_{i=1}^n p_i f(x_i)\geq\frac{\lambda'_-(1)}{\lambda_+(0)}f\left(\sum_{i=1}^n p_i x_i\right).$$

Remark 5. *Let $h(z)=\sum_{n=0}^\infty a_n z^n$ a power series with nonnegative coefficients $a_n\geq0$ for all $n\in\mathbb{N}$ and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$*

and $r \in (0, R)$. *Assume that the function* $f : I \to [0, \infty)$ *is differentiable on* $\mathring{I}$ *and* λ_r*-convex with* $\lambda_r : [0, \infty) \to [0, \infty)$,

$$\lambda_r(t) := \ln \left[\frac{h(r)}{h(r \exp(-t))} \right].$$

If $f : I \to [0, \infty)$ *is differentiable on* $\mathring{I}$ *and* λ_r*-convex, then for any* $u : \Omega \to [m, M] \subset \mathring{I}$ *so that* $f \circ u$, $u \in L_w(\Omega, \mu)$, *where* $w \geq 0$ μ*-a.e. (*μ*-almost everywhere) on* Ω *with* $\int_\Omega w d\mu = 1$ *we have*

$$\int_\Omega w \cdot (f \circ u) \, d\mu \geq \frac{e^{-1} h'(re^{-1}) h(r)}{h(re^{-1}) h'(r)} f\left(\int_\Omega w u d\mu \right). \tag{3.6}$$

Remark 6. *If the function* $f : I \to [0, \infty)$ *is differentiable on* $\mathring{I}$ *and* λ*-convex with* $\lambda : [0, \infty) \to [0, \infty)$, $\lambda(t) = 1 - \exp(-t)$, *then for any* $[a, b] \subset \mathring{I}$ *we have*

$$\frac{1}{b-a} \int_a^b f(t) \, dt \geq \frac{1}{e} f\left(\frac{a+b}{2} \right). \tag{3.7}$$

Also, for any $x_i \in \mathring{I}$ *and* $p_i \geq 0$, $i \in \{1, ..., n\}$ *with* $\sum_{i=1}^n p_i = 1$ *we have*

$$\sum_{i=1}^n p_i f(x_i) \geq \frac{1}{e} f\left(\sum_{i=1}^n p_i x_i \right). \tag{3.8}$$

Recall Slater's inequality for differentiable convex functions [56]:

Lemma 2. *Let* $f : I \to \mathbb{R}$ *be a nondecreasing (nonincreasing) differentiable convex function on* I, $x_i \in I$, $p_i \geq 0$ *with* $P_n = \sum_{i=1}^n p_i > 0$ *and assume that* $\sum_{i=1}^n p_i f'(x_i) \neq 0$. *Then one has the inequality*

$$f\left(\frac{\sum_{i=1}^n p_i x_i f'(x_i)}{\sum_{i=1}^n p_i f'(x_i)} \right) \geq \frac{1}{P_n} \sum_{i=1}^n p_i f(x_i). \tag{3.9}$$

As shown in [22, pp. 129-130], the monotonicity condition in Lemma 2 can be relaxed by assuming that

$$\frac{\sum_{i=1}^n p_i x_i f'(x_i)}{\sum_{i=1}^n p_i f'(x_i)} \in I.$$

We can state the following result that is similar to Slater's inequality:

Theorem 5. *Let* $\lambda : [0, \infty) \to [0, \infty)$ *be a subadditive function with* $\lambda(0) = 0$ *and having the lateral derivative* $\lambda_+(0)$, $\lambda'_-(1) \in (0, \infty)$. *If the function* $f : I \to [0, \infty)$ *is differentiable on* $\mathring{I}$ *and* λ*-convex, then for any* $u : \Omega \to [m, M] \subset \mathring{I}$ *so that* $f \circ u$, $u \cdot (f' \circ u)$, $f' \circ u \in L_w(\Omega, \mu)$, *where* $w \geq 0$ μ*-a.e. (*μ*-almost everywhere) on* Ω *with* $\int_\Omega w d\mu = 1$ *and*

$$\frac{\int_\Omega w u \cdot (f' \circ u) \, d\mu}{\int_\Omega w \cdot (f' \circ u) \, d\mu} \in [m, M],$$

we have

$$\frac{\lambda_+(0)}{\lambda'_-(1)} f\left(\frac{\int_\Omega w u \cdot (f' \circ u) \, d\mu}{\int_\Omega w \cdot (f' \circ u) \, d\mu} \right) \geq \int_\Omega w \cdot (f \circ u) \, d\mu. \tag{3.10}$$

Proof. Since the function $f : I \to [0, \infty)$ is differentiable on $\mathring{I}$ and λ-convex, then by (2.9) we have

$$\lambda_+ (0) f (x) - \lambda'_- (1) f (u (t)) \geq \lambda (1) f' (u (t)) (x - u (t)) \tag{3.11}$$

for any $x \in \mathring{I}$ and $t \in \Omega$.

If we multiply by $w (t) \geq 0$ and integrate we get

$$\begin{aligned} &\lambda_+ (0) f (x) - \lambda'_- (1) \int_\Omega w (t) f (u (t)) d\mu (t) \\ &\geq \lambda (1) x \int_\Omega w (t) f' (u (t)) d\mu (t) - \int_\Omega w (t) f' (u (t)) u (t) d\mu (t) , \end{aligned} \tag{3.12}$$

for any $x \in \mathring{I}$.

Since $\int_\Omega w (t) f' (u (t)) d\mu (t) \neq 0$ and

$$x_0 := \frac{\int_\Omega w (t) f' (u (t)) u (t) d\mu (t)}{\int_\Omega w (t) f' (u (t)) d\mu (t)} \in [m, M] ,$$

then by taking $x = x_0$ in (3.12) we get the desired result (3.10). □

The following Hermite-Hadamard type inequality holds:

Corollary 6. *Let $\lambda : [0, \infty) \to [0, \infty)$ be a subadditive function with $\lambda (0) = 0$ and having the lateral derivative $\lambda_+ (0) , \lambda'_- (1) \in (0, \infty)$. If the function $f : I \to [0, \infty)$ is differentiable on $\mathring{I}$ and λ-convex, and for $[a, b] \subset \mathring{I}$ we have*

$$\frac{\int_a^b t f' (t) dt}{f (b) - f (a)} = \frac{b f (b) - a f (a) - \int_a^b f (t) dt}{f (b) - f (a)} \in [a, b] , \tag{3.13}$$

then we have

$$\frac{\lambda_+ (0)}{\lambda'_- (1)} f \left(\frac{b f (b) - a f (a) - \int_a^b f (t) dt}{f (b) - f (a)} \right) \geq \frac{1}{b - a} \int_a^b f (t) dt. \tag{3.14}$$

The following discrete inequality also holds:

Corollary 7. *Let $\lambda : [0, \infty) \to [0, \infty)$ be a subadditive function with $\lambda (0) = 0$ and having the lateral derivative $\lambda_+ (0) , \lambda'_- (1) \in (0, \infty)$. If the function $f : I \to [0, \infty)$ is differentiable on $\mathring{I}$ and λ-convex, then for any $x_i \in \mathring{I}$ and $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$ and*

$$\frac{\sum_{i=1}^n p_i x_i f' (x_i)}{\sum_{i=1}^n p_i f' (x_i)} \in \mathring{I},$$

we have

$$\frac{\lambda_+ (0)}{\lambda'_- (1)} f \left(\frac{\sum_{i=1}^n p_i x_i f' (x_i)}{\sum_{i=1}^n p_i f' (x_i)} \right) \geq \sum_{i=1}^n p_i f (x_i) . \tag{3.15}$$

Remark 7. *The interested reader can obtain some particular inequalities of interest by taking λ_r-convex functions with $\lambda_r : [0, \infty) \to [0, \infty)$,*

$$\lambda_r (t) := \ln \left[\frac{h (r)}{h (r \exp (-t))} \right] ,$$

and h is as in Theorem 1. The details are omitted.

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