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Optimal estimations of some Seiffert type means by Lehmer means

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Abstract

Let us consider the logarithmic mean $\mathcal{L}$, the identric mean $\mathcal{I}$, the trigonometric means $\mathcal{P}$ and $\mathcal{T}$ defined by H. J. Seiffert, the hyperbolic mean $\mathcal{M}$ defined by E. Neuman and J. Sándor, and the Gini mean $\mathcal{J}$. The optimal estimations of these means by power means $\mathcal{A}_p$ and also some of the optimal estimations by Lehmer means $\mathcal{L}_q$ are known. We prove two new optimal estimation by Lehmer means $\mathcal{L}_0 < \mathcal{M} < \mathcal{L}_{1/6}$ and $\mathcal{L}_{1/2} < \mathcal{J} < \mathcal{L}_1$ and establish some connections between all these estimations.

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1 Introduction

A **mean** is a function $M : \mathbb{R}_+^2 \to \mathbb{R}_+$, with the property

$$\min(a, b) \leq M(a, b) \leq \max(a, b),\ \forall a, b > 0.$$

Each mean is **reflexive**, that is

$$M(a, a) = a,\ \forall a > 0.$$

This is also used as the definition of $M(a, a)$.

A mean is **symmetric** if

$$M(b, a) = M(a, b),\ \forall a, b > 0;$$

it is **homogeneous** (of degree 1) if

$$M(ta,tb) = t \cdot M(a,b),\ \forall a,b,t > 0.$$

We shall refer here to the following symmetric and homogeneous means:
- the Gini (or sum) means $\mathcal{S}_{p,q}$, defined by

$$\mathcal{S}_{p,q}(a,b) = \begin{cases} \left(\frac{a^p+b^p}{a^q+b^q}\right)^{\frac{1}{p-q}} & \text{if } p \neq q, \\ \left(a^{a^p} \cdot b^{b^p}\right)^{\frac{1}{a^p+b^p}} & \text{if } p = q; \end{cases}$$

- the power means $\mathcal{A}_p = \mathcal{S}_{p,0}$;
- the arithmetic mean $\mathcal{A} = \mathcal{A}_1$;
- the geometric mean $\mathcal{G} = \mathcal{A}_0 = \mathcal{S}_{0,0}$;
- the Lehmer means $\mathcal{L}_p = \mathcal{S}_{p+1,p}$;
- the logarithmic mean $\mathcal{L}$ defined by

$$\mathcal{L}(a,b) = \frac{a-b}{\ln a - \ln b}, a \neq b;$$

- the identric mean $\mathcal{I}$ defined by

$$\mathcal{I}(a,b) = \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{\frac{1}{a-b}}, a \neq b;$$

- the special Gini mean $\mathcal{J} = \mathcal{S}_{1,1}$;
- the first Seiffert mean $\mathcal{P}$, defined in [14] by

$$\mathcal{P}(a,b) = \frac{a-b}{2\sin^{-1}\frac{a-b}{a+b}}, a \neq b;$$

- the second Seiffert mean $\mathcal{T}$, defined in [15] by

$$\mathcal{T}(a,b) = \frac{a-b}{2\tan^{-1}\frac{a-b}{a+b}}, a \neq b;$$

- the Neuman-Sándor mean $\mathcal{M}$, defined in [9] by

$$\mathcal{M}(a,b) = \frac{a-b}{2\sinh^{-1}\frac{a-b}{a+b}}, a \neq b.$$

As $\mathcal{L}$ can be represented by

$$\mathcal{L}(a,b) = \frac{a-b}{2\tanh^{-1}\frac{a-b}{a+b}}, a \neq b,$$

the four means $\mathcal{P}$,$\mathcal{T}$,$\mathcal{M}$, and $\mathcal{L}$ are very similar and are called Seiffert-type means. Also the Gini mean $\mathcal{J}$, given by

$$\mathcal{J}(a,b) = \left(a^a b^b\right)^{\frac{1}{a+b}},$$

is very similar with the identric mean $\mathcal{I}$.

For two means M and N we write $M < N$ if $M(a,b) < N(a,b)$ for $a \neq b$. For instance, it is known that

$$\mathcal{L} < \mathcal{P} < \mathcal{I} < \mathcal{A} < \mathcal{M} < \mathcal{T}, \tag{1}$$

as it was shown in [14] and [9].

Some complicated means were estimated by families of simpler means. Consider a family of means F_p, $p \in \mathbb{R}$. It is an **increasing** family if

$$F_p < F_q \text{ for } p < q.$$

A lower (upper) **estimation** of a given mean M by this family of means assumes the determination of some real index p (respectively q) such that $F_p < M$ (respectively $M < F_q$). A lower estimation is **optimal** if p is the greatest index r such that $F_r < M$. Similarly an upper estimation is **optimal** if q is the smallest index r with the property that $M < F_r$.

Optimal estimations by power means where given for the logarithmic mean in [8]

$$\mathcal{A}_0 < \mathcal{L} < \mathcal{A}_{1/3}, \tag{2}$$

for the identric mean in [12]

$$\mathcal{A}_{2/3} < \mathcal{I} < \mathcal{A}_{\ln 2}, \tag{3}$$

for the Gini mean in [10]

$$\mathcal{A}_2 < \mathcal{J}, \tag{4}$$

and for the first Seiffert mean in [6]

$$\mathcal{A}_{\ln 2/\ln \pi} < \mathcal{P} < \mathcal{A}_{2/3}. \tag{5}$$

We consider also the following definition. Two means $M < N$ can be **separated** by a family of means F_p, $p \in \mathbb{R}$ if there is an index p such that $M < F_p < N$. For example, the first separation of the means $\mathcal{M} < \mathcal{T}$ by power means was given in [3]:

$$\mathcal{M} < \mathcal{A}_{3/2} < \mathcal{T}.$$

Using the above inequalities, the chain of means (1) was separated in [4] by power means with equidistant indices:

$$\mathcal{A}_0 < \mathcal{L} < \mathcal{A}_{1/3} < \mathcal{P} < \mathcal{A}_{2/3} < \mathcal{I} < \mathcal{A}_1 < \mathcal{M} < \mathcal{A}_{4/3} < \mathcal{T} < \mathcal{A}_{5/3} < \mathcal{A}_{6/3} < \mathcal{J}.$$

Finally, optimal estimations by power means were obtained independently in [5] and [16] for the second Seiffert mean:

$$\mathcal{A}_{\ln 2/\ln(\pi/2)} < \mathcal{T} < \mathcal{A}_{5/3} \tag{6}$$

and in [5], [17] and [2] for the Neuman-Sándor mean:

$$\mathcal{A}_{\ln 2/\ln(\ln(3+2\sqrt{2}))} < \mathcal{M} < \mathcal{A}_{4/3}. \tag{7}$$

Remark 1 *In [18] the author claims his priority in publishing these results. We think that it is more natural to consider that the results were proved simultaneously. Some relative advance comes from the way chosen to publish the results. Our paper [5] was submitted for publication on June 12, 2012 while [16] and [17] were posted on-line on June 24, 2012 and on August 4, 2012 respectively. Moreover, in [16] it is cited the paper [3] (yet unpublished at that time). To complete the informations, the paper [2] was submitted for publication on November 8, 2012.*

2 Estimations by Lehmer means

The first (lower) estimation of one of the above (complicated) means by a family of means was published in [7] for the logarithmic mean

$$\mathcal{L}_{-1/3} < \mathcal{L}. \tag{8}$$

In fact, in [7] it is proved that $\mathcal{R}_{1/3} < \mathcal{L}$, where

$$\mathcal{R}_p(a,b) = \frac{ab^p + ba^p}{a^p + b^p},$$

but it is easy to see that

$$\mathcal{R}_p = \mathcal{S}_{1-p,-p} = \mathcal{L}_{-p}.$$

The inequality (8) was shown in [1] to be optimal. A second estimation was given in [1] for the the identric mean:

$$\mathcal{L}_{-1/6} < \mathcal{I}. \tag{9}$$

Recently, in [19] were given optimal estimations for the Seiffert means

$$\mathcal{L}_{-1/6} < \mathcal{P} < \mathcal{L}_0 \tag{10}$$

and

$$\mathcal{L}_0 < \mathcal{T} < \mathcal{L}_{1/3}. \tag{11}$$

Remark 2 *In the paper [19] the authors refer to the result (9) from [1]. Therefore, it is strange that the same authors prove again in [13] (not referring this time at [1]) the optimal estimations (8) and (9) as their main results. We remark that they also prove that the evaluations $\mathcal{L} < \mathcal{L}_0 = \mathcal{A}$ and $\mathcal{I} < \mathcal{L}_0$ are optimal.*

As both the power means and the Lehmer means are Gini means, we want to compare the above estimations using the following result proved in [11]. Define the function

$$k(u,v) = \begin{cases} \frac{|u|-|v|}{u-v}, & \text{if } u \neq v, \\ sgn(u), & \text{if } u = v, \end{cases} .$$

Theorem 3 *The inequality $\mathcal{S}_{p,q} \leq \mathcal{S}_{r,s}$ holds if and only if $p+q \leq r+s$ and:*

1) if $0 \leq \min\{p,q,r,s\}$ then $\min\{p,q\} \leq \min\{r,s\}$;
2) if $\min\{p,q,r,s\} < 0 < \max\{p,q,r,s\}$ then $k(p,q) \leq k(r,s)$;
3) if $\max\{p,q,r,s\} \leq 0$ then $\max\{p,q\} \leq \max\{r,s\}$.

Applying this result to $\mathcal{A}_p = \mathcal{S}_{p,0}$ and $\mathcal{L}_p = \mathcal{S}_{p+1,p}$ we get the following

Corollary 4 *For $p \geq 0$ we have: 1) $\mathcal{A}_p \leq \mathcal{L}_q$ if and only if $p \geq 1$ and $q \geq (p-1)/2$; and 2) $\mathcal{L}_q \leq \mathcal{A}_p$ if and only if $0 \leq p \leq 1$ and $q \leq (p-1)/2$.*

We deduce that:

1) $\mathcal{A}_{5/3} \leq \mathcal{L}_{1/3}$ thus the second part of (6) implies the second part of (11);
2) $\mathcal{L}_{-1/6} \leq \mathcal{A}_{2/3}$ thus the first part of (3) implies (9) and the second part of (5) suggests the first part of (10);
3) $\mathcal{L}_{-1/3} \leq \mathcal{A}_{1/3}$ thus the second part of (2) suggests (8).

In all these cases must be proven that the estimations are optimal. Other implications among the previous inequalities cannot be deduced on this way. The involved means are not comparable. For instance, the first inequality of (6) is $\mathcal{A}_p < \mathcal{T}$ where $p = \ln 2/\ln(\pi/2) = 1.5349...$ We have $\mathcal{A}_p \leq \mathcal{L}_{(p-1)/2}$ but $\mathcal{L}_{(p-1)/2}$ and $\mathcal{T}$ are not comparable as follows from (11) as $(p-1)/2 < 1/3$.

Thus the lower optimal estimations of the mean $\mathcal{T}$ by power means and by Lehmer means are not comparable. The lower optimal estimations of the mean $\mathcal{L}$ by Lehmer means is better than that by power means as $\mathcal{A}_0 = \mathcal{G} = \mathcal{L}_{-1/2} < \mathcal{L}_{-1/3}$. In all the other cases the optimal estimations by power means are better than that by Lehmer means.

3 Main results

We use the above procedure to deduce the following results.

Theorem 5 *The optimal estimation of the Neuman-Sándor mean $\mathcal{M}$ by Lehmer means is given by*

$$\mathcal{L}_0 < \mathcal{M} < \mathcal{L}_{1/6}. \tag{12}$$

Proof. The first part of (12) is known from (1) as $\mathcal{L}_0 = \mathcal{A}$. To prove the second part we use the second part of (7): $\mathcal{M} < \mathcal{A}_{4/3}$ and the Corollary 4. We want now to prove that the estimations are optimal. First of all we have to prove that for any $\varepsilon > 0$, the mean $\mathcal{M}$ is not greater than $\mathcal{L}_\varepsilon$. As

$$\lim_{t\to\infty} \frac{\mathcal{L}_\varepsilon(t,1)}{\mathcal{M}(t,1)} = \lim_{t\to\infty} \frac{t^{\varepsilon+1}+1}{t^\varepsilon+1} \cdot \frac{2\sinh^{-1}\frac{t-1}{t+1}}{t-1} = 2\sinh^{-1}1 = 2\ln\left(1+\sqrt{2}\right) > 1,$$

for sufficient great t we have $\mathcal{L}_\varepsilon(t,1) > \mathcal{M}(t,1)$. Similarly we have to prove that for any $\varepsilon > 0$, the mean $\mathcal{M}$ is not less than $\mathcal{L}_{(1-\varepsilon)/6}$. For this we prove that for every $\varepsilon > 0$ there exists $t > 1$ such that

$$f(t) = \mathcal{M}(t^6,1) - \mathcal{L}_{(1-\varepsilon)/6}(t^6,1) > 0.$$

But

$$f(t)=\frac{t^6-1}{2\sinh^{-1}\frac{t^6-1}{t^6+1}}-\frac{t^{7-\varepsilon}+1}{t^{1-\varepsilon}+1}=\frac{\left(t^{7-\varepsilon}+1\right)g(t)}{2\left(t^{1-\varepsilon}+1\right)\sinh^{-1}\frac{t^6-1}{t^6+1}}$$

where

$$g(t)=\frac{\left(t^6-1\right)\left(t^{1-\varepsilon}+1\right)}{t^{7-\varepsilon}+1}-2\sinh^{-1}\frac{t^6-1}{t^6+1}.$$

Making use of the Taylor expansion we get

$$g(t)=9\varepsilon\left(t-1\right)^3+O((t-1)^4),$$

which gives $g(t)>0$ and so $f(t)>0$ for some t. ■

Remark 6 *We have used the computer algebra system Mathematica to get the Taylor expansion of g.*

We add also similar results for the mean $\mathcal{J}$.

Theorem 7 *The optimal estimations of the Gini mean $\mathcal{J}$ by Lehmer means are given by*

$$\mathcal{L}_{1/2}<\mathcal{J}<\mathcal{L}_1. \tag{13}$$

Proof. The results follows easily from Theorem 3 as $\mathcal{L}_p=\mathcal{S}_{p+1,p}$ and $\mathcal{J}=\mathcal{S}_{1,1}$. ■

Remark 8 *It follows that the estimations of the Gini mean $\mathcal{J}$ by Lehmer means are better than by power means. Indeed $\mathcal{A}_2<\mathcal{L}_{1/2}$ and there is no finite index p such that $\mathcal{J}<\mathcal{A}_p$.*

Remark 9 *It is easy to see that the optimal estimations by power means can be ordered as:*

$$\mathcal{A}_0<\mathcal{L}<\mathcal{A}_{1/3}<\mathcal{A}_{\ln 2/\ln\pi}<\mathcal{P}<\mathcal{A}_{2/3}<\mathcal{I}<\mathcal{A}_{\ln 2}<\mathcal{A}_{\ln 2/\ln(\ln(3+2\sqrt{2}))}$$
$$<\mathcal{M}<\mathcal{A}_{4/3}<\mathcal{A}_{\ln 2/\ln(\pi/2)}<\mathcal{T}<\mathcal{A}_{5/3}<\mathcal{A}_2<\mathcal{J}<\mathcal{A}_\infty. \tag{14}$$

To the contrary, the optimal estimations by Lehmer means can be ordered only in two chains as:

$$\mathcal{L}_{-1/3}<\mathcal{L}<\mathcal{L}_0<\mathcal{M}<\mathcal{L}_{1/6}<\mathcal{L}_{1/2}<\mathcal{J}<\mathcal{L}_1$$

and

$$\mathcal{L}_{-1/6}<\mathcal{P}<\mathcal{I}<\mathcal{L}_0<\mathcal{T}<\mathcal{L}_{1/3}.$$

Of course we can write also a single chain

$$\mathcal{L}_{-1/3}<\mathcal{L}<\mathcal{P}<\mathcal{I}<\mathcal{L}_0<\mathcal{M}<\mathcal{T}<\mathcal{L}_{1/3}<\mathcal{L}_{1/2}<\mathcal{J}<\mathcal{L}_1$$

but loosing the terms $\mathcal{L}_{-1/6}$ and $\mathcal{L}_{1/6}$ which are not comparable with $\mathcal{L}$ respectively $\mathcal{T}$. Moreover the means $\mathcal{L}<\mathcal{P}<\mathcal{I}$ and $\mathcal{M}<\mathcal{T}$ cannot be separated by Lehmer means.

4 Applications

Taking the inequalities (12) and (13) in the point $(t, 1)$, we get the following estimations

$$t+1 < \frac{t-1}{\sinh^{-1}\frac{t-1}{t+1}} < 2\cdot\frac{t^{7/6}+1}{t^{1/6}+1}, t > 1$$

and

$$t-\sqrt{t}+1 < t^{\frac{t}{t+1}} < \frac{t^2+1}{t+1}, t > 1.$$

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