

**JENSEN AND OSTROWSKI TYPE INEQUALITIES FOR
GENERAL LEBESGUE INTEGRAL WITH APPLICATIONS**

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ABSTRACT. Some inequalities related to Jensen and Ostrowski inequalities for general Lebesgue integral are obtained. Applications for f -divergence measure are provided as well.

1. INTRODUCTION

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω , a σ -algebra \mathcal{A} of parts of Ω and a countably additive and positive measure μ on \mathcal{A} with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_{\Omega} d\mu = 1$. Consider the Lebesgue space

$$L(\Omega, \mu) := \{f : \Omega \rightarrow \mathbb{R}, f \text{ is } \mu\text{-measurable and } \int_{\Omega} |f(t)| d\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_{\Omega} w d\mu$ instead of $\int_{\Omega} w(t) d\mu(t)$.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, S.S. Dragomir obtained in 2002 [29] the following result:

Theorem 1. *Let $\Phi : [m, M] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable convex function on (m, M) and $f : \Omega \rightarrow [m, M]$ so that $\Phi \circ f, f, \Phi' \circ f, (\Phi' \circ f) \cdot f \in L(\Omega, \mu)$. Then we have the inequality:*

$$\begin{aligned} (1.1) \quad 0 &\leq \int_{\Omega} \Phi \circ f d\mu - \Phi\left(\int_{\Omega} f d\mu\right) \\ &\leq \int_{\Omega} f \cdot (\Phi' \circ f) d\mu - \int_{\Omega} \Phi' \circ f d\mu \int_{\Omega} f d\mu \\ &\leq \frac{1}{2} [\Phi'(M) - \Phi'(m)] \int_{\Omega} \left| f - \int_{\Omega} f d\mu \right| d\mu. \end{aligned}$$

In the case of discrete measure, we have:

Corollary 1. *Let $\Phi : [m, M] \rightarrow \mathbb{R}$ be a differentiable convex function on (m, M) . If $x_i \in [m, M]$ and $w_i \geq 0$ ($i = 1, \dots, n$) with $W_n := \sum_{i=1}^n w_i = 1$, then one has*

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the counterpart of Jensen's weighted discrete inequality:

$$\begin{aligned}
(1.2) \quad 0 &\leq \sum_{i=1}^n w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^n w_i x_i\right) \\
&\leq \sum_{i=1}^n w_i \Phi'(x_i) x_i - \sum_{i=1}^n w_i \Phi'(x_i) \sum_{i=1}^n w_i x_i \\
&\leq \frac{1}{2} [\Phi'(M) - \Phi'(m)] \sum_{i=1}^n w_i \left| x_i - \sum_{j=1}^n w_j x_j \right|.
\end{aligned}$$

Remark 1. We notice that the inequality between the first and the second term in (1.2) was proved in 1994 by Dragomir & Ionescu, see [35].

If $f, g : \Omega \rightarrow \mathbb{R}$ are μ -measurable functions and $f, g, fg \in L(\Omega, \mu)$, then we may consider the Čebyšev functional

$$(1.3) \quad T(f, g) := \int_{\Omega} f g d\mu - \int_{\Omega} f d\mu \int_{\Omega} g d\mu.$$

The following result is known in the literature as the Grüss inequality

$$(1.4) \quad |T(f, g)| \leq \frac{1}{4} (\Gamma - \gamma) (\Delta - \delta),$$

provided

$$(1.5) \quad -\infty < \gamma \leq f(t) \leq \Gamma < \infty, \quad -\infty < \delta \leq g(t) \leq \Delta < \infty$$

for μ -a.e. $t \in \Omega$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

If we assume that $-\infty < \gamma \leq f(x) \leq \Gamma < \infty$ for μ -a.e. $x \in \Omega$, then by the Grüss inequality for $g = f$ and by the Schwarz's integral inequality, we have

$$(1.6) \quad \int_{\Omega} \left| f - \int_{\Omega} f d\mu \right| d\mu \leq \left[\int_{\Omega} f^2 d\mu - \left(\int_{\Omega} f d\mu \right)^2 \right]^{\frac{1}{2}} \leq \frac{1}{2} (\Gamma - \gamma).$$

On making use of the results (1.1) and (1.6), we can state the following string of reverse inequalities

$$\begin{aligned}
(1.7) \quad 0 &\leq \int_{\Omega} \Phi \circ f d\mu - \Phi\left(\int_{\Omega} f d\mu\right) \\
&\leq \int_{\Omega} f \cdot (\Phi' \circ f) d\mu - \int_{\Omega} \Phi' \circ f d\mu \int_{\Omega} f d\mu \\
&\leq \frac{1}{2} [\Phi'(M) - \Phi'(m)] \int_{\Omega} \left| f - \int_{\Omega} f d\mu \right| d\mu \\
&\leq \frac{1}{2} [\Phi'(M) - \Phi'(m)] \left[\int_{\Omega} f^2 d\mu - \left(\int_{\Omega} f d\mu \right)^2 \right]^{\frac{1}{2}} \\
&\leq \frac{1}{4} [\Phi'(M) - \Phi'(m)] (M - m),
\end{aligned}$$

provided that $\Phi : [m, M] \subset \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable convex function on (m, M) and $f : \Omega \rightarrow [m, M]$ so that $\Phi \circ f, f, \Phi' \circ f, f \cdot (\Phi' \circ f) \in L(\Omega, \mu)$, with $\int_{\Omega} d\mu = 1$.

The following reverse of the Jensen's inequality also holds [33]:

Theorem 2. *Let $\Phi : I \rightarrow \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \overset{\circ}{I}$, where $\overset{\circ}{I}$ is the interior of I . If $f : \Omega \rightarrow \mathbb{R}$ is μ -measurable, satisfies the bounds*

$$-\infty < m \leq f(t) \leq M < \infty \text{ for } \mu\text{-a.e. } t \in \Omega$$

and such that $f, \Phi \circ f \in L(\Omega, \mu)$, then

$$\begin{aligned} (1.8) \quad 0 &\leq \int_{\Omega} \Phi \circ f d\mu - \Phi \left(\int_{\Omega} f d\mu \right) \\ &\leq \left(M - \int_{\Omega} f d\mu \right) \left(\int_{\Omega} f d\mu - m \right) \frac{\Phi'_-(M) - \Phi'_+(m)}{M - m} \\ &\leq \frac{1}{4} (M - m) [\Phi'_-(M) - \Phi'_+(m)], \end{aligned}$$

where Φ'_- is the left and Φ'_+ is the right derivative of the convex function Φ .

For other reverse of Jensen inequality and applications to divergence measures see [33].

In 1938, A. Ostrowski [53], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a} \int_a^b \Phi(t) dt$ and the value $\Phi(x)$, $x \in [a, b]$.

Theorem 3. *Let $\Phi : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) such that $\Phi' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , i.e., $\|\Phi'\|_{\infty} := \sup_{t \in (a, b)} |\Phi'(t)| < \infty$.*

Then

$$(1.9) \quad \left| \Phi(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|\Phi'\|_{\infty} (b-a),$$

for all $x \in [a, b]$ and the constant $\frac{1}{4}$ is the best possible.

For various results related to Ostrowski's inequality see [6]-[9], [15]-[40], [42] and the references therein.

Motivated by the above results, in this paper we investigate the magnitude of the quantity

$$\int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \lambda \left(\int_{\Omega} g d\mu - x \right), \quad x \in [a, b],$$

for various assumptions on the absolutely continuous function Φ , which in the particular case of $x = \int_{\Omega} g d\mu$ provides some results connected with Jensen's inequality while in the case $\lambda = 0$ provides some generalizations of Ostrowski's inequality. Applications for divergence measures are provided as well.

2. SOME IDENTITIES

The following result holds:

Lemma 1. *Let $\Phi : I \rightarrow \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \overset{\circ}{I}$, the interior of I . If $g : \Omega \rightarrow [a, b]$ is Lebesgue μ -measurable on Ω and such that $\Phi \circ g$,*

$g \in L(\Omega, \mu)$, then we have the equality

$$(2.1) \quad \begin{aligned} & \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \lambda \left(\int_{\Omega} g d\mu - x \right) \\ &= \int_{\Omega} \left[(g-x) \int_0^1 (\Phi'((1-s)x + sg) - \lambda) ds \right] d\mu \end{aligned}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a, b]$.

In particular, we have

$$(2.2) \quad \int_{\Omega} \Phi \circ g d\mu - \Phi(x) = \int_{\Omega} \left[(g-x) \int_0^1 \Phi'((1-s)x + sg) ds \right] d\mu,$$

for any $x \in [a, b]$.

Proof. Since Φ is absolutely continuous on $[a, b]$, then for any $u, v \in [a, b]$ we have

$$(2.3) \quad \Phi(u) - \Phi(v) = (u-v) \int_0^1 \Phi'((1-s)v + su) ds.$$

This implies that

$$\Phi(g(t)) - \Phi(x) = (g(t) - x) \int_0^1 \Phi'((1-s)x + sg(t)) ds$$

for any $t \in \Omega$, or equivalently,

$$(2.4) \quad \Phi \circ g - \Phi(x) = (g-x) \int_0^1 \Phi'((1-s)x + sg) ds.$$

Since $\Phi : I \rightarrow \mathbb{C}$ is an absolutely continuous functions on $[a, b]$ the Lebesgue integral over μ in the right side of (2.1) exists for any $\lambda \in \mathbb{C}$ and $x \in [a, b]$.

Integrating (2.4) over the measure μ on Ω and since $\int_{\Omega} d\mu = 1$, then we have

$$(2.5) \quad \int_{\Omega} \Phi \circ g d\mu - \Phi(x) = \int_{\Omega} \left[(g-x) \int_0^1 \Phi'((1-s)x + sg) ds \right] d\mu.$$

Now, observe that for $\lambda \in \mathbb{C}$ we have

$$(2.6) \quad \begin{aligned} & \int_{\Omega} \left[(g-x) \int_0^1 (\Phi'((1-s)x + sg) - \lambda) ds \right] d\mu \\ &= \int_{\Omega} \left[(g-x) \left(\int_0^1 \Phi'((1-s)x + sg) ds - \lambda \right) \right] d\mu \\ &= \int_{\Omega} \left[(g-x) \int_0^1 \Phi'((1-s)x + sg) ds \right] d\mu - \lambda \int_{\Omega} (g-x) d\mu \\ &= \int_{\Omega} \left[(g-x) \int_0^1 \Phi'((1-s)x + sg) ds \right] d\mu - \lambda \left(\int_{\Omega} g d\mu - x \right). \end{aligned}$$

Making use of (2.5) and (2.6) we deduce the desired result (2.1). \square

Remark 2. With the assumptions of Lemma 1 we have

$$(2.7) \quad \begin{aligned} & \int_{\Omega} \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) \\ &= \int_{\Omega} \left[\left(g - \frac{a+b}{2}\right) \int_0^1 \Phi' \left((1-s) \frac{a+b}{2} + sg \right) ds \right] d\mu. \end{aligned}$$

Corollary 2. *With the assumptions of Lemma 1 we have*

$$(2.8) \quad \int_{\Omega} \Phi \circ g d\mu - \Phi \left(\int_{\Omega} g d\mu \right) \\ = \int_{\Omega} \left[\left(g - \int_{\Omega} g d\mu \right) \int_0^1 \Phi' \left((1-s) \int_{\Omega} g d\mu + sg \right) ds \right] d\mu.$$

Proof. We observe that since $g : \Omega \rightarrow [a, b]$ and $\int_{\Omega} d\mu = 1$ then $\int_{\Omega} g d\mu \in [a, b]$ and by taking $x = \int_{\Omega} g d\mu$ in (2.2) we get (2.8). \square

Corollary 3. *With the assumptions of Lemma 1 we have*

$$(2.9) \quad \int_{\Omega} \Phi \circ g d\mu - \frac{1}{b-a} \int_a^b \Phi(x) dx - \lambda \left(\int_{\Omega} g d\mu - \frac{a+b}{2} \right) \\ = \int_{\Omega} \left\{ \frac{1}{b-a} \int_a^b \left[(g-x) \int_0^1 (\Phi'((1-s)x + sg) - \lambda) ds \right] dx \right\} d\mu.$$

Proof. Follows by integrating the identity (2.1) over $x \in [a, b]$, dividing by $b-a > 0$ and using Fubini's theorem. \square

Corollary 4. *Let $\Phi : I \rightarrow \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \mathring{I}$, the interior of I . If $g, h : \Omega \rightarrow [a, b]$ are Lebesgue μ -measurable on Ω and such that $\Phi \circ g, \Phi \circ h, g, h \in L(\Omega, \mu)$, then we have the equality*

$$(2.10) \quad \int_{\Omega} \Phi \circ g d\mu - \int_{\Omega} \Phi \circ h d\mu - \lambda \left(\int_{\Omega} g d\mu - \int_{\Omega} h d\mu \right) \\ = \int_{\Omega} \int_{\Omega} \left[(g(t) - h(\tau)) \int_0^1 (\Phi'((1-s)h(\tau) + sg(t)) - \lambda) ds \right] \\ \times d\mu(t) d\mu(\tau)$$

for any $\lambda \in \mathbb{C}$ and $x \in [a, b]$.

In particular, we have

$$(2.11) \quad \int_{\Omega} \Phi \circ g d\mu - \int_{\Omega} \Phi \circ h d\mu \\ = \int_{\Omega} \int_{\Omega} \left[(g(t) - h(\tau)) \int_0^1 \Phi'((1-s)h(\tau) + sg(t)) ds \right] d\mu(t) d\mu(\tau),$$

for any $x \in [a, b]$.

Proof. From (2.1) we have for any $\tau \in \Omega$ that

$$\int_{\Omega} \Phi \circ g d\mu - \Phi(h(\tau)) - \lambda \left(\int_{\Omega} g d\mu - \Phi(h(\tau)) \right) \\ = \int_{\Omega} \left[(g - \Phi(h(\tau))) \int_0^1 (\Phi'((1-s)\Phi(h(\tau)) + sg) - \lambda) ds \right] d\mu$$

for any $\lambda \in \mathbb{C}$ and $x \in [a, b]$.

Integrating on Ω over $d\mu(\tau)$ and using Fubini's theorem we get the desired result (2.10). \square

Remark 3. The above inequality (2.10) can be extended for two measures as follows

$$(2.12) \quad \begin{aligned} & \int_{\Omega_1} \Phi \circ g d\mu_1 - \int_{\Omega_2} \Phi \circ h d\mu_2 - \lambda \left(\int_{\Omega_1} g d\mu_1 - \int_{\Omega_2} h d\mu_2 \right) \\ &= \int_{\Omega_1} \int_{\Omega_2} \left[(g(t) - h(\tau)) \int_0^1 (\Phi'((1-s)h(\tau) + sg(t)) - \lambda) ds \right] \\ & \quad \times d\mu_1(t) d\mu_2(\tau), \end{aligned}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a, b]$ and provided that $\Phi \circ g, g \in L(\Omega_1, \mu_1)$ while $\Phi \circ h, h \in L(\Omega_2, \mu_2)$.

Remark 4. If $w \geq 0$ μ -almost everywhere (μ -a.e.) on Ω with $\int_{\Omega} w d\mu > 0$, then by replacing $d\mu$ with $\frac{w d\mu}{\int_{\Omega} w d\mu}$ in (2.1) we have the weighted equality

$$(2.13) \quad \begin{aligned} & \frac{1}{\int_{\Omega} w d\mu} \int_{\Omega} w (\Phi \circ g) d\mu - \Phi(x) - \lambda \left(\frac{1}{\int_{\Omega} w d\mu} \int_{\Omega} w g d\mu - x \right) \\ &= \frac{1}{\int_{\Omega} w d\mu} \int_{\Omega} w \cdot \left[(g - x) \int_0^1 (\Phi'((1-s)x + sg) - \lambda) ds \right] d\mu \end{aligned}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a, b]$, provided $\Phi \circ g, g \in L_w(\Omega, \mu)$ where

$$L_w(\Omega, \mu) := \left\{ g \mid \int_{\Omega} w |g| d\mu < \infty \right\}.$$

The other equalities have similar weighted versions. However the details are omitted.

If we use the discrete measure, then for $\Phi : I \rightarrow \mathbb{C}$ an absolutely continuous functions on $[a, b] \subset \hat{I}$, the interior of I , $x_j \in [a, b]$ and $p_j \geq 0$ with $\sum_{j=1}^n p_j = 1$ we can state the following identity

$$(2.14) \quad \begin{aligned} & \sum_{j=1}^n p_j \Phi(x_j) - \Phi(x) - \lambda \left(\sum_{j=1}^n p_j x_j - x \right) \\ &= \sum_{j=1}^n p_j \left[(x_j - x) \int_0^1 (\Phi'((1-s)x + sx_j) - \lambda) ds \right] \end{aligned}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a, b]$.

In particular, we have

$$(2.15) \quad \sum_{j=1}^n p_j \Phi(x_j) - \Phi(x) = \sum_{j=1}^n p_j \left[(x_j - x) \int_0^1 \Phi'((1-s)x + sx_j) ds \right]$$

for any $x \in [a, b]$ and

$$(2.16) \quad \begin{aligned} & \sum_{j=1}^n p_j \Phi(x_j) - \Phi\left(\frac{a+b}{2}\right) \\ &= \sum_{j=1}^n p_j \left[\left(x_j - \frac{a+b}{2}\right) \int_0^1 \Phi'\left(\left(1-s\right)\frac{a+b}{2} + sx_j\right) ds \right] \end{aligned}$$

and

$$(2.17) \quad \begin{aligned} & \sum_{j=1}^n p_j \Phi(x_j) - \Phi\left(\sum_{k=1}^n p_k x_k\right) \\ &= \sum_{j=1}^n p_j \left[\left(x_j - \sum_{k=1}^n p_k x_k\right) \int_0^1 \Phi' \left((1-s) \sum_{k=1}^n p_k x_k + s x_j \right) ds \right]. \end{aligned}$$

If $x_j \in [a, b]$ and $p_j \geq 0$, $j \in \{1, \dots, n\}$ with $\sum_{j=1}^n p_j = 1$ and if $y_k \in [a, b]$ and $q_k \geq 0$, $k \in \{1, \dots, m\}$ with $\sum_{k=1}^m q_k = 1$, then we can state the following identity as well:

$$(2.18) \quad \begin{aligned} & \sum_{j=1}^n p_j \Phi(x_j) - \sum_{k=1}^m q_k \Phi(y_k) - \lambda \left(\sum_{j=1}^n p_j x_j - \sum_{k=1}^m q_k y_k \right) \\ &= \sum_{j=1}^n p_j \sum_{k=1}^m q_k \left[(x_j - y_k) \int_0^1 (\Phi'((1-s)y_k + s x_j) - \lambda) ds \right]. \end{aligned}$$

In particular, we have

$$(2.19) \quad \begin{aligned} & \sum_{j=1}^n p_j \Phi(x_j) - \sum_{k=1}^m q_k \Phi(y_k) \\ &= \sum_{j=1}^n p_j \sum_{k=1}^m q_k \left[(x_j - y_k) \int_0^1 \Phi'((1-s)y_k + s x_j) ds \right]. \end{aligned}$$

3. BOUNDS IN TERMS OF p -NORMS

We use the notations

$$\|k\|_{\Omega, p} := \begin{cases} \left(\int_{\Omega} |k(t)|^p d\mu(t) \right)^{1/p} < \infty, & p \geq 1, k \in L_p(\Omega, \mu); \\ \text{ess sup}_{t \in \Omega} |k(t)| < \infty, & p = \infty, k \in L_{\infty}(\Omega, \mu); \end{cases}$$

and

$$\|\Phi\|_{[0,1], p} := \begin{cases} \left(\int_0^1 |\Phi(s)|^p ds \right)^{1/p} < \infty, & p \geq 1, \Phi \in L_p(0, 1); \\ \text{ess sup}_{s \in [0,1]} |\Phi(s)| < \infty, & p = \infty, \Phi \in L_{\infty}(0, 1). \end{cases}$$

If we consider the identity function $\ell : [0, 1] \rightarrow [0, 1]$, $\ell(s) = s$ we have

$$\int_0^1 |\Phi'((1-s)x + sg(t)) - \lambda|^p ds = \|\Phi'((1-\ell)x + \ell g(t)) - \lambda\|_{[0,1], p}^p$$

and

$$\text{ess sup}_{s \in [0,1]} |\Phi'((1-s)x + sg(t)) - \lambda| = \|\Phi'((1-\ell)x + \ell g(t)) - \lambda\|_{[0,1], \infty}$$

for $t \in \Omega$.

Theorem 4. Let $\Phi : I \rightarrow \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \overset{\circ}{I}$, the interior of I . If $g : \Omega \rightarrow [a, b]$ is Lebesgue μ -measurable on Ω and such that $\Phi \circ g, g \in L(\Omega, \mu)$, then

$$(3.1) \quad \begin{aligned} & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \lambda \left(\int_{\Omega} g d\mu - x \right) \right| \\ & \leq \int_{\Omega} |g - x| \|\Phi'((1 - \ell)x + \ell g) - \lambda\|_{[0,1],1} d\mu \\ & \leq \begin{cases} \|g - x\|_{\Omega, \infty} \|\Phi'((1 - \ell)x + \ell g) - \lambda\|_{[0,1],1} \Big\|_{\Omega, 1} ; \\ \|g - x\|_{\Omega, p} \|\Phi'((1 - \ell)x + \ell g) - \lambda\|_{[0,1],1} \Big\|_{\Omega, q} \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \|g - x\|_{\Omega, 1} \|\Phi'((1 - \ell)x + \ell g) - \lambda\|_{[0,1],1} \Big\|_{\Omega, \infty} ; \end{cases} \end{aligned}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a, b]$.

In particular, we have

$$(3.2) \quad \begin{aligned} & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) \right| \\ & \leq \int_{\Omega} |g - x| \|\Phi'((1 - \ell)x + \ell g)\|_{[0,1],1} d\mu \\ & \leq \begin{cases} \|g - x\|_{\Omega, \infty} \|\Phi'((1 - \ell)x + \ell g)\|_{[0,1],1} \Big\|_{\Omega, 1} ; \\ \|g - x\|_{\Omega, p} \|\Phi'((1 - \ell)x + \ell g)\|_{[0,1],1} \Big\|_{\Omega, q} \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \|g - x\|_{\Omega, 1} \|\Phi'((1 - \ell)x + \ell g)\|_{[0,1],1} \Big\|_{\Omega, \infty} ; \end{cases} \end{aligned}$$

for any $x \in [a, b]$.

Proof. Taking the modulus in the equality (2.1), we have

$$(3.3) \quad \begin{aligned} & \left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \lambda \left(\int_{\Omega} g d\mu - x \right) \right| \\ & \leq \int_{\Omega} \left| (g - x) \int_0^1 (\Phi'((1 - s)x + sg) - \lambda) ds \right| d\mu \\ & \leq \int_{\Omega} |g - x| \int_0^1 |\Phi'((1 - s)x + sg) - \lambda| ds d\mu \\ & = \int_{\Omega} |g - x| \|\Phi'((1 - \ell)x + \ell g) - \lambda\|_{[0,1],1} d\mu \end{aligned}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a, b]$.

Utilising Hölder's inequality for the μ -measurable functions $F, G : \Omega \rightarrow \mathbb{C}$,

$$\left| \int_{\Omega} FG d\mu \right| \leq \left(\int_{\Omega} |F|^p d\mu \right)^{1/p} \left(\int_{\Omega} |G|^q d\mu \right)^{1/q}, \quad p > 1, \frac{1}{p} + \frac{1}{q} = 1$$

and

$$\left| \int_{\Omega} FGd\mu \right| \leq \operatorname{ess\,sup}_{t \in \Omega} |F(t)| \int_{\Omega} |G| d\mu$$

we get from (3.3) the desired result (3.1). \square

Remark 5. *If we take $x = \frac{a+b}{2}$ in (3.1), then we get*

$$(3.4) \quad \begin{aligned} & \left| \int_{\Omega} \Phi \circ gd\mu - \Phi \left(\frac{a+b}{2} \right) - \lambda \left(\int_{\Omega} gd\mu - \frac{a+b}{2} \right) \right| \\ & \leq \int_{\Omega} \left| g - \frac{a+b}{2} \right| \left\| \Phi' \left((1-\ell) \frac{a+b}{2} + \ell g \right) - \lambda \right\|_{[0,1],1} d\mu \\ & \leq \begin{cases} \|g - \frac{a+b}{2}\|_{\Omega, \infty} \left\| \Phi' \left((1-\ell) \frac{a+b}{2} + \ell g \right) - \lambda \right\|_{[0,1],1} \Big\|_{\Omega,1}; \\ \|g - \frac{a+b}{2}\|_{\Omega, p} \left\| \Phi' \left((1-\ell) \frac{a+b}{2} + \ell g \right) - \lambda \right\|_{[0,1],1} \Big\|_{\Omega, q} \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \|g - \frac{a+b}{2}\|_{\Omega, 1} \left\| \Phi' \left((1-\ell) \frac{a+b}{2} + \ell g \right) - \lambda \right\|_{[0,1],1} \Big\|_{\Omega, \infty}; \end{cases} \end{aligned}$$

for any $\lambda \in \mathbb{C}$ and, in particular, for $\lambda = 0$ we have

$$(3.5) \quad \begin{aligned} & \left| \int_{\Omega} \Phi \circ gd\mu - \Phi \left(\frac{a+b}{2} \right) \right| \\ & \leq \int_{\Omega} \left| g - \frac{a+b}{2} \right| \left\| \Phi' \left((1-\ell) \frac{a+b}{2} + \ell g \right) \right\|_{[0,1],1} d\mu \\ & \leq \begin{cases} \|g - \frac{a+b}{2}\|_{\Omega, \infty} \left\| \Phi' \left((1-\ell) \frac{a+b}{2} + \ell g \right) \right\|_{[0,1],1} \Big\|_{\Omega,1}; \\ \|g - \frac{a+b}{2}\|_{\Omega, p} \left\| \Phi' \left((1-\ell) \frac{a+b}{2} + \ell g \right) \right\|_{[0,1],1} \Big\|_{\Omega, q} \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \|g - \frac{a+b}{2}\|_{\Omega, 1} \left\| \Phi' \left((1-\ell) \frac{a+b}{2} + \ell g \right) \right\|_{[0,1],1} \Big\|_{\Omega, \infty}; \end{cases} \end{aligned}$$

If we take $x = \int_{\Omega} gd\mu$ in (3.1), then we get

$$(3.6) \quad \begin{aligned} & \left| \int_{\Omega} \Phi \circ gd\mu - \Phi \left(\int_{\Omega} gd\mu \right) \right| \\ & \leq \int_{\Omega} \left| g - \int_{\Omega} gd\mu \right| \left\| \Phi' \left((1-\ell) \int_{\Omega} gd\mu + \ell g \right) - \lambda \right\|_{[0,1],1} d\mu \\ & \leq \begin{cases} \|g - \int_{\Omega} gd\mu\|_{\Omega, \infty} \left\| \Phi' \left((1-\ell) \int_{\Omega} gd\mu + \ell g \right) - \lambda \right\|_{[0,1],1} \Big\|_{\Omega,1}; \\ \|g - \int_{\Omega} gd\mu\|_{\Omega, p} \left\| \Phi' \left((1-\ell) \int_{\Omega} gd\mu + \ell g \right) - \lambda \right\|_{[0,1],1} \Big\|_{\Omega, q} \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \|g - \int_{\Omega} gd\mu\|_{\Omega, 1} \left\| \Phi' \left((1-\ell) \int_{\Omega} gd\mu + \ell g \right) - \lambda \right\|_{[0,1],1} \Big\|_{\Omega, \infty}; \end{cases} \end{aligned}$$

for any $\lambda \in \mathbb{C}$ and, in particular, for $\lambda = 0$ we have

$$(3.7) \quad \left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left(\int_{\Omega} g d\mu \right) \right| \leq \int_{\Omega} \left| g - \int_{\Omega} g d\mu \right| \left\| \Phi' \left((1-\ell) \int_{\Omega} g d\mu + \ell g \right) \right\|_{[0,1],1} d\mu$$

$$\leq \begin{cases} \|g - \int_{\Omega} g d\mu\|_{\Omega, \infty} \left\| \Phi' \left((1-\ell) \int_{\Omega} g d\mu + \ell g \right) \right\|_{[0,1],1} \Big\|_{\Omega,1}; \\ \|g - \int_{\Omega} g d\mu\|_{\Omega, p} \left\| \Phi' \left((1-\ell) \int_{\Omega} g d\mu + \ell g \right) \right\|_{[0,1],1} \Big\|_{\Omega, q} \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \|g - \int_{\Omega} g d\mu\|_{\Omega, 1} \left\| \Phi' \left((1-\ell) \int_{\Omega} g d\mu + \ell g \right) \right\|_{[0,1],1} \Big\|_{\Omega, \infty}. \end{cases}$$

Corollary 5. Let $\Phi : I \rightarrow \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \hat{I}$, the interior of I . If $g : \Omega \rightarrow [a, b]$ is Lebesgue μ -measurable on Ω and such that $\Phi \circ g, g \in L(\Omega, \mu)$, then

$$(3.8) \quad \left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) \right| \leq \|\Phi'\|_{[a,b], \infty} \int_{\Omega} |g - x| d\mu$$

for any $x \in [a, b]$.

In particular, we have

$$(3.9) \quad \left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left(\frac{a+b}{2} \right) \right| \leq \|\Phi'\|_{[a,b], \infty} \int_{\Omega} \left| g - \frac{a+b}{2} \right| d\mu$$

and

$$(3.10) \quad \left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left(\int_{\Omega} g d\mu \right) \right| \leq \|\Phi'\|_{[a,b], \infty} \int_{\Omega} \left| g - \int_{\Omega} g d\mu \right| d\mu.$$

Proof. We have from (3.1) that

$$(3.11) \quad \left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) \right| \leq \int_{\Omega} |g - x| \left(\int_0^1 |\Phi'((1-s)x + sg)| ds \right) d\mu$$

for any $x \in [a, b]$.

However, for any $t \in \Omega$ and almost every $s \in [0, 1]$ we have

$$|\Phi'((1-s)x + sg(t))| \leq \text{ess sup}_{u \in [a,b]} |\Phi'(u)| = \|\Phi'\|_{[a,b], \infty},$$

for any $x \in [a, b]$.

Making use of (3.11) we get (3.8). □

Remark 6. We remark that, the quantity from Corollary 5

$$\delta_{\mu}(g, x) := \int_{\Omega} |g - x| d\mu$$

cannot be computed in general.

However, in the case when $\Omega = [a, b]$, $g : [a, b] \rightarrow [a, b]$, $g(t) = t$ and $\mu(t) = \frac{1}{b-a}dt$, we have

$$\begin{aligned} \delta_\mu(g, x) &:= \frac{1}{b-a} \int_a^b |t-x| dt = \frac{1}{b-a} \left[\int_a^x (x-t) dt + \int_x^b (t-x) dt \right] \\ &= \frac{1}{b-a} \left[(x-a)^2 + (b-x)^2 \right] \\ &= \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a), \end{aligned}$$

where $x \in [a, b]$.

Utilising the inequality (3.8) we get Ostrowski's inequality

$$(3.12) \quad \left| \frac{1}{b-a} \int_a^b \Phi(t) dt - \Phi(x) \right| \leq \|\Phi'\|_{[a,b],\infty} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a)$$

for any $x \in [a, b]$.

From the inequalities (3.9) and (3.10) we get the midpoint inequality

$$(3.13) \quad \left| \frac{1}{b-a} \int_a^b \Phi(t) dt - \Phi\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{4} \|\Phi'\|_{[a,b],\infty} (b-a).$$

Remark 7. If we consider the dispersion or the standard variation

$$\sigma_\mu(g) := \left(\int_\Omega \left(g - \int_\Omega g d\mu \right)^2 d\mu \right)^{1/2} = \left(\int_\Omega g^2 d\mu - \left(\int_\Omega g d\mu \right)^2 \right)^{1/2},$$

then by (3.10) we have the inequalities

$$(3.14) \quad \begin{aligned} \left| \int_\Omega \Phi \circ g d\mu - \Phi \left(\int_\Omega g d\mu \right) \right| &\leq \|\Phi'\|_{[a,b],\infty} \delta_\mu \left(g, \int_\Omega g d\mu \right) \\ &\leq \|\Phi'\|_{[a,b],\infty} \sigma_\mu(g). \end{aligned}$$

In general, we have by Cauchy-Bunyakovsky-Schwarz's inequality that

$$(3.15) \quad \delta_\mu(g, x) := \int_\Omega |g-x| d\mu \leq \left(\int_\Omega (g-x)^2 d\mu \right)^{1/2}.$$

Since

$$\begin{aligned} &\int_\Omega (g-x)^2 d\mu \\ &= \int_\Omega \left(g - \int_\Omega g d\mu + \int_\Omega g d\mu - x \right)^2 d\mu \\ &= \int_\Omega \left(g - \int_\Omega g d\mu \right)^2 d\mu + 2 \int_\Omega \left(g - \int_\Omega g d\mu \right) \left(\int_\Omega g d\mu - x \right) d\mu \\ &\quad + \int_\Omega \left(\int_\Omega g d\mu - x \right)^2 d\mu \\ &= \int_\Omega \left(g - \int_\Omega g d\mu \right)^2 d\mu + \left(\int_\Omega g d\mu - x \right)^2 \end{aligned}$$

for any $x \in [a, b]$, then by (3.8) and (3.15) we get the inequalities

$$(3.16) \quad \left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) \right| \leq \|\Phi'\|_{[a,b],\infty} \delta_{\mu}(g, x) \\ \leq \|\Phi'\|_{[a,b],\infty} \left[\sigma_{\mu}^2(g) + \left(\int_{\Omega} g d\mu - x \right)^2 \right]^{1/2}$$

for any $x \in [a, b]$.

If we use the discrete measure, then from (3.16) we have

$$(3.17) \quad \left| \sum_{j=1}^n p_j \Phi(x_j) - \Phi(x) \right| \\ \leq \|\Phi'\|_{[a,b],\infty} \sum_{j=1}^n p_j |x_j - x| \\ \leq \|\Phi'\|_{[a,b],\infty} \left[\sum_{j=1}^n p_j x_j^2 - \left(\sum_{j=1}^n p_j x_j \right)^2 + \left(\sum_{j=1}^n p_j x_j - x \right)^2 \right]^{1/2},$$

for any $x \in [a, b]$, where $x_j \in [a, b]$ and $p_j \geq 0$ with $\sum_{j=1}^n p_j = 1$.

In particular, we have

$$(3.18) \quad \left| \sum_{j=1}^n p_j \Phi(x_j) - \Phi\left(\frac{a+b}{2}\right) \right| \leq \|\Phi'\|_{[a,b],\infty} \sum_{j=1}^n p_j \left| x_j - \frac{a+b}{2} \right| \\ \leq \frac{1}{2} (b-a) \|\Phi'\|_{[a,b],\infty}$$

and

$$(3.19) \quad \left| \sum_{j=1}^n p_j \Phi(x_j) - \Phi\left(\sum_{k=1}^n p_k x_k\right) \right| \\ \leq \|\Phi'\|_{[a,b],\infty} \sum_{j=1}^n p_j \left| x_j - \sum_{k=1}^n p_k x_k \right| \\ \leq \|\Phi'\|_{[a,b],\infty} \left[\sum_{j=1}^n p_j x_j^2 - \left(\sum_{j=1}^n p_j x_j \right)^2 \right]^{1/2} \leq \frac{1}{2} (b-a) \|\Phi'\|_{[a,b],\infty}.$$

4. INEQUALITIES FOR BOUNDED DERIVATIVES

Now, for $\gamma, \Gamma \in \mathbb{C}$ and $[a, b]$ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_{[a,b]}(\gamma, \Gamma) \\ := \left\{ f : [a, b] \rightarrow \mathbb{C} \mid \operatorname{Re} \left[(\Gamma - f(t)) \left(\overline{f(t)} - \bar{\gamma} \right) \right] \geq 0 \text{ for almost every } t \in [a, b] \right\}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma, \Gamma) := \left\{ f : [a, b] \rightarrow \mathbb{C} \mid \left| f(t) - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \text{ for a.e. } t \in [a, b] \right\}.$$

The following representation result may be stated.

Proposition 1. *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have that $\bar{U}_{[a,b]}(\gamma, \Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma, \Gamma)$ are nonempty, convex and closed sets and*

$$(4.1) \quad \bar{U}_{[a,b]}(\gamma, \Gamma) = \bar{\Delta}_{[a,b]}(\gamma, \Gamma).$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left| z - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma|$$

if and only if

$$\operatorname{Re}[(\Gamma - z)(\bar{z} - \bar{\gamma})] \geq 0.$$

This follows by the equality

$$\frac{1}{4} |\Gamma - \gamma|^2 - \left| z - \frac{\gamma + \Gamma}{2} \right|^2 = \operatorname{Re}[(\Gamma - z)(\bar{z} - \bar{\gamma})]$$

that holds for any $z \in \mathbb{C}$.

The equality (3.1) is thus a simple consequence of this fact. \square

On making use of the complex numbers field properties we can also state that:

Corollary 6. *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have that*

$$(4.2) \quad \bar{U}_{[a,b]}(\gamma, \Gamma) = \{ f : [a, b] \rightarrow \mathbb{C} \mid (\operatorname{Re} \Gamma - \operatorname{Re} f(t))(\operatorname{Re} f(t) - \operatorname{Re} \gamma) \\ + (\operatorname{Im} \Gamma - \operatorname{Im} f(t))(\operatorname{Im} f(t) - \operatorname{Im} \gamma) \geq 0 \text{ for a.e. } t \in [a, b] \}.$$

Now, if we assume that $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$(4.3) \quad \bar{S}_{[a,b]}(\gamma, \Gamma) := \{ f : [a, b] \rightarrow \mathbb{C} \mid \operatorname{Re}(\Gamma) \geq \operatorname{Re} f(t) \geq \operatorname{Re}(\gamma) \\ \text{and } \operatorname{Im}(\Gamma) \geq \operatorname{Im} f(t) \geq \operatorname{Im}(\gamma) \text{ for a.e. } t \in [a, b] \}.$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma, \Gamma)$ is closed, convex and

$$(4.4) \quad \emptyset \neq \bar{S}_{[a,b]}(\gamma, \Gamma) \subseteq \bar{U}_{[a,b]}(\gamma, \Gamma).$$

The following result holds:

Theorem 5. *Let $\Phi : I \rightarrow \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \tilde{I}$, the interior of I . For some $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, assume that $\Phi' \in \bar{U}_{[a,b]}(\gamma, \Gamma) = \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$. If $g : \Omega \rightarrow [a, b]$ is Lebesgue μ -measurable on Ω and such that $\Phi \circ g, g \in L(\Omega, \mu)$, then we have the inequality*

$$(4.5) \quad \left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \frac{\gamma + \Gamma}{2} \left(\int_{\Omega} g d\mu - x \right) \right| \leq \frac{1}{2} |\Gamma - \gamma| \int_{\Omega} |g - x| d\mu$$

for any $x \in [a, b]$.

In particular, we have

$$(4.6) \quad \left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left(\frac{a+b}{2} \right) - \frac{\gamma+\Gamma}{2} \left(\int_{\Omega} g d\mu - \frac{a+b}{2} \right) \right| \\ \leq \frac{1}{2} |\Gamma - \gamma| \int_{\Omega} \left| g - \frac{a+b}{2} \right| d\mu \leq \frac{1}{4} (b-a) |\Gamma - \gamma|$$

and

$$(4.7) \quad \left| \int_{\Omega} \Phi \circ g d\mu - \Phi \left(\int_{\Omega} g d\mu \right) \right| \leq \frac{1}{2} |\Gamma - \gamma| \int_{\Omega} \left| g - \int_{\Omega} g d\mu \right| d\mu \\ \leq \frac{1}{2} |\Gamma - \gamma| \left(\int_{\Omega} g^2 d\mu - \left(\int_{\Omega} g d\mu \right)^2 \right)^{1/2} \\ \leq \frac{1}{4} (b-a) |\Gamma - \gamma|.$$

Proof. By the equality (2.1) for $\lambda = \frac{\gamma+\Gamma}{2}$ we have

$$(4.8) \quad \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \frac{\gamma+\Gamma}{2} \left(\int_{\Omega} g d\mu - x \right) \\ = \int_{\Omega} \left[(g-x) \int_0^1 \left(\Phi'((1-s)x + sg) - \frac{\gamma+\Gamma}{2} \right) ds \right] d\mu.$$

Since $\Phi' \in \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$ we have

$$(4.9) \quad \left| \Phi'((1-s)x + sg(t)) - \frac{\gamma+\Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma|$$

for a.e. $s \in [0, 1]$ and for any $x \in [a, b]$ and any $t \in \Omega$.

Integrating (4.9) over s on $[0, 1]$ we get

$$(4.10) \quad \int_0^1 \left| \Phi'((1-s)x + sg(t)) - \frac{\gamma+\Gamma}{2} \right| ds \leq \frac{1}{2} |\Gamma - \gamma|$$

for any $x \in [a, b]$ and any $t \in \Omega$.

Taking the modulus in (4.8) we get via (4.10) that

$$(4.11) \quad \left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \frac{\gamma+\Gamma}{2} \left(\int_{\Omega} g d\mu - x \right) \right| \\ \leq \int_{\Omega} \left[|g-x| \int_0^1 \left(\Phi'((1-s)x + sg) - \frac{\gamma+\Gamma}{2} \right) ds \right] d\mu \\ \leq \int_{\Omega} \left[|g-x| \int_0^1 \left| \Phi'((1-s)x + sg(t)) - \frac{\gamma+\Gamma}{2} \right| ds \right] d\mu \\ \leq \frac{1}{2} |\Gamma - \gamma| \int_{\Omega} |g-x| d\mu$$

and the proof of (4.5) is completed. \square

Corollary 7. Let $\Phi : I \rightarrow \mathbb{R}$ be a convex function on $[a, b] \subset \overset{\circ}{I}$, the interior of I . If $g : \Omega \rightarrow [a, b]$ is Lebesgue μ -measurable on Ω and such that $\Phi \circ g, g \in L(\Omega, \mu)$,

then we have the inequality

$$(4.12) \quad \left| \int_{\Omega} \Phi \circ g d\mu - \Phi(x) - \frac{\Phi'_+(a) + \Phi'_-(b)}{2} \left(\int_{\Omega} g d\mu - x \right) \right| \leq \frac{1}{2} [\Phi'_-(b) - \Phi'_+(a)] \int_{\Omega} |g - x| d\mu$$

for any $x \in [a, b]$.

In particular, we have

$$(4.13) \quad \left| \int_{\Omega} \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \frac{\Phi'_+(a) + \Phi'_-(b)}{2} \left(\int_{\Omega} g d\mu - \frac{a+b}{2} \right) \right| \leq \frac{1}{2} [\Phi'_-(b) - \Phi'_+(a)] \int_{\Omega} \left| g - \frac{a+b}{2} \right| d\mu \leq \frac{1}{2} (b-a) [\Phi'_-(b) - \Phi'_+(a)]$$

and

$$(4.14) \quad 0 \leq \int_{\Omega} \Phi \circ g d\mu - \Phi\left(\int_{\Omega} g d\mu\right) \leq \frac{1}{2} [\Phi'_-(b) - \Phi'_+(a)] \int_{\Omega} \left| g - \int_{\Omega} g d\mu \right| d\mu.$$

The inequality (4.14) is not as good as (1.1).

The discrete case is as follows:

Remark 8. Let $\Phi : I \rightarrow \mathbb{C}$ be an absolutely continuous functions on $[a, b] \subset \mathring{I}$, the interior of I . For some $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, assume that $\Phi' \in \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$. If $x_j \in [a, b]$ and $p_j \geq 0$ with $\sum_{j=1}^n p_j = 1$ then we have the inequality

$$(4.15) \quad \left| \sum_{j=1}^n p_j \Phi(x_j) - \Phi(x) - \frac{\gamma + \Gamma}{2} \left(\sum_{k=1}^n p_k x_k - x \right) \right| \leq \frac{1}{2} |\Gamma - \gamma| \sum_{j=1}^n p_j |x_j - x|$$

for any $x \in [a, b]$.

In particular, we have

$$(4.16) \quad \left| \sum_{j=1}^n p_j \Phi(x_j) - \Phi\left(\frac{a+b}{2}\right) - \frac{\gamma + \Gamma}{2} \left(\sum_{k=1}^n p_k x_k - \frac{a+b}{2} \right) \right| \leq \frac{1}{2} |\Gamma - \gamma| \sum_{j=1}^n p_j \left| x_j - \frac{a+b}{2} \right| \leq \frac{1}{4} (b-a) |\Gamma - \gamma|$$

and

$$(4.17) \quad \left| \sum_{j=1}^n p_j \Phi(x_j) - \Phi\left(\sum_{k=1}^n p_k x_k\right) \right| \leq \frac{1}{2} |\Gamma - \gamma| \sum_{j=1}^n p_j \left| x_j - \sum_{k=1}^n p_k x_k \right| \leq \frac{1}{2} |\Gamma - \gamma| \left(\sum_{j=1}^n p_j x_j^2 - \left(\sum_{k=1}^n p_k x_k \right)^2 \right)^{1/2} \leq \frac{1}{4} (b-a) |\Gamma - \gamma|.$$

If $\Phi : I \rightarrow \mathbb{R}$ is a convex function on $[a, b]$, then we have

$$(4.18) \quad \left| \sum_{j=1}^n p_j \Phi(x_j) - \Phi(x) - \frac{\Phi'_+(a) + \Phi'_-(b)}{2} \left(\sum_{k=1}^n p_k x_k - x \right) \right| \\ \leq \frac{1}{2} [\Phi'_-(b) - \Phi'_+(a)] \sum_{j=1}^n p_j |x_j - x|$$

for any $x \in [a, b]$.

In particular, we have

$$(4.19) \quad \left| \sum_{j=1}^n p_j \Phi(x_j) - \Phi\left(\frac{a+b}{2}\right) - \frac{\Phi'_+(a) + \Phi'_-(b)}{2} \left(\sum_{k=1}^n p_k x_k - \frac{a+b}{2} \right) \right| \\ \leq \frac{1}{2} [\Phi'_-(b) - \Phi'_+(a)] \sum_{j=1}^n p_j \left| x_j - \frac{a+b}{2} \right| \leq \frac{1}{4} (b-a) [\Phi'_-(b) - \Phi'_+(a)].$$

5. APPLICATIONS FOR f -DIVERGENCE

One of the important issues in many applications of Probability Theory is finding an appropriate measure of *distance* (or *difference* or *discrimination*) between two probability distributions. A number of divergence measures for this purpose have been proposed and extensively studied by Jeffreys [45], Kullback and Leibler [50], Rényi [56], Havrda and Charvat [43], Kapur [48], Sharma and Mittal [59], Burbea and Rao [5], Rao [55], Lin [51], Csiszár [12], Ali and Silvey [1], Vajda [65], Shioya and Da-te [60] and others (see for example [52] and the references therein).

These measures have been applied in a variety of fields such as: anthropology [55], genetics [52], finance, economics, and political science [58], [63], [64], biology [54], the analysis of contingency tables [41], approximation of probability distributions [11], [49], signal processing [46], [47] and pattern recognition [4], [10]. A number of these measures of distance are specific cases of Csiszár f -divergence and so further exploration of this concept will have a flow on effect to other measures of distance and to areas in which they are applied.

Assume that a set Ω and the σ -finite measure μ are given. Consider the set of all probability densities on μ to be $\mathcal{P} := \{p|p : \Omega \rightarrow \mathbb{R}, p(t) \geq 0, \int_{\Omega} p(t) d\mu(t) = 1\}$. The Kullback-Leibler divergence [50] is well known among the information divergences. It is defined as:

$$(5.1) \quad D_{KL}(p, q) := \int_{\Omega} p(t) \ln \left[\frac{p(t)}{q(t)} \right] d\mu(t), \quad p, q \in \mathcal{P},$$

where \ln is to base e .

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence. These are the: *variation distance* D_v , *Hellinger distance* D_H [44], χ^2 -*divergence* D_{χ^2} , α -*divergence* D_{α} , *Bhattacharyya distance* D_B [3], *Harmonic distance* $D_{H\alpha}$, *Jeffrey's distance* D_J [45], *triangular discrimination* D_{Δ} [62], etc... They are defined as follows:

$$(5.2) \quad D_v(p, q) := \int_{\Omega} |p(t) - q(t)| d\mu(t), \quad p, q \in \mathcal{P};$$

$$(5.3) \quad D_H(p, q) := \int_{\Omega} \left| \sqrt{p(t)} - \sqrt{q(t)} \right| d\mu(t), \quad p, q \in \mathcal{P};$$

$$(5.4) \quad D_{\chi^2}(p, q) := \int_{\Omega} p(t) \left[\left(\frac{q(t)}{p(t)} \right)^2 - 1 \right] d\mu(t), \quad p, q \in \mathcal{P};$$

$$(5.5) \quad D_{\alpha}(p, q) := \frac{4}{1-\alpha^2} \left[1 - \int_{\Omega} [p(t)]^{\frac{1-\alpha}{2}} [q(t)]^{\frac{1+\alpha}{2}} d\mu(t) \right], \quad p, q \in \mathcal{P};$$

$$(5.6) \quad D_B(p, q) := \int_{\Omega} \sqrt{p(t)q(t)} d\mu(t), \quad p, q \in \mathcal{P};$$

$$(5.7) \quad D_{Ha}(p, q) := \int_{\Omega} \frac{2p(t)q(t)}{p(t)+q(t)} d\mu(t), \quad p, q \in \mathcal{P};$$

$$(5.8) \quad D_J(p, q) := \int_{\Omega} [p(t) - q(t)] \ln \left[\frac{p(t)}{q(t)} \right] d\mu(t), \quad p, q \in \mathcal{P};$$

$$(5.9) \quad D_{\Delta}(p, q) := \int_{\Omega} \frac{[p(t) - q(t)]^2}{p(t) + q(t)} d\mu(t), \quad p, q \in \mathcal{P}.$$

For other divergence measures, see the paper [48] by Kapur or the book on line [61] by Taneja.

Csiszár f -divergence is defined as follows [13]

$$(5.10) \quad I_f(p, q) := \int_{\Omega} p(t) f \left[\frac{q(t)}{p(t)} \right] d\mu(t), \quad p, q \in \mathcal{P},$$

where f is convex on $(0, \infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u = 1$. By appropriately defining this convex function, various divergences are derived. Most of the above distances (5.1)-(5.9), are particular instances of Csiszár f -divergence. There are also many others which are not in this class (see for example [61]). For the basic properties of Csiszár f -divergence see [13], [14] and [65].

The following result holds:

Proposition 2. *Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a convex function with the property that $f(1) = 0$. Assume that $p, q \in \mathcal{P}$ and there exists the constants $0 < r < 1 < R < \infty$ such that*

$$(5.11) \quad r \leq \frac{q(t)}{p(t)} \leq R \text{ for } \mu\text{-a.e. } t \in \Omega.$$

If $x \in [r, R]$, then we have the inequalities

$$(5.12) \quad \begin{aligned} |I_f(p, q) - f(x)| &\leq \|f'\|_{[r, R], \infty} D_{v, x}(p, q) \\ &\leq \|f'\|_{[r, R], \infty} \left[D_{\chi^2}(p, q) + (x-1)^2 \right]^{1/2}, \end{aligned}$$

where

$$D_{v, x}(p, q) := \int_{\Omega} |p(t) - xq(t)| d\mu(t), \quad p, q \in \mathcal{P}.$$

In particular, we have

$$(5.13) \quad \left| I_f(p, q) - f\left(\frac{r+R}{2}\right) \right| \leq \|f'\|_{[r, R], \infty} D_{v, \frac{r+R}{2}}(p, q) \\ \leq \|f'\|_{[r, R], \infty} \left[D_{\chi^2}(p, q) + \left(\frac{r+R}{2} - 1\right)^2 \right]^{1/2}$$

and

$$(5.14) \quad 0 \leq I_f(p, q) \leq \|f'\|_{[r, R], \infty} D_v(p, q) \leq \|f'\|_{[r, R], \infty} [D_{\chi^2}(p, q)]^{1/2}.$$

Proof. Utilising the inequality (3.16) for the convex function f we have:

$$\left| \int_{\Omega} p(t) f\left[\frac{q(t)}{p(t)}\right] d\mu(t) - f(x) \right| \\ \leq \|f'\|_{[r, R], \infty} \int_{\Omega} \left| \frac{q(t)}{p(t)} - x \right| p(t) d\mu(t) \\ \leq \|f'\|_{[r, R], \infty} \left[\int_{\Omega} \left(\frac{q(t)}{p(t)}\right)^2 p(t) d\mu(t) - \left(\int_{\Omega} \frac{q(t)}{p(t)} p(t) d\mu(t)\right)^2 \right. \\ \left. + \left(\int_{\Omega} \frac{q(t)}{p(t)} p(t) d\mu(t) - x\right)^2 \right]^{1/2},$$

which is equivalent to (5.12). □

We also have:

Proposition 3. *With the assumptions of Proposition 2 we have*

$$(5.15) \quad \left| I_f(p, q) - f(x) - \frac{f'_+(r) + f'_-(R)}{2} (1-x) \right| \\ \leq \frac{1}{2} [f'_-(R) - f'_+(r)] D_{v, x}(p, q)$$

for any $x \in [r, R]$.

In particular, we have

$$(5.16) \quad \left| I_f(p, q) - f\left(\frac{r+R}{2}\right) - \frac{f'_+(r) + f'_-(R)}{2} \left(1 - \frac{r+R}{2}\right) \right| \\ \leq \frac{1}{2} [f'_-(R) - f'_+(r)] D_{v, \frac{r+R}{2}}(p, q)$$

and

$$(5.17) \quad 0 \leq I_f(p, q) \leq \frac{1}{2} [f'_-(R) - f'_+(r)] D_v(p, q).$$

Proof. Utilising the inequality (4.12) we have

$$\left| \int_{\Omega} p(t) f\left[\frac{q(t)}{p(t)}\right] d\mu(t) - f(x) - \frac{f'_+(r) + f'_-(R)}{2} \left(\int_{\Omega} \frac{q(t)}{p(t)} p(t) d\mu(t) - x\right) \right| \\ \leq \frac{1}{2} [f'_-(R) - f'_+(r)] \int_{\Omega} \left| \frac{q(t)}{p(t)} - x \right| p(t) d\mu(t)$$

for any $x \in [a, b]$, which is equivalent to (5.15). □

If we consider the convex function $f : (0, \infty) \rightarrow \mathbb{R}$, $f(t) = t \ln t$ then

$$\begin{aligned} I_f(p, q) &:= \int_{\Omega} p(t) \frac{q(t)}{p(t)} \ln \left[\frac{q(t)}{p(t)} \right] d\mu(t) = \int_{\Omega} q(t) \ln \left[\frac{q(t)}{p(t)} \right] d\mu(t) \\ &= D_{KL}(q, p). \end{aligned}$$

We have $f'(t) = \ln t + 1$ and by (5.17) we get

$$(5.18) \quad 0 \leq D_{KL}(q, p) \leq \ln \sqrt{\frac{R}{r}} D_v(p, q),$$

provided that

$$r \leq \frac{q(t)}{p(t)} \leq R \text{ for } \mu\text{-a.e. } t \in \Omega.$$

If we consider the convex function $f : (0, \infty) \rightarrow \mathbb{R}$, $f(t) = -\ln t$ then

$$\begin{aligned} I_f(p, q) &:= - \int_{\Omega} p(t) \ln \left[\frac{q(t)}{p(t)} \right] d\mu(t) = \int_{\Omega} p(t) \ln \left[\frac{p(t)}{q(t)} \right] d\mu(t) \\ &= D_{KL}(p, q). \end{aligned}$$

We have $f'(t) = -\frac{1}{t}$ and by (5.17) we get

$$(5.19) \quad 0 \leq D_{KL}(p, q) \leq \frac{1}{2} \frac{R-r}{rR} D_v(p, q)$$

provided that

$$r \leq \frac{q(t)}{p(t)} \leq R \text{ for } \mu\text{-a.e. } t \in \Omega.$$

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