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NEW JENSEN AND OSTROWSKI TYPE INEQUALITIES FOR GENERAL LEBESGUE INTEGRAL WITH APPLICATIONS

S. S. DRAGOMIR[1,2]

Abstract. Some new inequalities related to Jensen and Ostrowski inequalities for general Lebesgue integral are obtained. Applications for f-divergence measure are provided as well.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ − algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu = 1$. Consider the Lebesgue space

$$L(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)|\, d\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(t)\, d\mu(t)$.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, S.S. Dragomir obtained in 2002 [29] the following result:

Theorem 1. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M) and $f : \Omega \to [m, M]$ so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) \cdot f \in L(\Omega, \mu)$. Then we have the inequality:*

$$0 \le \int_\Omega \Phi \circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \le \int_\Omega f \cdot (\Phi' \circ f)\, d\mu - \int_\Omega \Phi' \circ f d\mu \int_\Omega f d\mu \le \frac{1}{2}[\Phi'(M) - \Phi'(m)] \int_\Omega \left| f - \int_\Omega f d\mu \right| d\mu. \tag{1.1}$$

In the case of discrete measure, we have:

Corollary 1. *Let $\Phi : [m, M] \to \mathbb{R}$ be a differentiable convex function on (m, M). If $x_i \in [m, M]$ and $w_i \ge 0$ $(i = 1, \ldots, n)$ with $W_n := \sum_{i=1}^n w_i = 1$, then one has*

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the counterpart of Jensen's weighted discrete inequality:

$$
\begin{aligned}
(1.2) \qquad 0 &\leq \sum_{i=1}^{n} w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} w_i x_i\right) \\
&\leq \sum_{i=1}^{n} w_i \Phi'(x_i) x_i - \sum_{i=1}^{n} w_i \Phi'(x_i) \sum_{i=1}^{n} w_i x_i \\
&\leq \frac{1}{2}\left[\Phi'(M) - \Phi'(m)\right] \sum_{i=1}^{n} w_i \left| x_i - \sum_{j=1}^{n} w_j x_j \right|.
\end{aligned}
$$

Remark 1. *We notice that the inequality between the first and the second term in (1.2) was proved in 1994 by Dragomir & Ionescu, see* [36].

If $f, g : \Omega \to \mathbb{R}$ are $\mu-$measurable functions and $f,\ g,\ fg \in L(\Omega, \mu)$, then we may consider the *Čebyšev functional*

$$
(1.3) \qquad T(f,g) := \int_{\Omega} fg d\mu - \int_{\Omega} f d\mu \int_{\Omega} g d\mu.
$$

The following result is known in the literature as the *Grüss inequality*

$$
(1.4) \qquad |T(f,g)| \leq \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta),
$$

provided

$$
(1.5) \qquad -\infty < \gamma \leq f(t) \leq \Gamma < \infty, \ \ -\infty < \delta \leq g(t) \leq \Delta < \infty
$$

for $\mu -$ a.e. $t \in \Omega$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

If we assume that $-\infty < \gamma \leq f(t) \leq \Gamma < \infty$ for $\mu -$ a.e. $t \in \Omega$, then by the Grüss inequality for $g = f$ and by the Schwarz's integral inequality, we have

$$
(1.6) \qquad \int_{\Omega} \left| f - \int_{\Omega} f d\mu \right| d\mu \leq \left[\int_{\Omega} f^2 d\mu - \left(\int_{\Omega} f d\mu \right)^2 \right]^{\frac{1}{2}} \leq \frac{1}{2}(\Gamma - \gamma).
$$

On making use of the results (1.1) and (1.6), we can state the following string of reverse inequalities

$$
\begin{aligned}
(1.7) \qquad 0 &\leq \int_{\Omega} \Phi \circ f d\mu - \Phi\left(\int_{\Omega} f d\mu\right) \\
&\leq \int_{\Omega} f \cdot (\Phi' \circ f) d\mu - \int_{\Omega} \Phi' \circ f d\mu \int_{\Omega} f d\mu \\
&\leq \frac{1}{2}\left[\Phi'(M) - \Phi'(m)\right] \int_{\Omega} \left| f - \int_{\Omega} f d\mu \right| d\mu \\
&\leq \frac{1}{2}\left[\Phi'(M) - \Phi'(m)\right] \left[\int_{\Omega} f^2 d\mu - \left(\int_{\Omega} f d\mu \right)^2 \right]^{\frac{1}{2}} \\
&\leq \frac{1}{4}\left[\Phi'(M) - \Phi'(m)\right](M - m),
\end{aligned}
$$

provided that $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ is a differentiable convex function on (m, M) and $f : \Omega \to [m, M]$ so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ f \cdot (\Phi' \circ f) \in L(\Omega, \mu)$, with $\int_{\Omega} d\mu = 1$.

The following reverse of the Jensen's inequality also holds [33]:

Theorem 2. *Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, where $\mathring{I}$ is the interior of I. If $f : \Omega \to \mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty < m \leq f(t) \leq M < \infty \text{ for } \mu\text{-a.e. } t \in \Omega$$

and such that f, $\Phi \circ f \in L(\Omega, \mu)$, then

$$\begin{aligned} 0 &\leq \int_\Omega \Phi \circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \\ &\leq \left(M - \int_\Omega f d\mu\right)\left(\int_\Omega f d\mu - m\right) \frac{\Phi'_-(M) - \Phi'_+(m)}{M - m} \\ &\leq \frac{1}{4}(M - m)\left[\Phi'_-(M) - \Phi'_+(m)\right], \end{aligned} \tag{1.8}$$

where Φ'_- is the left and Φ'_+ is the right derivative of the convex function Φ.

For other reverse of Jensen inequality and applications to divergence measures see [33].

In 1938, A. Ostrowski [54], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b \Phi(t)\, dt$ and the value $\Phi(x)$, $x \in [a, b]$.

For various results related to Ostrowski's inequality see [6]-[9], [15]-[41], [43] and the references therein.

Theorem 3. *Let $\Phi : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) such that $\Phi' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e., $\|\Phi'\|_\infty := \sup_{t \in (a,b)} |\Phi'(t)| < \infty$. Then*

$$\left|\Phi(x) - \frac{1}{b-a}\int_a^b f(t)\, dt\right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right] \|\Phi'\|_\infty (b - a), \tag{1.9}$$

for all $x \in [a, b]$ and the constant $\frac{1}{4}$ is the best possible.

Now, for $\gamma, \Gamma \in \mathbb{C}$ and $[a, b]$ an interval of real numbers, define the sets of complex-valued functions [34]

$$\begin{aligned} &\bar{U}_{[a,b]}(\gamma, \Gamma) \\ &:= \left\{f : [a, b] \to \mathbb{C} | \operatorname{Re}\left[(\Gamma - f(t))\left(\overline{f(t)} - \overline{\gamma}\right)\right] \geq 0 \text{ for almost every } t \in [a, b]\right\} \end{aligned}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma, \Gamma) := \left\{f : [a, b] \to \mathbb{C} | \ \left|f(t) - \frac{\gamma + \Gamma}{2}\right| \leq \frac{1}{2}|\Gamma - \gamma| \text{ for a.e. } t \in [a, b]\right\}.$$

The following representation result may be stated [34].

Proposition 1. *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have that $\bar{U}_{[a,b]}(\gamma, \Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma, \Gamma)$ are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma, \Gamma) = \bar{\Delta}_{[a,b]}(\gamma, \Gamma). \tag{1.10}$$

On making use of the complex numbers field properties we can also state that:

Corollary 2. *For any* $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, *we have that*

$$\bar{U}_{[a,b]}(\gamma,\Gamma) = \{f : [a,b] \to \mathbb{C} \mid (\operatorname{Re}\Gamma - \operatorname{Re} f(t))(\operatorname{Re} f(t) - \operatorname{Re}\gamma) + (\operatorname{Im}\Gamma - \operatorname{Im} f(t))(\operatorname{Im} f(t) - \operatorname{Im}\gamma) \geq 0 \text{ for a.e. } t \in [a,b]\}. \tag{1.11}$$

Now, if we assume that $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\bar{S}_{[a,b]}(\gamma,\Gamma) := \{f : [a,b] \to \mathbb{C} \mid \operatorname{Re}(\Gamma) \geq \operatorname{Re} f(t) \geq \operatorname{Re}(\gamma) \text{ and } \operatorname{Im}(\Gamma) \geq \operatorname{Im} f(t) \geq \operatorname{Im}(\gamma) \text{ for a.e. } t \in [a,b]\}. \tag{1.12}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma,\Gamma)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\gamma,\Gamma) \subseteq \bar{U}_{[a,b]}(\gamma,\Gamma). \tag{1.13}$$

The following result holds [34]:

Theorem 4. *Let* $\Phi : I \to \mathbb{C}$ *be an absolutely continuous functions on* $[a,b] \subset \mathring{I}$, *the interior of* I. *For some* $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, *assume that* $\Phi' \in \bar{U}_{[a,b]}(\gamma,\Gamma)\left(= \bar{\Delta}_{[a,b]}(\gamma,\Gamma)\right)$. *If* $g : \Omega \to [a,b]$ *is Lebesgue* μ*-measurable on* Ω *and such that* $\Phi \circ g$, $g \in L(\Omega,\mu)$, *then we have the inequality*

$$\left| \int_\Omega \Phi \circ g d\mu - \Phi(x) - \frac{\gamma+\Gamma}{2}\left(\int_\Omega g d\mu - x\right)\right| \leq \frac{1}{2}|\Gamma-\gamma| \int_\Omega |g-x|\, d\mu \tag{1.14}$$

for any $x \in [a,b]$.

In particular, we have

$$\begin{aligned} &\left| \int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \frac{\gamma+\Gamma}{2}\left(\int_\Omega g d\mu - \frac{a+b}{2}\right)\right| \\ &\leq \frac{1}{2}|\Gamma-\gamma| \int_\Omega \left|g - \frac{a+b}{2}\right| d\mu \leq \frac{1}{4}(b-a)|\Gamma-\gamma| \end{aligned} \tag{1.15}$$

and

$$\begin{aligned} \left| \int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| &\leq \frac{1}{2}|\Gamma-\gamma| \int_\Omega \left|g - \int_\Omega g d\mu\right| d\mu \\ &\leq \frac{1}{2}|\Gamma-\gamma| \left(\int_\Omega g^2 d\mu - \left(\int_\Omega g d\mu\right)^2\right)^{1/2} \\ &\leq \frac{1}{4}(b-a)|\Gamma-\gamma|. \end{aligned} \tag{1.16}$$

Motivated by the above results, in this paper we provide more upper bounds for the quantity

$$\left| \int_\Omega \Phi \circ g d\mu - \Phi(x) - \lambda\left(\int_\Omega g d\mu - x\right)\right|, \ x \in [a,b],$$

under various assumptions on the absolutely continuous function Φ, which in the particular case of $x = \int_\Omega g d\mu$ provides some results connected with Jensen's inequality while in the case $\lambda = 0$ provides some generalizations of Ostrowski's inequality. Applications for divergence measures are provided as well.

2. Some Identities

The following result holds [34]:

Lemma 1. *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a,b] \subset \mathring{I}$, the interior of I. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g$, $g \in L(\Omega,\mu)$, then we have the equality*

$$\int_\Omega \Phi \circ g d\mu - \Phi(x) - \lambda \left(\int_\Omega g d\mu - x \right) = \int_\Omega \left[(g-x) \int_0^1 \left(\Phi'((1-s)x + sg) - \lambda \right) ds \right] d\mu \tag{2.1}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a,b]$.

In particular, we have

$$\int_\Omega \Phi \circ g d\mu - \Phi(x) = \int_\Omega \left[(g-x) \int_0^1 \Phi'((1-s)x + sg) ds \right] d\mu, \tag{2.2}$$

for any $x \in [a,b]$.

Remark 2. *With the assumptions of Lemma 1 we have*

$$\int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) = \int_\Omega \left[\left(g - \frac{a+b}{2}\right) \int_0^1 \Phi'\left((1-s)\frac{a+b}{2} + sg \right) ds \right] d\mu. \tag{2.3}$$

Corollary 3. *With the assumptions of Lemma 1 we have*

$$\int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu \right) = \int_\Omega \left[\left(g - \int_\Omega g d\mu \right) \int_0^1 \Phi'\left((1-s)\int_\Omega g d\mu + sg \right) ds \right] d\mu. \tag{2.4}$$

Proof. We observe that since $g : \Omega \to [a,b]$ and $\int_\Omega d\mu = 1$ then $\int_\Omega g d\mu \in [a,b]$ and by taking $x = \int_\Omega g d\mu$ in (2.2) we get (2.4). □

Corollary 4. *With the assumptions of Lemma 1 we have*

$$\int_\Omega \Phi \circ g d\mu - \frac{1}{b-a}\int_a^b \Phi(x) dx - \lambda \left(\int_\Omega g d\mu - \frac{a+b}{2} \right) = \int_\Omega \left\{ \frac{1}{b-a} \int_a^b \left[(g-x) \int_0^1 \left(\Phi'((1-s)x + sg) - \lambda \right) ds \right] dx \right\} d\mu. \tag{2.5}$$

Proof. Follows by integrating the identity (2.1) over $x \in [a,b]$, dividing by $b-a > 0$ and using Fubini's theorem. □

Corollary 5. *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a,b] \subset \mathring{I}$, the interior of I. If $g, h : \Omega \to [a,b]$ are Lebesgue μ-measurable on Ω and such that*

$\Phi\circ g,\ \Phi\circ h,\ g,\ h\in L(\Omega,\mu)$, then we have the equality

$$\int_\Omega \Phi\circ g d\mu-\int_\Omega \Phi\circ h d\mu-\lambda\left(\int_\Omega g d\mu-\int_\Omega h d\mu\right) \tag{2.6}$$
$$=\int_\Omega\int_\Omega\left[(g(t)-h(\tau))\int_0^1(\Phi'((1-s)h(\tau)+sg(t))-\lambda)ds\right] \times d\mu(t)d\mu(\tau)$$

for any $\lambda\in\mathbb{C}$ and $x\in[a,b]$.

In particular, we have

$$\int_\Omega \Phi\circ g d\mu-\int_\Omega \Phi\circ h d\mu \tag{2.7}$$
$$=\int_\Omega\int_\Omega\left[(g(t)-h(\tau))\int_0^1\Phi'((1-s)h(\tau)+sg(t))ds\right]d\mu(t)d\mu(\tau),$$

for any $x\in[a,b]$.

Remark 3. *The above inequality (2.6) can be extended for two measures as follows*

$$\int_{\Omega_1} \Phi\circ g d\mu_1-\int_{\Omega_2} \Phi\circ h d\mu_2-\lambda\left(\int_{\Omega_1} g d\mu_1-\int_{\Omega_2} h d\mu_2\right) \tag{2.8}$$
$$=\int_{\Omega_1}\int_{\Omega_2}\left[(g(t)-h(\tau))\int_0^1(\Phi'((1-s)h(\tau)+sg(t))-\lambda)ds\right] \times d\mu_1(t)d\mu_2(\tau),$$

for any $\lambda\in\mathbb{C}$ and $x\in[a,b]$ and provided that $\Phi\circ g,\ g\in L(\Omega_1,\mu_1)$ while $\Phi\circ h, h\in L(\Omega_2,\mu_2)$.

Remark 4. *If $w\geq 0$ μ-almost everywhere (μ-a.e.) on Ω with $\int_\Omega wd\mu>0$, then by replacing $d\mu$ with $\frac{wd\mu}{\int_\Omega wd\mu}$ in (2.1) we have the weighted equality*

$$\frac{1}{\int_\Omega wd\mu}\int_\Omega w(\Phi\circ g)d\mu-\Phi(x)-\lambda\left(\frac{1}{\int_\Omega wd\mu}\int_\Omega wgd\mu-x\right) \tag{2.9}$$
$$=\frac{1}{\int_\Omega wd\mu}\int_\Omega w\cdot\left[(g-x)\int_0^1(\Phi'((1-s)x+sg)-\lambda)ds\right]d\mu$$

for any $\lambda\in\mathbb{C}$ and $x\in[a,b]$, provided $\Phi\circ g,\ g\in L_w(\Omega,\mu)$ where

$$L_w(\Omega,\mu):=\left\{g|\ \int_\Omega w|g|d\mu<\infty\right\}.$$

The other equalities have similar weighted versions. However the details are omitted.

3. Inequalities for Derivatives of Bounded Variation

The following result holds:

Theorem 5. *Let $\Phi: I\to\mathbb{C}$ be an absolutely continuous functions on $[a,b]\subset \mathring{I}$, the interior of I and with the property that the derivative Φ' is of bounded variation* *on* $[a,b]$. *If* $g:\Omega\to[a,b]$ *is Lebesgue* μ*-measurable on* Ω *and such that* $\Phi\circ g$, $g\in L(\Omega,\mu)$, *then we have*

$$\left|\int_\Omega \Phi\circ g d\mu-\Phi(x)-\frac{\Phi'(a)+\Phi'(b)}{2}\left(\int_\Omega g d\mu-x\right)\right| \le \frac{1}{2}\bigvee_a^b(\Phi')\int_\Omega |g-x|\,d\mu \tag{3.1}$$

for any $x\in[a,b]$.

In particular, we have

$$\left|\int_\Omega \Phi\circ g d\mu-\Phi\left(\frac{a+b}{2}\right)-\frac{\Phi'(a)+\Phi'(b)}{2}\left(\int_\Omega g d\mu-\frac{a+b}{2}\right)\right| \le \frac{1}{2}\bigvee_a^b(\Phi')\int_\Omega \left|g-\frac{a+b}{2}\right|d\mu \le \frac{1}{2}(b-a)\bigvee_a^b(\Phi') \tag{3.2}$$

and

$$\left|\int_\Omega \Phi\circ g d\mu-\Phi\left(\int_\Omega g d\mu\right)\right| \le \frac{1}{2}\bigvee_a^b(\Phi')\int_\Omega \left|g-\int_\Omega g d\mu\right|d\mu \le \frac{1}{2}\bigvee_a^b(\Phi')\left(\int_\Omega g^2 d\mu-\left(\int_\Omega g d\mu\right)^2\right)^{1/2} \le \frac{1}{4}(b-a)\bigvee_a^b(\Phi'). \tag{3.3}$$

Proof. From the identity (2.1) we have

$$\int_\Omega \Phi\circ g d\mu-\Phi(x)-\frac{\Phi'(a)+\Phi'(b)}{2}\left(\int_\Omega g d\mu-x\right) = \int_\Omega\left[(g-x)\int_0^1\left(\Phi'((1-s)x+sg)-\frac{\Phi'(a)+\Phi'(b)}{2}\right)ds\right]d\mu \tag{3.4}$$

for any $x\in[a,b]$.

Taking the modulus in (3.4) we get

$$\left|\int_\Omega \Phi\circ g d\mu-\Phi(x)-\frac{\Phi'(a)+\Phi'(b)}{2}\left(\int_\Omega g d\mu-x\right)\right| \le \int_\Omega\left|(g-x)\int_0^1\left(\Phi'((1-s)x+sg)-\frac{\Phi'(a)+\Phi'(b)}{2}\right)\right|dsd\mu \le \int_\Omega |g-x|\int_0^1\left|\Phi'((1-s)x+sg)-\frac{\Phi'(a)+\Phi'(b)}{2}\right|dsd\mu \tag{3.5}$$

for any $x\in[a,b]$.

Since Φ' is of bounded variation on $[a,b]$, then for any $s\in[0,1]$, $x\in[a,b]$ and $t\in\Omega$ we have

$$\begin{aligned}
&\left|\Phi'((1-s)x+sg(t))-\frac{\Phi'(a)+\Phi'(b)}{2}\right| \\
&=\frac{1}{2}\left|\Phi'((1-s)x+sg(t))-\Phi'(a)+\Phi'((1-s)x+sg(t))-\Phi'(b)\right| \\
&\leq\frac{1}{2}\left[\left|\Phi'((1-s)x+sg(t))-\Phi'(a)\right|+\left|\Phi'(b)-\Phi'((1-s)x+sg(t))\right|\right] \\
&\leq\frac{1}{2}\bigvee_a^b(\Phi').
\end{aligned}$$

Then we have

$$\begin{aligned}
&\int_\Omega|g-x|\int_0^1\left|\Phi'((1-s)x+sg)-\frac{\Phi'(a)+\Phi'(b)}{2}\right|dsd\mu \\
&\leq\frac{1}{2}\bigvee_a^b(\Phi')\int_\Omega|g-x|\,d\mu
\end{aligned}\tag{3.6}$$

for any $x\in[a,b]$.

Making use of (3.5) and (3.6) we deduce the desired result (3.1). □

Remark 5. *Let $\Phi:I\to\mathbb{C}$ be an absolutely continuous functions on $[a,b]\subset\mathring{I}$, the interior of I and with the property that the derivative Φ' is of bounded variation on $[a,b]$. If $x_i\in[m,M]$ and $w_i\geq0$ $(i=1,\dots,n)$ with $W_n:=\sum_{i=1}^n w_i=1$, then one has the weighted discrete inequality:*

$$\begin{aligned}
&\left|\sum_{i=1}^n w_i\Phi(x_i)-\Phi(x)-\frac{\Phi'(a)+\Phi'(b)}{2}\left(\sum_{i=1}^n w_ix_i-x\right)\right| \\
&\leq\frac{1}{2}\bigvee_a^b(\Phi')\sum_{i=1}^n w_i|x_i-x|
\end{aligned}\tag{3.7}$$

for any $x\in[a,b]$.

In particular, we have

$$\begin{aligned}
&\left|\sum_{i=1}^n w_i\Phi(x_i)-\Phi\left(\frac{a+b}{2}\right)-\frac{\Phi'(a)+\Phi'(b)}{2}\left(\sum_{i=1}^n w_ix_i-\frac{a+b}{2}\right)\right| \\
&\leq\frac{1}{2}\bigvee_a^b(\Phi')\sum_{i=1}^n w_i\left|x_i-\frac{a+b}{2}\right|\leq\frac{1}{4}(b-a)\bigvee_a^b(\Phi')
\end{aligned}\tag{3.8}$$

and

$$\left|\sum_{i=1}^{n} w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} w_i x_i\right)\right| \leq \frac{1}{2} \bigvee_a^b (\Phi') \sum_{i=1}^{n} w_i \left|x_i - \sum_{i=1}^{n} w_i x_i\right| \tag{3.9}$$

$$\leq \frac{1}{2} \bigvee_a^b (\Phi') \left(\sum_{j=1}^{n} w_i x_j^2 - \left(\sum_{k=1}^{n} w_k x_k\right)^2\right)^{1/2}$$

$$\leq \frac{1}{4}(b-a) \bigvee_a^b (\Phi').$$

4. Inequalities for Lipschitzian Derivatives

The following result holds:

Theorem 6. *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a,b] \subset \mathring{I}$, the interior of I and with the property that the derivative Φ' is Lipschitzian with the constant $K > 0$ on $[a,b]$. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g,\ g \in L(\Omega,\mu)$, then we have*

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'(x)\left(\int_\Omega g d\mu - x\right)\right| \tag{4.1}$$

$$\leq \frac{1}{2} K \left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - x\right)^2\right]$$

for any $x \in [a,b]$, where $\sigma_\mu(g)$ is the dispersion or the standard variation, namely

$$\sigma_\mu(g) := \left(\int_\Omega \left(g - \int_\Omega g d\mu\right)^2 d\mu\right)^{1/2} = \left(\int_\Omega g^2 d\mu - \left(\int_\Omega g d\mu\right)^2\right)^{1/2}.$$

In particular, we have

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \Phi'\left(\frac{a+b}{2}\right)\left(\int_\Omega g d\mu - \frac{a+b}{2}\right)\right| \tag{4.2}$$

$$\leq \frac{1}{2} K \left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - \frac{a+b}{2}\right)^2\right]$$

and

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| \leq \frac{1}{2} K \sigma_\mu^2(g) \leq \frac{1}{8} K (b-a)^2. \tag{4.3}$$

Proof. From the identity (2.1) we have for $\lambda = \Phi'(x)$ that

$$\int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'(x)\left(\int_\Omega g d\mu - x\right) \tag{4.4}$$

$$= \int_\Omega \left[(g - x) \int_0^1 (\Phi'((1-s)x + sg) - \Phi'(x)) ds\right] d\mu$$

for any $x \in [a,b]$.

Taking the modulus in (4.4) we get

$$\begin{aligned}(4.5)\qquad & \left|\int_\Omega \Phi\circ g d\mu-\Phi(x)-\Phi'(x)\left(\int_\Omega gd\mu-x\right)\right| \\ & \le \int_\Omega |g-x|\left|\int_0^1 \left(\Phi'((1-s)x+sg)-\Phi'(x)\right)ds\right| d\mu \\ & \le \int_\Omega \left[|g-x|\int_0^1 \left|\left(\Phi'((1-s)x+sg)-\Phi'(x)\right)\right| ds\right] d\mu \\ & \le K\int_\Omega \left[|g-x|\int_0^1 s|g-x|\,ds\right] d\mu=\frac{1}{2}K\int_\Omega (g-x)^2 d\mu\end{aligned}$$

for any $x\in[a,b]$.

However,

$$\begin{aligned}&\int_\Omega (g-x)^2 d\mu \\ &=\int_\Omega\left(g-\int_\Omega gd\mu+\int_\Omega gd\mu-x\right)^2 d\mu \\ &=\int_\Omega\left(g-\int_\Omega gd\mu\right)^2 d\mu+2\int_\Omega\left(g-\int_\Omega gd\mu\right)\left(\int_\Omega gd\mu-x\right)d\mu \\ &+\int_\Omega\left(\int_\Omega gd\mu-x\right)^2 d\mu \\ &=\int_\Omega\left(g-\int_\Omega gd\mu\right)^2 d\mu+\left(\int_\Omega gd\mu-x\right)^2\end{aligned}$$

for any $x\in[a,b]$, and by (4.5) we get the desired result (4.1). □

Corollary 6. *Let $\Phi: I\to\mathbb{C}$ be a twice differentiable functions on $[a,b]\subset \mathring{I}$ with $\|\Phi''\|_{[a,b],\infty}:=ess\sup_{t\in[a,b]}|\Phi''(t)|<\infty$. Then the inequalities (4.1)-(4.3) hold for $K=\|\Phi''\|_{[a,b],\infty}$.*

Remark 6. *Let $\Phi: I\to\mathbb{C}$ be an absolutely continuous functions on $[a,b]\subset \mathring{I}$ and with the property that the derivative Φ' is Lipschitzian with the constant $K>0$ on $[a,b]$. If $x_i\in[m,M]$ and $w_i\ge 0$ $(i=1,\dots,n)$ with $W_n:=\sum_{i=1}^n w_i=1$, then one has the weighted discrete inequality:*

$$\begin{aligned}(4.6)\qquad & \left|\sum_{i=1}^n w_i\Phi(x_i)-\Phi(x)-\Phi'(x)\left(\sum_{i=1}^n w_ix_i-x\right)\right| \\ & \le \frac{1}{2}K\left[\sigma_w^2(\mathbf{x})+\left(\sum_{i=1}^n w_ix_i-x\right)^2\right]\end{aligned}$$

for any $x\in[a,b]$, where

$$\sigma_w(\mathbf{x}):=\left(\sum_{i=1}^n w_i\left(x_i-\sum_{k=1}^n w_kx_k\right)^2\right)^{1/2}=\left(\sum_{i=1}^n w_ix_i^2-\left(\sum_{k=1}^n w_kx_k\right)^2\right)^{1/2}.$$

The following lemma may be stated:

Lemma 2. *Let $u : [a,b] \to \mathbb{R}$ and $l, L \in \mathbb{R}$ with $L > l$. The following statements are equivalent:*

(i) *The function $u - \frac{l+L}{2} \cdot e$, where $e(t) = t$, $t \in [a,b]$ is $\frac{1}{2}(L-l)$ −Lipschitzian;*
(ii) *We have the inequalities*

$$l \leq \frac{u(t) - u(s)}{t-s} \leq L \quad \text{for each } t, s \in [a,b] \quad \text{with } t \neq s; \tag{4.7}$$

(iii) *We have the inequalities*

$$l(t-s) \leq u(t) - u(s) \leq L(t-s) \quad \text{for each } t, s \in [a,b] \quad \text{with } t > s. \tag{4.8}$$

Following [52], we can introduce the definition of (l, L)-Lipschitzian functions:

Definition 1. *The function $u : [a,b] \to \mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) from Lemma 2 is said to be (l, L)-Lipschitzian on $[a,b]$.*

If $L > 0$ and $l = -L$, then $(-L, L)$ −Lipschitzian means L-Lipschitzian in the classical sense.

Utilising *Lagrange's mean value theorem*, we can state the following result that provides examples of (l, L)-Lipschitzian functions.

Proposition 2. *Let $u : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $-\infty < l = \inf_{t\in[a,b]} u'(t)$ and $\sup_{t\in[a,b]} u'(t) = L < \infty$, then u is (l, L)-Lipschitzian on $[a,b]$.*

The following result holds.

Corollary 7. *Let $\Phi : I \to \mathbb{R}$ be an absolutely continuous functions on $[a,b] \subset \mathring{I}$, with the property that the derivative Φ' is (l, L)-Lipschitzian on $[a,b]$, where $l, L \in \mathbb{R}$ with $L > l$. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g$, $g \in L(\Omega, \mu)$, then we have*

$$\begin{aligned} &\left| \int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'(x) \left(\int_\Omega g d\mu - x \right) \right. \\ &\left. -\frac{1}{4}(L+l) \left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - x \right)^2 \right] \right| \\ &\leq \frac{1}{4}(L-l) \left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - x \right)^2 \right] \end{aligned} \tag{4.9}$$

for any $x \in [a,b]$.

In particular, we have

$$\begin{aligned} &\left| \int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \Phi'\left(\frac{a+b}{2}\right) \left(\int_\Omega g d\mu - \frac{a+b}{2} \right) \right. \\ &\left. -\frac{1}{4}(L+l) \left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - \frac{a+b}{2} \right)^2 \right] \right| \\ &\leq \frac{1}{4}(L-l) \left[\sigma_\mu^2(g) + \left(\int_\Omega g d\mu - \frac{a+b}{2} \right)^2 \right] \end{aligned} \tag{4.10}$$

and

$$\left|\int_{\Omega}\Phi\circ gd\mu-\Phi\left(\int_{\Omega}gd\mu\right)-\frac{1}{4}\left(L+l\right)\sigma_{\mu}^{2}\left(g\right)\right|\leq\frac{1}{4}\left(L-l\right)\sigma_{\mu}^{2}\left(g\right) \tag{4.11}$$
$$\leq\frac{1}{16}\left(L-l\right)\left(b-a\right)^{2}.$$

Proof. Consider the auxiliary function $\Psi:[a,b]\to\mathbb{R}$ given by

$$\Psi\left(x\right)=\Phi\left(x\right)-\frac{1}{4}\left(L+l\right)x^{2}.$$

We observe that Ψ is differentiable and

$$\Psi'\left(x\right)=\Phi'\left(x\right)-\frac{1}{2}\left(L+l\right)x.$$

Since Φ' is (l,L)-Lipschitzian on $[a,b]$ it follows that Ψ' is Lipschitzian with the constant $\frac{1}{2}\left(L-l\right)$, so we can apply Theorem 6 for Ψ, i.e. we have the inequality

$$\left|\int_{\Omega}\Psi\circ gd\mu-\Psi\left(x\right)-\Psi'\left(x\right)\left(\int_{\Omega}gd\mu-x\right)\right| \tag{4.12}$$
$$\leq\frac{1}{4}\left(L-l\right)\left[\sigma_{\mu}^{2}\left(g\right)+\left(\int_{\Omega}gd\mu-x\right)^{2}\right].$$

However

$$\begin{aligned}&\int_{\Omega}\Psi\circ gd\mu-\Psi\left(x\right)-\Psi'\left(x\right)\left(\int_{\Omega}gd\mu-x\right)\\&=\int_{\Omega}\Phi\circ gd\mu-\Phi\left(x\right)-\Phi'\left(x\right)\left(\int_{\Omega}gd\mu-x\right)\\&-\frac{1}{4}\left(L+l\right)\left[\int_{\Omega}g^{2}d\mu-x^{2}-2x\left(\int_{\Omega}gd\mu-x\right)\right]\\&=\int_{\Omega}\Phi\circ gd\mu-\Phi\left(x\right)-\Phi'\left(x\right)\left(\int_{\Omega}gd\mu-x\right)\\&-\frac{1}{4}\left(L+l\right)\left[\sigma_{\mu}^{2}\left(g\right)+\left(\int_{\Omega}gd\mu-x\right)^{2}\right]\end{aligned}$$

and by (4.12) we get the desired result (4.9). □

Remark 7. *We observe that if the function Φ is twice differentiable on $\mathring{I}$ and for $[a,b]\subset\mathring{I}$ we have*

$$-\infty<l\leq\Phi''\left(x\right)\leq L<\infty\text{ for any }x\in\left[a,b\right],$$

then Φ' is (l,L)-Lipschitzian on $[a,b]$ and the inequalities (4.9)-(4.11) hold true.

The following result also holds:

Theorem 7. *Let $\Phi:I\to\mathbb{C}$ be an absolutely continuous functions on $[a,b]\subset\mathring{I}$, the interior of I and with the property that the derivative Φ' is Lipschitzian with the constant $K>0$ on $[a,b]$. If $g:\Omega\to[a,b]$ is Lebesgue μ-measurable on Ω and* *such that* $\Phi \circ g,\ g \in L(\Omega, \mu)$, *then we have*

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'\left(\int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - x\right)\right| \tag{4.13}$$
$$\leq \frac{1}{2}K\left[\left|x - \int_\Omega g d\mu\right| \int_\Omega |g - x| d\mu + \int_\Omega |g - x| \left|g - \int_\Omega g d\mu\right| d\mu\right]$$
$$\leq \frac{1}{2}K\left[\left|x - \int_\Omega g d\mu\right| + \left\|g - \int_\Omega g d\mu\right\|_{\Omega,\infty}\right] \int_\Omega |g - x| d\mu$$

for any $x \in [a, b]$, *where*

$$\left\|g - \int_\Omega g d\mu\right\|_{\Omega,\infty} := ess\sup_{t\in\Omega}\left|g(t) - \int_\Omega g d\mu\right| < \infty.$$

In particular, we have

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \Phi'\left(\int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - \frac{a+b}{2}\right)\right| \tag{4.14}$$
$$\leq \frac{1}{2}K\left[\left|\frac{a+b}{2} - \int_\Omega g d\mu\right| \int_\Omega \left|g - \frac{a+b}{2}\right| d\mu\right.$$
$$\left. + \int_\Omega \left|g - \frac{a+b}{2}\right| \left|g - \int_\Omega g d\mu\right| d\mu\right]$$
$$\leq \frac{1}{2}K\left[\left|\frac{a+b}{2} - \int_\Omega g d\mu\right| + \left\|g - \int_\Omega g d\mu\right\|_{\Omega,\infty}\right] \int_\Omega \left|g - \frac{a+b}{2}\right| d\mu.$$

Proof. From the identity (2.1) we have for $\lambda = \Phi'\left(\int_\Omega g d\mu\right)$ that

$$\int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'\left(\int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - x\right) \tag{4.15}$$
$$= \int_\Omega \left[(g - x) \int_0^1 \left(\Phi'((1-s)x + sg) - \Phi'\left(\int_\Omega g d\mu\right)\right) ds\right] d\mu$$

for any $x \in [a, b]$.

Taking the modulus in (4.15) we get

$$\left|\int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'\left(\int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - x\right)\right| \tag{4.16}$$
$$\leq \int_\Omega |g - x| \left|\int_0^1 \left(\Phi'((1-s)x + sg) - \Phi'\left(\int_\Omega g d\mu\right)\right) ds\right| d\mu$$
$$\leq \int_\Omega \left[|g - x| \int_0^1 \left|\left(\Phi'((1-s)x + sg) - \Phi'\left(\int_\Omega g d\mu\right)\right)\right| ds\right] d\mu$$
$$\leq K \int_\Omega \left[|g - x| \int_0^1 \left|(1-s)x + sg - \int_\Omega g d\mu\right| ds\right] d\mu$$
$$= K \int_\Omega \left[|g - x| \int_0^1 \left|(1-s)x + sg - (1-s)\int_\Omega g d\mu - s\int_\Omega g d\mu\right| ds\right] d\mu$$
$$:= B.$$

Using the triangle inequality we have for any $t \in \Omega$

$$\begin{aligned}&\int_0^1 \left|(1-s)x + sg(t) - (1-s)\int_\Omega g d\mu - s\int_\Omega g d\mu\right| ds \\ &\leq \int_0^1 (1-s)\left|x - \int_\Omega g d\mu\right| ds + \int_0^1 s\left|g(t) - \int_\Omega g d\mu\right| ds \\ &= \frac{1}{2}\left[\left|x - \int_\Omega g d\mu\right| + \left|g(t) - \int_\Omega g d\mu\right|\right]\end{aligned}$$

and then

$$\begin{aligned}B &\leq \frac{1}{2}K\int_\Omega |g-x|\left[\left|x - \int_\Omega g d\mu\right| + \left|g(t) - \int_\Omega g d\mu\right|\right] d\mu \\ &= \frac{1}{2}K\left[\left|x - \int_\Omega g d\mu\right|\int_\Omega |g-x|\, d\mu + \int_\Omega |g-x|\left|g - \int_\Omega g d\mu\right| d\mu\right].\end{aligned} \tag{4.17}$$

Making use of (4.16) and (4.17) we deduce the desired result (4.13). □

Corollary 8. *Let $\Phi : I \to \mathbb{R}$ be an absolutely continuous functions on $[a,b] \subset \mathring{I}$, with the property that the derivative Φ' is (l,L)-Lipschitzian on $[a,b]$, where $l, L \in \mathbb{R}$ with $L > l$. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g$, $g \in L(\Omega,\mu)$, then we have*

$$\begin{aligned}&\left|\int_\Omega \Phi \circ g d\mu - \Phi(x) - \Phi'\left(\int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - x\right)\right. \\ &\left.-\frac{1}{4}(L+l)\left[\sigma_\mu^2(g) - \left(x - \int_\Omega g d\mu\right)^2\right]\right| \\ &\leq \frac{1}{4}(L-l)\left[\left|x - \int_\Omega g d\mu\right|\int_\Omega |g-x|\, d\mu + \int_\Omega |g-x|\left|g - \int_\Omega g d\mu\right| d\mu\right] \\ &\leq \frac{1}{4}(L-l)\left[\left|x - \int_\Omega g d\mu\right| + \left\|g - \int_\Omega g d\mu\right\|_{\Omega,\infty}\right]\int_\Omega |g-x|\, d\mu\end{aligned} \tag{4.18}$$

for any $x \in [a,b]$.

In particular, we have

$$\begin{aligned}&\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \Phi'\left(\int_\Omega g d\mu\right)\left(\int_\Omega g d\mu - \frac{a+b}{2}\right)\right. \\ &\left.-\frac{1}{4}(L+l)\left[\sigma_\mu^2(g) - \left(\frac{a+b}{2} - \int_\Omega g d\mu\right)^2\right]\right| \\ &\leq \frac{1}{4}(L-l)\left[\left|\frac{a+b}{2} - \int_\Omega g d\mu\right|\int_\Omega \left|g - \frac{a+b}{2}\right| d\mu\right. \\ &\left.+\int_\Omega \left|g - \frac{a+b}{2}\right|\left|g - \int_\Omega g d\mu\right| d\mu\right] \\ &\leq \frac{1}{4}(L-l)\left[\left|\frac{a+b}{2} - \int_\Omega g d\mu\right| + \left\|g - \int_\Omega g d\mu\right\|_{\Omega,\infty}\right]\int_\Omega \left|g - \frac{a+b}{2}\right| d\mu.\end{aligned} \tag{4.19}$$

5. Applications for f-Divergence

One of the important issues in many applications of Probability Theory is finding an appropriate measure of *distance* (or *difference* or *discrimination*) between two probability distributions. A number of divergence measures for this purpose have been proposed and extensively studied by Jeffreys [46], Kullback and Leibler [51], Rényi [57], Havrda and Charvat [44], Kapur [49], Sharma and Mittal [60], Burbea and Rao [5], Rao [56], Lin [52], Csiszár [12], Ali and Silvey [1], Vajda [66], Shioya and Da-te [61] and others (see for example [53] and the references therein).

These measures have been applied in a variety of fields such as: anthropology [56], genetics [53], finance, economics, and political science [59], [64], [65], biology [55], the analysis of contingency tables [42], approximation of probability distributions [11], [50], signal processing [47], [48] and pattern recognition [4], [10]. A number of these measures of distance are specific cases of Csiszár f-divergence and so further exploration of this concept will have a flow on effect to other measures of distance and to areas in which they are applied.

Assume that a set Ω and the σ-finite measure μ are given. Consider the set of all probability densities on μ to be $\mathcal{P} := \left\{p | p : \Omega \to \mathbb{R},\, p(t) \geq 0,\, \int_\Omega p(t)\, d\mu(t) = 1\right\}$. The Kullback-Leibler divergence [51] is well known among the information divergences. It is defined as:

$$D_{KL}(p,q) := \int_\Omega p(t) \ln \left[\frac{p(t)}{q(t)}\right] d\mu(t), \quad p,q \in \mathcal{P}, \tag{5.1}$$

where ln is to base e.

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence. These are the: *variation distance* D_v, *Hellinger distance* D_H [45], χ^2-*divergence* D_{χ^2}, α-*divergence* D_α, *Bhattacharyya distance* D_B [3], *Harmonic distance* D_{Ha}, *Jeffrey's distance* D_J [46], *triangular discrimination* D_Δ [63], etc... They are defined as follows:

$$D_v(p,q) := \int_\Omega |p(t) - q(t)|\, d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.2}$$

$$D_H(p,q) := \int_\Omega \left|\sqrt{p(t)} - \sqrt{q(t)}\right| d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.3}$$

$$D_{\chi^2}(p,q) := \int_\Omega p(t) \left[\left(\frac{q(t)}{p(t)}\right)^2 - 1\right] d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.4}$$

$$D_\alpha(p,q) := \frac{4}{1-\alpha^2}\left[1 - \int_\Omega [p(t)]^{\frac{1-\alpha}{2}} [q(t)]^{\frac{1+\alpha}{2}} d\mu(t)\right], \quad p,q \in \mathcal{P}; \tag{5.5}$$

$$D_B(p,q) := \int_\Omega \sqrt{p(t)\, q(t)}\, d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.6}$$

$$D_{Ha}(p,q) := \int_\Omega \frac{2p(t)\, q(t)}{p(t) + q(t)} d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.7}$$

$$D_J(p,q) := \int_\Omega [p(t) - q(t)] \ln \left[\frac{p(t)}{q(t)}\right] d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.8}$$

$$D_{\Delta}(p,q) := \int_{\Omega} \frac{[p(t)-q(t)]^2}{p(t)+q(t)} d\mu(t), \quad p,q \in \mathcal{P}. \tag{5.9}$$

For other divergence measures, see the paper [49] by Kapur or the book on line [62] by Taneja.

Csiszár f-divergence is defined as follows [13]

$$I_f(p,q) := \int_{\Omega} p(t) f\left[\frac{q(t)}{p(t)}\right] d\mu(t), \quad p,q \in \mathcal{P}, \tag{5.10}$$

where f is convex on $(0,\infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u=1$. By appropriately defining this convex function, various divergences are derived. Most of the above distances (5.1)-(5.9), are particular instances of Csiszár f-divergence. There are also many others which are not in this class (see for example [62]). For the basic properties of Csiszár f-divergence see [13], [14] and [66].

The following result holds:

Proposition 3. *Let $f:(0,\infty)\to\mathbb{R}$ be a twice differentiable convex function with the property that $f(1)=0$ and there exists the constants γ,Γ so that*

$$-\infty < \gamma \le f(t) \le \Gamma < \infty.$$

Assume that $p,q\in\mathcal{P}$ and there exists the constants $0<r<1<R<\infty$ such that

$$r \le \frac{q(t)}{p(t)} \le R \text{ for } \mu\text{-a.e. } t\in\Omega. \tag{5.11}$$

If $x\in[r,R]$, then we have the inequality

$$\begin{aligned} &\left| I_f(p,q) - f(x) - f'(x)(1-x) - \frac{1}{4}(L+l)\left[D_{\chi^2}(p,q) + (1-x)^2\right]\right| \\ &\le \frac{1}{4}(L-l)\left[D_{\chi^2}(p,q)+(1-x)^2\right]. \end{aligned} \tag{5.12}$$

In particular, we have

$$\begin{aligned} &\left| I_f(p,q) - f\left(\frac{r+R}{2}\right) - f'\left(\frac{r+R}{2}\right)\left(1-\frac{r+R}{2}\right)\right. \\ &\left. -\frac{1}{4}(L+l)\left[D_{\chi^2}(p,q) + \left(1-\frac{r+R}{2}\right)^2\right]\right| \\ &\le \frac{1}{4}(L-l)\left[D_{\chi^2}(p,q) + \left(1-\frac{r+R}{2}\right)^2\right] \end{aligned} \tag{5.13}$$

and

$$\left| I_f(p,q) - \frac{1}{4}(L+l) D_{\chi^2}(p,q)\right| \le \frac{1}{4}(L-l) D_{\chi^2}(p,q). \tag{5.14}$$

Proof. From (4.9) we have

$$\left| \int_{\Omega} p(t) f\left(\frac{q(t)}{p(t)}\right) d\mu(t) - f(x) - f'(x)(1-x) \right.$$
$$\left. -\frac{1}{4}(L+l)\left[\int_{\Omega} p(t)\left(\frac{q(t)}{p(t)}\right)^2 d\mu(t) - 1 + (1-x)^2\right] \right|$$
$$\leq \frac{1}{4}(L-l)\left[\int_{\Omega} p(t)\left(\frac{q(t)}{p(t)}\right)^2 d\mu(t) - 1 + (1-x)^2\right]$$

for any $x \in [r, R]$, which is equivalent to (5.12). □

Utilising Corollary 8 we can state the following result as well:

Proposition 4. *With the assumptions in Proposition 3, we have*

$$\left| I_f(p,q) - f(x) - f'(1)(1-x) - \frac{1}{4}(L+l)\left[D_{\chi^2}(p,q) - (1-x)^2\right] \right| \tag{5.15}$$
$$\leq \frac{1}{4}(L-l)\left[|x-1| \int_{\Omega} |q - xp|\, d\mu + \int_{\Omega} |q - xp| \left|\frac{q}{p} - 1\right| d\mu\right]$$
$$\leq \frac{1}{4}(L-l)\left[|x-1| + \left\|\frac{q}{p} - 1\right\|_{\Omega,\infty}\right] \int_{\Omega} |q - xp|\, d\mu$$

for any $x \in [r, R]$.

If we consider the convex function $f : (0,\infty) \to \mathbb{R}$, $f(t) = t \ln t$ then

$$I_f(p,q) := \int_{\Omega} p(t) \frac{q(t)}{p(t)} \ln\left[\frac{q(t)}{p(t)}\right] d\mu(t) = \int_{\Omega} q(t) \ln\left[\frac{q(t)}{p(t)}\right] d\mu(t)$$
$$= D_{KL}(q,p).$$

We have $f'(t) = \ln t + 1$ and $f''(t) = \frac{1}{t}$ and then we can choose $l = \frac{1}{R}$ and $L = \frac{1}{r}$. Applying the inequality (5.14) we get

$$\left| D_{KL}(q,p) - \left(\frac{R+r}{4rR}\right) D_{\chi^2}(p,q) \right| \leq \frac{R-r}{4rR} D_{\chi^2}(p,q). \tag{5.16}$$

If we consider the convex function $f : (0,\infty) \to \mathbb{R}$, $f(t) = -\ln t$ then

$$I_f(p,q) := -\int_{\Omega} p(t) \ln\left[\frac{q(t)}{p(t)}\right] d\mu(t) = \int_{\Omega} p(t) \ln\left[\frac{p(t)}{q(t)}\right] d\mu(t)$$
$$= D_{KL}(p,q).$$

We have $f'(t) = -\frac{1}{t}$ and $f''(t) = \frac{1}{t^2}$ and then we can choose $l = \frac{1}{R^2}$ and $L = \frac{1}{r^2}$. Applying the inequality (5.14) we get

$$\left| D_{KL}(p,q) - \frac{R^2 + r^2}{4R^2 r^2} D_{\chi^2}(p,q) \right| \leq \frac{R^2 - r^2}{4R^2 r^2} D_{\chi^2}(p,q). \tag{5.17}$$

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa