Received 11/02/14

INEQUALITIES FOR THE AREA BALANCE OF ABSOLUTELY CONTINUOUS FUNCTIONS

S. S. DRAGOMIR[1,2]

Abstract. We introduce the *area balance* function associated to a Lebesgue integrable function $f:[a,b]\to\mathbb{C}$ by

$$AB_f(a,b,\cdot):[a,b]\to\mathbb{C},\ AB_f(a,b,x):=\frac{1}{2}\left[\int_x^b f(t)\,dt-\int_a^x f(t)\,dt\right].$$

We show amongst other that, if $f:I\to\mathbb{C}$ is an absolutely continuous function on the interval I and $[a,b]\subset\mathring{I}$, where $\mathring{I}$ is the interior of I and such that f' is of bounded variation on $[a,b]$, then we have the inequality

$$\left|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)-\frac{f'(a)+f'(b)}{4}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]\right|\le\frac{1}{4}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\bigvee_a^b(f')$$

for any $x\in[a,b]$.

If there exists the real numbers m,M such that

$$m\le f'(t)\le M \text{ for a.e. } t\in[a,b],$$

then also

$$\left|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)-\frac{m+M}{4}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]\right|\le\frac{1}{4}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right](M-m)$$

for any $x\in[a,b]$.

1. Introduction

For a *Lebesgue integrable* function $f:[a,b]\to\mathbb{C}$ and a number $x\in(a,b)$ we can naturally ask how far the integral $\int_x^b f(t)\,dt$ is from the integral $\int_a^x f(t)\,dt$. If f is nonnegative and continuous on $[a,b]$, then the above question has the geometrical interpretation of comparing the area under the curve generated by f at the right of the point x with the area at the left of x. The point x will be called a *median point*, if

$$\int_x^b f(t)\,dt=\int_a^x f(t)\,dt.$$

1991 *Mathematics Subject Classification.* 26D15; 25D10.

Key words and phrases. Functions of bounded variation, Lipschitzian functions, Convex functions, Integral inequalities.

Due to the above geometrical interpretation, we can introduce the *area balance* function associated to a Lebesgue integrable function $f:[a,b]\to\mathbb{C}$ defined as

$$AB_f(a,b,\cdot):[a,b]\to\mathbb{C},\ AB_f(a,b,x):=\frac{1}{2}\left[\int_x^b f(t)\,dt-\int_a^x f(t)\,dt\right].$$

Utilising the *cumulative function* notation $F:[a,b]\to\mathbb{C}$ given by

$$F(x):=\int_a^x f(t)\,dt$$

then we observe that

$$AB_f(a,b,x)=\frac{1}{2}F(b)-F(x),\ \ x\in[a,b].$$

If f is a *probability density*, i.e. f is nonnegative and $\int_a^b f(t)\,dt=1$, then

$$AB_f(a,b,x)=\frac{1}{2}-F(x),\ \ x\in[a,b].$$

In this paper we obtain some inequalities concerning the area balance for absolutely continuous. Applications for differentiable functions whose derivatives are Lipschitzian functions are provided. Bounds involving the *Jensen difference*

$$\frac{g(a)+g(b)}{2}-g\left(\frac{a+b}{2}\right)$$

are also established.

We notice that Jensen difference is closely related to the Hermite-Hadamard type inequalities where various bounds for the quantities

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt$$

and

$$\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)$$

are provided, see [1]-[6] and [8]-[18].

2. Preliminary Results

The following representation result holds:

Theorem 1. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$. Then we have the representation*

$$AB_f(a,b,x)=\left(\frac{a+b}{2}-x\right)f(x)+\frac{1}{2}\left[\int_a^x (t-a)f'(t)\,dt+\int_x^b (b-t)f'(t)\,dt\right] \tag{2.1}$$

and

$$AB_f(a,b,x)=\frac{bf(b)+af(a)}{2}-\frac{f(b)+f(a)}{2}x-\frac{1}{2}\int_a^b |t-x|\,f'(t)\,dt \tag{2.2}$$

for any $x \in [a,b]$, where the integrals in the right hand side are taken in the Lebesgue sense.

Proof. Since f is absolutely continuous on $[a,b]$, then f is differentiable almost everywhere (a.e.) on $[a,b]$ and the Lebesgue integrals in the right hand side of the equations (2.1) and (2.2) exist.

Utilising the integration by parts formula for the Lebesgue integral, we have

$$\begin{aligned} (2.3)\qquad & \int_a^x (t-a) f'(t)\,dt + \int_x^b (b-t) f'(t)\,dt \\ &= (t-a) f(t)\big|_a^x - \int_a^x f(t)\,dt + (b-t) f(t)\big|_x^b + \int_x^b f(t)\,dt \\ &= (x-a) f(x) - \int_a^x f(t)\,dt - (b-x) f(x) + \int_x^b f(t)\,dt \\ &= (2x-a-b) f(x) + 2AB_f(a,b,x) \end{aligned}$$

for any $x \in [a,b]$.

Dividing (2.3) by 2 and rearranging the equation, we deduce (2.1).

Integrating by parts, we also have

$$\begin{aligned} (2.4)\qquad & \int_a^b |t-x| f'(t)\,dt \\ &= \int_a^x (x-t) f'(t)\,dt + \int_x^b (t-x) f'(t)\,dt \\ &= (x-t) f(t)\big|_a^x + \int_a^x f(t)\,dt + (t-x) f(t)\big|_x^b - \int_x^b f(t)\,dt \\ &= -(x-a) f(a) + (b-x) f(b) - 2AB_f(a,b,x) \\ &= bf(b) + af(a) - [f(b)+f(a)]\,x - 2AB_f(a,b,x) \end{aligned}$$

for any $x \in [a,b]$.

Dividing (2.4) by 2 and rearranging the equation, we deduce (2.2). □

Corollary 1. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f'(t)\ge 0$ for a.e. $t\in[a,b]$, then*

$$\begin{aligned} (2.5)\qquad \frac{bf(b)+af(a)}{2} - \frac{f(b)+f(a)}{2}x &\ge AB_f(a,b,x) \\ &\ge \left(\frac{a+b}{2}-x\right) f(x) \end{aligned}$$

for any $x\in[a,b]$.

In particular,

$$(2.6)\qquad \frac{1}{4}(b-a)[f(b)-f(a)] \ge AB_f\left(a,b,\frac{a+b}{2}\right) \ge 0.$$

The constant $\frac{1}{4}$ is a best possible constant in the sense that it cannot be replaced by a smaller quantity.

Proof. The inequalities (2.5) follow from the representations (2.1) and (2.2) by taking into account that $f'(t)\ge 0$ for a.e. $t\in[a,b]$.

The inequality (2.6) follows by (2.5) for $x=\frac{a+b}{2}$.

Assume that the first inequality in (2.6) holds for a constant $C > 0$, i.e.

$$C(b-a)[f(b)-f(a)] \geq AB_f\left(a,b,\frac{a+b}{2}\right) \tag{2.7}$$

Consider the function $f_n : [-1,1] \to \mathbb{R}$ given by

$$f_n(t) = \begin{cases} 0 & \text{if } t \in [-1,0] \\ nt & \text{if } t \in \left(0,\frac{1}{n}\right) \\ 1 & \text{if } t \in \left[\frac{1}{n},1\right] \end{cases}$$

where $n \geq 2$, a natural number. This functions is absolutely continuous and $f_n'(t) \geq 0$ for any $t \in (-1,1)$. We have for $a = -1, b = 1$

$$C(b-a)[f_n(b)-f_n(a)] = 2C$$

and

$$\begin{aligned} AB_{f_n}\left(a,b,\frac{a+b}{2}\right) &= \frac{1}{2}\left[\int_0^1 f_n(t)\,dt - \int_{-1}^0 f_n(t)\,dt\right] \\ &= \frac{1}{2}\left(\int_0^{\frac{1}{n}} ntdt + \int_{\frac{1}{n}}^1 1dt\right) \\ &= \frac{1}{2}\left(\frac{1}{2n}+1-\frac{1}{n}\right) = \frac{1}{2}\left(1-\frac{1}{2n}\right). \end{aligned}$$

Replacing these values in (2.7) we get

$$2C \geq \frac{1}{2}\left(1-\frac{1}{2n}\right) \tag{2.8}$$

for any $n \geq 2$.

Taking the limit for $n \to \infty$ in (2.8) we get $C \geq \frac{1}{4}$, which proves that $\frac{1}{4}$ is best possible in the first inequality in (2.6) □

Remark 1. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f'(t) \geq 0$ for a.e. $t \in [a,b]$, then $AB_f(a,b,x) \geq 0$ for $x \in \left[a,\frac{a+b}{2}\right]$ $\left(\left[\frac{a+b}{2},b\right]\right)$.*

Moreover, if $f(b) \neq -f(a)$ and

$$\frac{bf(b)+af(a)}{f(b)+f(a)} \in [a,b] \tag{2.9}$$

then

$$AB_f\left(a,b,\frac{bf(b)+af(a)}{f(b)+f(a)}\right) \leq 0. \tag{2.10}$$

Also, if $f(a), f(b) > 0$, then (2.9) holds and the inequality (2.10) is valid.

Corollary 2. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$ and $\gamma\in\mathbb{C}$. Then we have the representation*

$$\begin{aligned}(2.11)\qquad & AB_f(a,b,x)\\ &= \frac{1}{2}\gamma\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]+\left(\frac{a+b}{2}-x\right)f(x)\\ &+\frac{1}{2}\left[\int_a^x (t-a)(f'(t)-\gamma)\,dt+\int_x^b (b-t)(f'(t)-\gamma)\,dt\right]\end{aligned}$$

and

$$\begin{aligned}(2.12)\qquad AB_f(a,b,x) &= \frac{bf(b)+af(a)}{2}-\frac{f(b)+f(a)}{2}x\\ &-\frac{1}{2}\gamma\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]\\ &-\frac{1}{2}\int_a^b |t-x|(f'(t)-\gamma)\,dt\end{aligned}$$

for any $x\in[a,b]$.

Proof. Let $e(t)=t, t\in[a,b]$. If we write the equality (2.1) for the function $f-\gamma e$ we have

$$\begin{aligned}(2.13)\qquad & AB_{f-\gamma e}(a,b,x)\\ &= \left(\frac{a+b}{2}-x\right)(f(x)-\gamma x)\\ &+\frac{1}{2}\left[\int_a^x (t-a)(f'(t)-\gamma)\,dt+\int_x^b (b-t)(f'(t)-\gamma)\,dt\right]\end{aligned}$$

for any $x\in[a,b]$.

Observe that

$$AB_{f-\gamma e}(a,b,x)=AB_f(a,b,x)-\gamma AB_e(a,b,x)$$

and

$$\begin{aligned}AB_e(a,b,x) &= \frac{1}{2}\left(\int_x^b t\,dt-\int_a^x t\,dt\right)\\ &= \frac{1}{2}\left(\frac{b^2-x^2}{2}-\frac{x^2-a^2}{2}\right)=\frac{1}{2}\left(\frac{a^2+b^2}{2}-x^2\right).\end{aligned}$$

From (2.13) we have

$$
\begin{aligned}
(2.14)\qquad & AB_f(a,b,x) \\
&= \left(\frac{a+b}{2}-x\right)(f(x)-\gamma x)+\frac{1}{2}\gamma\left(\frac{a^2+b^2}{2}-x^2\right) \\
&+\frac{1}{2}\left[\int_a^x (t-a)(f'(t)-\gamma)\,dt+\int_x^b (b-t)(f'(t)-\gamma)\,dt\right] \\
&= \left(\frac{a+b}{2}-x\right)f(x)+\frac{1}{2}\gamma\left(\frac{a^2+b^2}{2}-x^2\right)-\gamma\left(\frac{a+b}{2}-x\right)x \\
&= \frac{1}{2}\gamma\left[x^2-(a+b)x+\frac{a^2+b^2}{2}\right]+\left(\frac{a+b}{2}-x\right)f(x) \\
&+\frac{1}{2}\left[\int_a^x (t-a)(f'(t)-\gamma)\,dt+\int_x^b (b-t)(f'(t)-\gamma)\,dt\right]
\end{aligned}
$$

for any $x\in[a,b]$.

Since

$$x^2-(a+b)x+\frac{a^2+b^2}{2}=\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2$$

then from (2.14) we deduce the desired equality (2.11).

From (2.12) we have

$$
\begin{aligned}
AB_{f-\gamma e}(a,b,x) &= \frac{bf(b)+af(a)}{2}-\gamma\frac{b^2+a^2}{2}-\frac{f(b)+f(a)}{2}x+\gamma\frac{a+b}{2}x \\
&-\frac{1}{2}\int_a^b |t-x|(f'(t)-\gamma)\,dt
\end{aligned}
$$

and since

$$AB_{f-\gamma e}(a,b,x)=AB_f(a,b,x)-\gamma AB_e(a,b,x)$$

then

$$
\begin{aligned}
AB_f(a,b,x) &= \frac{1}{2}\gamma\left(\frac{a^2+b^2}{2}-x^2\right)+\frac{bf(b)+af(a)}{2} \\
&-\gamma\frac{b^2+a^2}{2}-\frac{f(b)+f(a)}{2}x+\gamma\frac{a+b}{2}x \\
&-\frac{1}{2}\int_a^b |t-x|(f'(t)-\gamma)\,dt \\
&= \frac{bf(b)+af(a)}{2}-\frac{f(b)+f(a)}{2}x \\
&-\frac{1}{2}\gamma\left[x^2-(a+b)x+\frac{a^2+b^2}{2}\right]-\frac{1}{2}\int_a^b |t-x|(f'(t)-\gamma)\,dt
\end{aligned}
$$

which proves the desired equality (2.12). □

Remark 2. *We have the following equalities*

$$AB_f\left(a,b,\frac{a+b}{2}\right) = \frac{1}{8}\gamma(b-a)^2 + \frac{1}{2}\left[\int_a^{\frac{a+b}{2}}(t-a)(f'(t)-\gamma)\,dt + \int_{\frac{a+b}{2}}^b (b-t)(f'(t)-\gamma)\,dt\right] \tag{2.15}$$

and

$$AB_f\left(a,b,\frac{a+b}{2}\right) = \frac{1}{4}(b-a)[f(b)-f(a)] - \frac{1}{8}\gamma(b-a)^2 - \frac{1}{2}\int_a^b \left|t-\frac{a+b}{2}\right|(f'(t)-\gamma)\,dt \tag{2.16}$$

for any $\gamma \in \mathbb{C}$.

3. Bounds for Absolutely Continuous Functions

Now, for $\gamma, \Gamma \in \mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_{[a,b]}(\gamma,\Gamma) := \left\{f:[a,b]\to\mathbb{C} \,|\, \operatorname{Re}\left[(\Gamma - f(t))\left(\overline{f(t)}-\overline{\gamma}\right)\right] \ge 0 \text{ for each } t\in[a,b]\right\}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma) := \left\{f:[a,b]\to\mathbb{C} \,\middle|\, \left|f(t)-\frac{\gamma+\Gamma}{2}\right| \le \frac{1}{2}|\Gamma-\gamma| \text{ for each } t\in[a,b]\right\}.$$

The following representation result may be stated.

Proposition 1. *For any* $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, *we have that* $\bar{U}_{[a,b]}(\gamma,\Gamma)$ *and* $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ *are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma). \tag{3.1}$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left|z-\frac{\gamma+\Gamma}{2}\right| \le \frac{1}{2}|\Gamma-\gamma|$$

if and only if

$$\operatorname{Re}\left[(\Gamma-z)(\bar{z}-\bar{\gamma})\right] \ge 0.$$

This follows by the equality

$$\frac{1}{4}|\Gamma-\gamma|^2 - \left|z-\frac{\gamma+\Gamma}{2}\right|^2 = \operatorname{Re}\left[(\Gamma-z)(\bar{z}-\bar{\gamma})\right]$$

that holds for any $z \in \mathbb{C}$.

The equality (3.1) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 3. *For any* $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, *we have that*

$$\bar{U}_{[a,b]}(\gamma,\Gamma) = \{f:[a,b]\to\mathbb{C} \mid (\operatorname{Re}\Gamma - \operatorname{Re} f(t))(\operatorname{Re} f(t) - \operatorname{Re}\gamma) + (\operatorname{Im}\Gamma - \operatorname{Im} f(t))(\operatorname{Im} f(t) - \operatorname{Im}\gamma) \ge 0 \textit{ for each } t\in[a,b]\}. \tag{3.2}$$

Now, if we assume that $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\bar{S}_{[a,b]}(\gamma,\Gamma) := \{f : [a,b] \to \mathbb{C} \mid \operatorname{Re}(\Gamma) \geq \operatorname{Re} f(t) \geq \operatorname{Re}(\gamma) \text{ and } \operatorname{Im}(\Gamma) \geq \operatorname{Im} f(t) \geq \operatorname{Im}(\gamma) \text{ for each } t \in [a,b]\}. \tag{3.3}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma,\Gamma)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\gamma,\Gamma) \subseteq \bar{U}_{[a,b]}(\gamma,\Gamma). \tag{3.4}$$

Theorem 2. *Let $f : [a,b] \to \mathbb{C}$ be an absolutely continuous function on $[a,b]$. If there exists $\gamma,\Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$ such that $f' \in \bar{U}_{[a,b]}(\gamma,\Gamma)$ then*

$$\left| AB_f(a,b,x) - \left(\frac{a+b}{2} - x\right) f(x) - \frac{\gamma+\Gamma}{4}\left[\left(x - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right]\right| \leq \frac{|\Gamma-\gamma|}{4}\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] \tag{3.5}$$

and

$$\left| AB_f(a,b,x) - \frac{bf(b)+af(a)}{2} + \frac{f(b)+f(a)}{2}x + \frac{\gamma+\Gamma}{4}\left[\left(x - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right]\right| \leq \frac{|\Gamma-\gamma|}{4}\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] \tag{3.6}$$

for any $x \in [a,b]$.

Proof. From the equality (2.11) we have

$$AB_f(a,b,x) - \frac{\gamma+\Gamma}{4}\left[\left(x - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right] - \left(\frac{a+b}{2} - x\right) f(x) = \frac{1}{2}\left[\int_a^x (t-a)\left(f'(t) - \frac{\gamma+\Gamma}{2}\right) dt + \int_x^b (b-t)\left(f'(t) - \frac{\gamma+\Gamma}{2}\right) dt\right] \tag{3.7}$$

for any $x \in [a,b]$.

If $f' \in \bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$, then by taking the modulus in (3.7) we get

$$\begin{aligned}
&\left| AB_f(a,b,x) - \left(\frac{a+b}{2} - x\right) f(x) \right. \\
&\left. - \frac{\gamma+\Gamma}{4}\left[\left(x - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right]\right| \\
&= \frac{1}{2}\left|\int_a^x (t-a)\left(f'(t) - \frac{\gamma+\Gamma}{2}\right)dt + \int_x^b (b-t)\left(f'(t) - \frac{\gamma+\Gamma}{2}\right)dt\right| \\
&\le \frac{1}{2}\left[\left|\int_a^x (t-a)\left(f'(t) - \frac{\gamma+\Gamma}{2}\right)dt\right| + \left|\int_x^b (b-t)\left(f'(t) - \frac{\gamma+\Gamma}{2}\right)dt\right|\right] \\
&\le \frac{1}{2}\left[\int_a^x (t-a)\left|f'(t) - \frac{\gamma+\Gamma}{2}\right|dt + \int_x^b (b-t)\left|f'(t) - \frac{\gamma+\Gamma}{2}\right|dt\right] \\
&\le \frac{|\Gamma-\gamma|}{4}\left[\int_a^x (t-a)\,dt + \int_x^b (b-t)\,dt\right] \\
&= \frac{|\Gamma-\gamma|}{4}\left[\frac{(x-a)^2 + (b-x)^2}{2}\right] = \frac{|\Gamma-\gamma|}{4}\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right],
\end{aligned}$$

for any $x \in [a,b]$, which proves the inequality (3.5).

From the equality (2.12) we have

$$\begin{aligned}
&AB_f(a,b,x) - \frac{bf(b) + af(a)}{2} + \frac{f(b) + f(a)}{2}x \\
&+ \frac{\gamma+\Gamma}{4}\left[\left(x - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right] \\
&= -\frac{1}{2}\int_a^b |t-x|\left(f'(t) - \frac{\gamma+\Gamma}{2}\right)dt
\end{aligned} \tag{3.8}$$

for any $x \in [a,b]$.

Taking the modulus in (3.8) and using the fact that $f' \in \bar{U}_{[a,b]}(\gamma,\Gamma) = \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ we have

$$\begin{aligned}
&\left| AB_f(a,b,x) - \frac{bf(b) + af(a)}{2} + \frac{f(b) + f(a)}{2}x \right. \\
&\left. + \frac{\gamma+\Gamma}{4}\left[\left(x - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right]\right| \\
&\le \frac{1}{2}\int_a^b |t-x|\left|f'(t) - \frac{\gamma+\Gamma}{2}\right|dt \\
&\le \frac{|\Gamma-\gamma|}{4}\int_a^b |t-x|\,dt = \frac{|\Gamma-\gamma|}{4}\left[\int_a^x (x-t)\,dt + \int_x^b (t-x)\,dt\right] \\
&= \frac{|\Gamma-\gamma|}{4}\left[\frac{(x-a)^2 + (b-x)^2}{2}\right] = \frac{|\Gamma-\gamma|}{4}\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right]
\end{aligned}$$

for any $x \in [a,b]$, which proves the desired inequality (3.6). □

Remark 3. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$. If there exists the real numbers m, M such that*

$$m \le f'(t) \le M \text{ for a.e. } t \in [a, b],$$

then

$$\begin{aligned}(3.9)\quad & \left| AB_f(a,b,x) - \left(\frac{a+b}{2} - x\right) f(x) \right. \\ & \left. - \frac{m+M}{4} \left[\left(x - \frac{a+b}{2} \right)^2 + \frac{1}{4}(b-a)^2 \right] \right| \\ & \le \frac{M-m}{4} \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right]\end{aligned}$$

and

$$\begin{aligned}(3.10)\quad & \left| AB_f(a,b,x) - \frac{bf(b) + af(a)}{2} + \frac{f(b)+f(a)}{2} x \right. \\ & \left. + \frac{m+M}{4} \left[\left(x - \frac{a+b}{2} \right)^2 + \frac{1}{4}(b-a)^2 \right] \right| \\ & \le \frac{M-m}{4} \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right]\end{aligned}$$

for any $x \in [a, b]$.

Corollary 4. *With the assumptions of Theorem 2 we have*

$$(3.11)\quad \left| AB_f(a,b,x) - \frac{\gamma + \Gamma}{16}(b-a)^2 \right| \le \frac{|\Gamma - \gamma|}{16}(b-a)^2$$

and

$$\begin{aligned}(3.12)\quad & \left| \frac{1}{4}(b-a)[f(b) - f(a)] - \frac{\gamma+\Gamma}{16}(b-a)^2 - AB_f\left(a, b, \frac{a+b}{2}\right) \right| \\ & \le \frac{|\Gamma - \gamma|}{16}(b-a)^2.\end{aligned}$$

Theorem 3. *Let $f : I \to \mathbb{R}$ be an absolutely continuous function on the interval I and $[a, b] \subset \mathring{I}$, where $\mathring{I}$ is the interior of I and such that f' is of bounded variation on $[a, b]$. Then we have the inequalities*

$$\begin{aligned}(3.13)\quad & \left| AB_f(a,b,x) - \left(\frac{a+b}{2} - x\right) f(x) \right. \\ & \left. - \frac{f'(a) + f'(b)}{4} \left[\left(x - \frac{a+b}{2} \right)^2 + \frac{1}{4}(b-a)^2 \right] \right| \\ & \le \frac{1}{4} \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \bigvee_a^b (f')\end{aligned}$$

and

$$\begin{aligned}(3.14)\qquad & \left| AB_f(a,b,x) - \frac{bf(b)+af(a)}{2} + \frac{f(b)+f(a)}{2}x \right. \\ & \left. + \frac{f'(a)+f'(b)}{4}\left[\left(x-\frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right]\right| \\ & \leq \frac{1}{4}\left[\frac{1}{4}(b-a)^2 + \left(x-\frac{a+b}{2}\right)^2\right]\bigvee_a^b (f')\end{aligned}$$

for any $x \in [a,b]$.

Proof. From (2.11) for $\gamma = \frac{f'(a)+f'(b)}{2}$ we have the representation

$$\begin{aligned}(3.15)\qquad & AB_f(a,b,x) \\ & - \frac{f'(a)+f'(b)}{4}\left[\left(x-\frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right] - \left(\frac{a+b}{2}-x\right)f(x) \\ & = \frac{1}{2}\left[\int_a^x (t-a)\left(f'(t) - \frac{f'(a)+f'(b)}{2}\right)dt \right. \\ & \left. + \int_x^b (b-t)\left(f'(t) - \frac{f'(a)+f'(b)}{2}\right)dt\right]\end{aligned}$$

for any $x \in [a,b]$.

Taking the modulus in (3.15) we get

$$\begin{aligned}(3.16)\qquad & \left| AB_f(a,b,x) - \left(\frac{a+b}{2}-x\right)f(x) \right. \\ & \left. - \frac{f'(a)+f'(b)}{4}\left[\left(x-\frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right]\right| \\ & \leq \frac{1}{2}\left[\int_a^x (t-a)\left|f'(t) - \frac{f'(a)+f'(b)}{2}\right|dt \right. \\ & \left. + \int_x^b (b-t)\left|f'(t) - \frac{f'(a)+f'(b)}{2}\right|dt\right]\end{aligned}$$

for any $x \in [a,b]$.

For $t \in [a,x]$ we have

$$\begin{aligned}\left|f'(t) - \frac{f'(a)+f'(b)}{2}\right| &= \left|\frac{f'(t)-f'(a)+f'(t)-f'(b)}{2}\right| \\ &\leq \frac{1}{2}\left[\left|f'(t)-f'(a)\right| + \left|f'(b)-f'(t)\right|\right] \\ &\leq \frac{1}{2}\bigvee_a^b (f')\end{aligned}$$

and similarly, for $t \in [x,b]$ we have

$$\left|f'(t) - \frac{f'(a)+f'(b)}{2}\right| \leq \frac{1}{2}\bigvee_a^b (f')$$

and then by (3.16) we get

$$\begin{aligned}&\left|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)\right.\\&\left.-\frac{f'(a)+f'(b)}{4}\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]\right|\\&\le\frac{1}{4}\left[\int_a^x(t-a)\,dt+\int_x^b(b-t)\,dt\right]\bigvee_a^b(f')\\&=\frac{1}{4}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\bigvee_a^b(f')\end{aligned}$$

for $t\in[a,b]$, and the inequality (3.13) is proved.

The second inequality goes along a similar way and we omit the details. □

Corollary 5. *With the assumptions of Theorem 3 we have*

$$\left|AB_f(a,b,x)-\frac{f'(a)+f'(b)}{16}(b-a)^2\right|\le\frac{1}{16}(b-a)^2\bigvee_a^b(f')\tag{3.17}$$

and

$$\begin{aligned}&\left|\frac{1}{4}(b-a)[f(b)-f(a)]-\frac{f'(a)+f'(b)}{16}(b-a)^2-AB_f\left(a,b,\frac{a+b}{2}\right)\right|\\&\le\frac{1}{16}(b-a)^2\bigvee_a^b(f').\end{aligned}\tag{3.18}$$

4. Bounds for Lipschitzian Derivatives

We say that v is *Lipschitzian* with the constant $L>0$, if

$$|v(t)-v(s)|\le L|t-s|$$

for any $t,s\in[a,b]$.

Theorem 4. *Let $f:I\to\mathbb{R}$ be an absolutely continuous function on the interval I and $[a,b]\subset\mathring{I}$, where $\mathring{I}$ is the interior of I and such that f' is Lipschitzian with the constant $K>0$ on $[a,b]$. Then we have the inequalities*

$$\begin{aligned}&\left|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)\right.\\&\left.-\frac{1}{2}f'(x)\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\right|\\&\le\frac{1}{12}(b-a)K\left[3\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]\end{aligned}\tag{4.1}$$

for any $x\in[a,b]$.

In particular, we have

$$\left|AB_f\left(a,b,\frac{a+b}{2}\right)-\frac{1}{8}f'\left(\frac{a+b}{2}\right)(b-a)^2\right|\le\frac{1}{48}K(b-a)^3.\tag{4.2}$$

The constant $\frac{1}{48}$ is best possible in (4.2).

Proof. We have from the equality (2.11) that

$$\begin{aligned} &AB_f(a,b,x) \\ &-\left(\frac{a+b}{2}-x\right)f(x)-\frac{1}{2}f'(x)\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right] \\ &=\frac{1}{2}\left[\int_a^x (t-a)\left[f'(t)-f'(x)\right]dt+\int_x^b (b-t)\left[f'(t)-f'(x)\right]dt\right] \end{aligned} \tag{4.3}$$

for any $x \in [a,b]$.

Taking the modulus on (4.3) we have

$$\begin{aligned} &\left|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)\right. \\ &\left.-\frac{1}{2}f'(x)\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\right| \\ &\le \frac{1}{2}\left[\int_a^x (t-a)\left|f'(t)-f'(x)\right|dt+\int_x^b (b-t)\left|f'(t)-f'(x)\right|dt\right] \\ &\le \frac{1}{2}K\left[\int_a^x (t-a)(x-t)\,dt+\int_x^b (b-t)(b-x)\,dt\right] \end{aligned} \tag{4.4}$$

for any $x \in [a,b]$.

Since a simple calculation shows that

$$\int_c^d (t-c)(d-t)\,dt=\frac{1}{6}(d-c)^3,$$

then

$$\begin{aligned} &\int_a^x (t-a)(x-t)\,dt+\int_x^b (b-t)(b-x)\,dt \\ &=\frac{1}{6}\left[(x-a)^3+(b-x)^3\right] \\ &=\frac{1}{6}(b-a)\left[3\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right] \end{aligned}$$

for any $x \in [a,b]$.

Utilising (4.4) we get the desired inequality (4.1).

Consider the function $f:[a,b]\to\mathbb{R}$,

$$f(t):=\begin{cases} -\left(t-\frac{a+b}{2}\right)^2 & \text{if } t\in\left[a,\frac{a+b}{2}\right) \\ \left(t-\frac{a+b}{2}\right)^2 & \text{if } t\in\left[\frac{a+b}{2},b\right]. \end{cases}$$

Then f is differentiable and

$$\begin{aligned} f'(t) &= \begin{cases} -2\left(t-\frac{a+b}{2}\right) & \text{if } t\in\left[a,\frac{a+b}{2}\right) \\ 2\left(t-\frac{a+b}{2}\right) & \text{if } t\in\left[\frac{a+b}{2},b\right]. \end{cases} \\ &= 2\left|t-\frac{a+b}{2}\right| \end{aligned}$$

for $t \in [a,b]$.

Since

$$\begin{aligned} |f'(t) - f'(s)| &= 2\left|\left|t - \frac{a+b}{2}\right| - \left|s - \frac{a+b}{2}\right|\right| \\ &\leq 2|t-s| \end{aligned}$$

for any $t, s \in [a,b]$, we conclude that f' is Lipschitzian with the constant $K = 2$.

We have

$$\begin{aligned} AB_f\left(a,b,\frac{a+b}{2}\right) &= \frac{1}{2}\left[\int_{\frac{a+b}{2}}^{b} f(t)\,dt - \int_a^{\frac{a+b}{2}} f(t)\,dt\right] \\ &= \frac{1}{2}\left[\int_{\frac{a+b}{2}}^{b} \left(t - \frac{a+b}{2}\right)^2 dt + \int_a^{\frac{a+b}{2}} \left(t - \frac{a+b}{2}\right)^2 dt\right] \\ &= \frac{1}{2}\int_a^b \left(t - \frac{a+b}{2}\right)^2 dt = \frac{1}{24}(b-a)^3. \end{aligned}$$

If we replace these values in (4.2) we get in both sides the same quantity $\frac{1}{24}(b-a)^3$. □

The following result also holds:

Theorem 5. *With the assumptions of Theorem 4 we have the inequalities*

$$\begin{aligned} &\left|AB_f(a,b,x) - \frac{bf(b)+af(a)}{2} + \frac{f(b)+f(a)}{2}x \right. \\ &\left. + \frac{1}{2}f'(x)\left[\left(x - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right]\right| \\ &\leq \frac{1}{12}(b-a)K\left[3\left(x - \frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right] \end{aligned} \tag{4.5}$$

for any $x \in [a,b]$.

In particular, we have

$$\begin{aligned} &\left|\frac{1}{4}(b-a)[f(b)-f(a)] - \frac{1}{8}f'\left(\frac{a+b}{2}\right)(b-a)^2 - AB_f\left(a,b,\frac{a+b}{2}\right)\right| \\ &\leq \frac{1}{48}K(b-a)^3. \end{aligned} \tag{4.6}$$

The proof is similar to the above Theorem 4 and the details are omitted.

5. Inequalities for p-Norms

For a Lebesgue measurable function $f : [c,d] \to \mathbb{C}$ we introduce the p-Lebesgue norms as

$$\|f\|_{[c,d],p} := \left(\int_a^b |f(t)|^p\,dt\right)^{1/p} \text{ if } p \geq 1$$

and

$$\|f\|_{[c,d],\infty} := ess \sup_{t\in[c,d]} |f(t)|$$

provided these quantities are finite. We denote $f \in L_p[c,d]$ and $f \in L_\infty[c,d]$.

Proposition 2. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$. Then we have the inequalities*

$$\left|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)\right| \le \frac{1}{2}\left[\int_a^x (t-a)\left|f'(t)\right|dt+\int_x^b (b-t)\left|f'(t)\right|dt\right]:=B_1(x) \tag{5.1}$$

and

$$\left|\frac{bf(b)+af(a)}{2}-\frac{f(b)+f(a)}{2}x-AB_f(a,b,x)\right| \le \frac{1}{2}\int_a^b |t-x|\left|f'(t)\right|dt:=B_2(x) \tag{5.2}$$

for any $x\in[a,b]$.

Moreover, we have

$$B_1(x) \le \frac{1}{2}\times\begin{cases}\frac{1}{2}(x-a)^2\|f'\|_{[a,x],\infty} & \text{if } f'\in L_\infty[a,x]\\ \frac{1}{(\alpha+1)^{1/\alpha}}(x-a)^{1+1/\alpha}\|f'\|_{[a,x],\beta} & \text{if } f'\in L_\beta[a,x],\ \frac{1}{\alpha}+\frac{1}{\beta}=1,\alpha>1\\ (x-a)\|f'\|_{[a,x],1}\end{cases} + \frac{1}{2}\times\begin{cases}\frac{1}{2}(b-x)^2\|f'\|_{[x,b],\infty} & \text{if } f'\in L_\infty[x,b]\\ \frac{1}{(\gamma+1)^{1/\gamma}}(b-x)^{1+1/\gamma}\|f'\|_{[x,b],\delta} & \text{if } f'\in L_\delta[x,b],\ \frac{1}{\gamma}+\frac{1}{\delta}=1,\gamma>1\\ (b-x)\|f'\|_{[x,b],1}\end{cases} \tag{5.3}$$

and

$$B_2(x) \le \frac{1}{2}\times\begin{cases}\frac{1}{2}(x-a)^2\|f'\|_{[a,x],\infty} & \text{if } f'\in L_\infty[a,x]\\ \frac{1}{(\alpha+1)^{1/\alpha}}(x-a)^{1+1/\alpha}\|f'\|_{[a,x],\beta} & \text{if } f'\in L_\beta[a,x],\ \frac{1}{\alpha}+\frac{1}{\beta}=1,\alpha>1\\ (x-a)\|f'\|_{[a,x],1}\end{cases} + \frac{1}{2}\times\begin{cases}\frac{1}{2}(b-x)^2\|f'\|_{[x,b],\infty} & \text{if } f'\in L_\infty[x,b]\\ \frac{1}{(\gamma+1)^{1/\gamma}}(b-x)^{1+1/\gamma}\|f'\|_{[x,b],\delta} & \text{if } f'\in L_\delta[x,b],\ \frac{1}{\gamma}+\frac{1}{\delta}=1,\gamma>1\\ (b-x)\|f'\|_{[x,b],1}\end{cases} \tag{5.4}$$

for any $x\in[a,b]$.

Proof. From (2.1) and (2.2) we have by taking the modulus

$$\begin{aligned}(5.5)\qquad & \left|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)\right| \\ & \leq \frac{1}{2}\left[\left|\int_a^x (t-a)f'(t)\,dt\right|+\left|\int_x^b (b-t)f'(t)\,dt\right|\right] \\ & \leq \frac{1}{2}\left[\int_a^x (t-a)\left|f'(t)\right|dt+\int_x^b (b-t)\left|f'(t)\right|dt\right]\end{aligned}$$

and

$$\begin{aligned}(5.6)\qquad & \left|\frac{bf(b)+af(a)}{2}-\frac{f(b)+f(a)}{2}x-AB_f(a,b,x)\right| \\ & \leq \frac{1}{2}\int_a^b |t-x|\left|f'(t)\right|dt \\ & = \frac{1}{2}\left[\int_a^x (x-t)\left|f'(t)\right|dt+\int_x^b (t-x)\left|f'(t)\right|dt\right]\end{aligned}$$

for any $x\in[a,b]$.

Using the Hölder inequality we have

$$\begin{aligned} & B_1(x) \\ \leq\ & \frac{1}{2}\times\begin{cases}\frac{1}{2}(x-a)^2\left\|f'\right\|_{[a,x],\infty} & \text{if } f'\in L_\infty[a,x] \\ \frac{1}{(\alpha+1)^{1/\alpha}}(x-a)^{1+1/\alpha}\left\|f'\right\|_{[a,x],\beta} & \text{if } f'\in L_\beta[a,x],\ \frac{1}{\alpha}+\frac{1}{\beta}=1,\alpha>1 \\ (x-a)\left\|f'\right\|_{[a,x],1}\end{cases} \\ & +\frac{1}{2}\times\begin{cases}\frac{1}{2}(b-x)^2\left\|f'\right\|_{[x,b],\infty} & \text{if } f'\in L_\infty[x,b] \\ \frac{1}{(\gamma+1)^{1/\gamma}}(b-x)^{1+1/\gamma}\left\|f'\right\|_{[x,b],\delta} & \text{if } f'\in L_\delta[x,b],\ \frac{1}{\gamma}+\frac{1}{\delta}=1,\gamma>1 \\ (b-x)\left\|f'\right\|_{[x,b],1}\end{cases}\end{aligned}$$

and a similar inequality for B_2. □

Remark 4. *We observe that*

$$\begin{aligned}(5.7)\qquad B_1(x) & \leq \frac{1}{4}(x-a)^2\left\|f'\right\|_{[a,x],\infty}+\frac{1}{4}(b-x)^2\left\|f'\right\|_{[x,b],\infty} \\ & \leq \left[\frac{1}{4}(x-a)^2+\frac{1}{4}(b-x)^2\right]\max\left\{\left\|f'\right\|_{[a,x],\infty},\left\|f'\right\|_{[x,b],\infty}\right\} \\ & = \frac{1}{2}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\left\|f'\right\|_{[a,b],\infty}\end{aligned}$$

therefore

$$\left|AB_f(a,b,x)-\left(\frac{a+b}{2}-x\right)f(x)\right| \le \frac{1}{2}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_{[a,b],\infty} \tag{5.8}$$

for any $x\in[a,b]$.

Similarly,

$$\left|\frac{bf(b)+af(a)}{2}-\frac{f(b)+f(a)}{2}x-AB_f(a,b,x)\right| \le \frac{1}{2}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_{[a,b],\infty} \tag{5.9}$$

for any $x\in[a,b]$.

In particular, we have

$$\left|AB_f\left(a,b,\frac{a+b}{2}\right)\right| \le \frac{1}{8}(b-a)^2\|f'\|_{[a,b],\infty} \tag{5.10}$$

and

$$\left|\frac{1}{4}(b-a)[f(b)-f(a)]-AB_f(a,b,x)\right| \le \frac{1}{8}(b-a)^2\|f'\|_{[a,b],\infty}. \tag{5.11}$$

6. Applications for Twice Differentiable Functions

If we write the equalities (2.11) and (2.12) for the function $f=g'$, where $g:I\to\mathbb{R}$ is a differentiable function on the interior of the interval I with the derivative absolutely continuous on $[a,b]\subset\mathring{I}$, then we get

$$\begin{aligned}AB_{g'}(a,b,x) &= \frac{1}{2}\gamma\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]+\left(\frac{a+b}{2}-x\right)g'(x)\\ &+\frac{1}{2}\left[\int_a^x (t-a)(g''(t)-\gamma)\,dt+\int_x^b (b-t)(g''(t)-\gamma)\,dt\right]\end{aligned} \tag{6.1}$$

and

$$\begin{aligned}AB_{g'}(a,b,x) &= \frac{bg'(b)+ag'(a)}{2}-\frac{g'(b)+g'(a)}{2}x\\ &-\frac{1}{2}\gamma\left[\left(x-\frac{a+b}{2}\right)^2+\frac{1}{4}(b-a)^2\right]\\ &-\frac{1}{2}\int_a^b |t-x|(g''(t)-\gamma)\,dt\end{aligned} \tag{6.2}$$

and since

$$AB_f(a,b,x)=\frac{1}{2}F(b)-F(x),$$

where $F(x) := \int_a^x f(t)\,dt$, then

$$AB_{g'}(a,b,x) = \frac{1}{2}[g(b)-g(a)] - g(x) + g(a) = \frac{g(a)+g(b)}{2} - g(x)$$

and by (6.1) and (6.2) we get the representations

$$g(x) = \frac{g(a)+g(b)}{2} - \frac{1}{2}\gamma\left[\left(x-\frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right] - \left(\frac{a+b}{2}-x\right)g'(x) - \frac{1}{2}\left[\int_a^x (t-a)(g''(t)-\gamma)\,dt + \int_x^b (b-t)(g''(t)-\gamma)\,dt\right] \tag{6.3}$$

and

$$g(x) = \frac{g(a)+g(b)}{2} - \frac{bg'(b)+ag'(a)}{2} + \frac{g'(b)+g'(a)}{2}x + \frac{1}{2}\gamma\left[\left(x-\frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right] + \frac{1}{2}\int_a^b |t-x|\,(g''(t)-\gamma)\,dt \tag{6.4}$$

for any $x \in [a,b]$.

If we assume that $g'' \in \bar{U}_{[a,b]}(\psi,\Psi)$ for some $\psi, \Psi \in \mathbb{C}$, $\psi \neq \Psi$, then, as above, we have the inequalities

$$\left| g(x) - \frac{g(a)+g(b)}{2} + \frac{\psi+\Psi}{4}\left[\left(x-\frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right] + \left(\frac{a+b}{2}-x\right)g'(x)\right| \leq \frac{|\Psi-\psi|}{4}\left[\frac{1}{4}(b-a)^2 + \left(x-\frac{a+b}{2}\right)^2\right] \tag{6.5}$$

and

$$\left| g(x) - \frac{g(a)+g(b)}{2} + \frac{bg'(b)+ag'(a)}{2} - \frac{g'(b)+g'(a)}{2}x - \frac{\psi+\Psi}{4}\left[\left(x-\frac{a+b}{2}\right)^2 + \frac{1}{4}(b-a)^2\right]\right| \leq \frac{|\Psi-\psi|}{4}\left[\frac{1}{4}(b-a)^2 + \left(x-\frac{a+b}{2}\right)^2\right] \tag{6.6}$$

for any $x \in [a,b]$.

We have the particular inequalities

$$\left|g\left(\frac{a+b}{2}\right)-\frac{g(a)+g(b)}{2}+\frac{\psi+\Psi}{16}(b-a)^{2}\right| \leq \frac{|\Psi-\psi|}{16}(b-a)^{2} \tag{6.7}$$

and

$$\left|g\left(\frac{a+b}{2}\right)-\frac{g(a)+g(b)}{2}+\frac{1}{4}(b-a)\left[g'(b)-g'(a)\right] -\frac{\psi+\Psi}{16}(b-a)^{2}\right| \leq \frac{|\Psi-\psi|}{16}(b-a)^{2} \tag{6.8}$$

Other similar results may be stated, however we do not present the details here.

References

[1] A. G. AZPEITIA, Convex functions and the Hadamard inequality. *Rev. Colombiana Mat.* **28** (1994), no. 1, 7–12.

[2] S. S. DRAGOMIR, A mapping in connection to Hadamard's inequalities, *An. Öster. Akad. Wiss. Math.-Natur.,* (Wien), **128**(1991), 17-20. MR 934:26032. ZBL No. 747:26015.

[3] S. S. DRAGOMIR, Two mappings in connection to Hadamard's inequalities, *J. Math. Anal. Appl.,* **167**(1992), 49-56. MR:934:26038, ZBL No. 758:26014.

[4] S. S. DRAGOMIR, On Hadamard's inequalities for convex functions, *Mat. Balkanica,* **6**(1992), 215-222. MR: 934: 26033.

[5] S. S. DRAGOMIR, An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure & Appl. Math.*, 3(2002), No. 3, Art. 35. [Online: http://www.emis.de/journals/JIPAM/article187.html?sid=187].

[6] S. S. DRAGOMIR, Bounds for the normalized Jensen functional, *Bull. Austral. Math. Soc.* **74**(3)(2006), 471-476.

[7] S. S. DRAGOMIR, Refinements of the generalised trapezoid and Ostrowski inequalities for functions of bounded variation. *Arch. Math.* (Basel) **91** (2008), no. 5, 450–460.

[8] S. S. DRAGOMIR and I. GOMM, Bounds for two mappings associated to the Hermite-Hadamard inequality, Preprint, *RGMIA Res. Rep. Coll.*, **14**(2011), to appear.

[9] S. S. DRAGOMIR, D. S. MILOŠEVIĆ and J. SÁNDOR, On some refinements of Hadamard's inequalities and applications, *Univ. Belgrad, Publ. Elek. Fak. Sci. Math.,* **4**(1993), 21-24.

[10] S. S. DRAGOMIR and C. E. M. PEARCE, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, 2000. [Online http://rgmia.org/monographs/hermite_hadamard.html].

[11] A. GUESSAB and G. SCHMEISSER, Sharp integral inequalities of the Hermite-Hadamard type. *J. Approx. Theory* **115** (2002), no. 2, 260–288.

[12] E. KILIANTY and S. S. DRAGOMIR, Hermite-Hadamard's inequality and the p-HH-norm on the Cartesian product of two copies of a normed space, *Math. Inequal. Appl.* **13** (2010), no. 1, 1–32.

[13] M. MERKLE, Remarks on Ostrowski's and Hadamard's inequality, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.* **10** (1999), 113–117.

[14] C. E. M. PEARCE and A. M. RUBINOV, P-functions, quasi-convex functions, and Hadamard type inequalities, *J. Math. Anal. Appl.* **240** (1999), no. 1, 92–104.

[15] J. PEČARIĆ and A. VUKELIĆ, Hadamard and Dragomir-Agarwal inequalities, the Euler formulae and convex functions. *Functional Equations, Inequalities and Applications*, 105–137, Kluwer Acad. Publ., Dordrecht, 2003.

[16] G. TOADER, Superadditivity and Hermite-Hadamard's inequalities, *Studia Univ. Babeş-Bolyai Math.* **39** (1994), no. 2, 27–32.

[17] G.-S. YANG and M.-C. HONG, A note on Hadamard's inequality, *Tamkang J. Math.* **28** (1997), no. 1, 33–37.

[18] G.-S. YANG and K.-L. TSENG, On certain integral inequalities related to Hermite-Hadamard inequalities, *J. Math. Anal. Appl.* **239** (1999), no. 1, 180–187.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa