Received 17/03/14

Fractional Monotone Approximation Theory

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Let $f \in C^p([-1,1])$, $p \geq 0$ and let L be a linear left fractional differential operator such that $L(f) \geq 0$ throughout $[0,1]$. We can find a sequence of polynomials Q_n of degree $\leq n$ such that $L(Q_n) \geq 0$ over $[0,1]$, furthermore f is approximated uniformly by Q_n. The degree of this restricted approximations is given by an inequalities using the modulus of continuity of $f^{(p)}$.

2010 AMS Mathematics Subject Classification : 26A33, 41A10, 41A17, 41A25, 41A29.
Keywords and Phrases: Monotone Approximation, Caputo fractional derivative, fractional linear differential operator, modulus of continuity.

1 Introduction

The topic of monotone approximation started in [5] has become a major trend in approximation theory. A typical problem in this subject is: given a positive integer k, approximate a given function whose kth derivative is ≥ 0 by polynomials having this property.

In [2] the authors replaced the kth derivative with a linear differential operator of order k. We mention this motivating result.

Theorem 1 *Let h, k, p be integers, $0 \leq h \leq k \leq p$ and let f be a real function, $f^{(p)}$ continuous in $[-1,1]$ with modulus of continuity $\omega_1\left(f^{(p)}, x\right)$ there. Let $a_j(x)$, $j = h, h+1, ..., k$ be real functions, defined and bounded on $[-1,1]$ and assume $a_h(x)$ is either $\geq$ some number $\alpha > 0$ or $\leq$ some number $\beta < 0$ throughout $[-1,1]$. Consider the operator*

$$L = \sum_{j=h}^{k} a_j(x) \left[\frac{d^j}{dx^j}\right] \tag{1}$$

and suppose, throughout $[-1,1]$,

$$L(f) \geq 0. \tag{2}$$

Then, for every integer $n \geq 1$, *there is a real polynomial* $Q_n(x)$ *of degree* $\leq n$ *such that*

$$L(Q_n) \geq 0 \text{ throughout } [-1,1] \tag{3}$$

and

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq C n^{k-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right), \tag{4}$$

where C *is independent of* n *or* f.

In this article we extend Theorem 1 to the fractional level. Now L is a linear left Caputo fractional differential operator. Here the monotonicity property is only true on the critical interval $[0,1]$. Quantitative unifom approximation remains true on all of $[-1,1]$.

To the best of our knowledge this is the first time fractional monotone Approximation Theory is introduced.

We need and make

Definition 2 *([3], p. 50) Let* $\alpha > 0$ *and* $\lceil \alpha \rceil = m$, *(*$\lceil \cdot \rceil$ *ceiling of the number). Consider* $f \in C^m([-1,1])$. *We define the left Caputo fractional derivative of* f *of order* α *as follows:*

$$\left(D_{*-1}^{\alpha} f \right)(x) = \frac{1}{\Gamma(m-\alpha)} \int_{-1}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{5}$$

for any $x \in [-1,1]$, *where* Γ *is the gamma function.*

We set

$$D_{*-1}^{0} f(x) = f(x),$$
$$D_{*-1}^{m} f(x) = f^{(m)}(x), \ \ \forall\, x \in [-1,1]. \tag{6}$$

2 Main Result

We present

Theorem 3 *Let* h, k, p *be integers,* $0 \leq h \leq k \leq p$ *and let* f *be a real function,* $f^{(p)}$ *continuous in* $[-1,1]$ *with modulus of continuity* $\omega_1\left(f^{(p)}, \delta\right)$, $\delta > 0$, *there. Let* $\alpha_j(x)$, $j = h, h+1, \ldots, k$ *be real functions, defined and bounded on* $[-1,1]$ *and assume for* $x \in [0,1]$ *that* $\alpha_h(x)$ *is either* $\geq$ *some number* $\alpha > 0$ *or* $\leq$ *some number* $\beta < 0$. *Let the real numbers* $\alpha_0 = 0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < \ldots < \alpha_p \leq p$.

Here $D_{*-1}^{\alpha_j} f$ stands for the left Caputo fractional derivative of f of order α_j anchored at -1. Consider the linear left fractional differential operator

$$L := \sum_{j=h}^{k} \alpha_j(x)\left[D_{*-1}^{\alpha_j}\right] \tag{7}$$

and suppose, throughout $[0,1]$,

$$L(f) \geq 0. \tag{8}$$

Then, for any $n \in \mathbb{N}$, there exists a real polynomial $Q_n(x)$ of degree $\leq n$ such that

$$L(Q_n) \geq 0 \quad \textit{throughout} \quad [0,1], \tag{9}$$

and

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq C n^{k-p} \omega_1\left(f^{(p)}, \frac{1}{n}\right), \tag{10}$$

where C is independent of n or f.

Proof. Let $n \in \mathbb{N}$. By a theorem of Trigub [6, 7], given a real function g, with $g^{(p)}$ continuous in $[-1,1]$, there exists a real polynomial $q_n(x)$ of degree $\leq n$ such that

$$\max_{-1 \leq x \leq 1} \left| g^{(j)}(x) - q_n^{(j)}(x) \right| \leq R_p n^{j-p} \omega_1\left(g^{(p)}, \frac{1}{n}\right), \tag{11}$$

$j = 0, 1, ..., p$, where R_p is independent of n or g.

Here $h, k, p \in \mathbb{Z}_+$, $0 \leq h \leq k \leq p$.

Let $\alpha_j > 0$, $j = 1, ..., p$, such that $0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < \alpha_3 \leq 3... < ... < \alpha_p \leq p$. That is $\lceil \alpha_j \rceil = j$, $j = 1, ..., p$.

We consider the left Caputo fractional derivatives

$$\left(D_{*-1}^{\alpha_j} g\right)(x) = \frac{1}{\Gamma(j - \alpha_j)} \int_{-1}^{x} (x-t)^{j-\alpha_j-1} g^{(j)}(t)\, dt, \tag{12}$$

$$\left(D_{*-1}^{j} g\right)(x) = g^{(j)}(x),$$

and

$$\left(D_{*-1}^{\alpha_j} q_n\right)(x) = \frac{1}{\Gamma(j - \alpha_j)} \int_{-1}^{x} (x-t)^{j-\alpha_j-1} q_n^{(j)}(t)\, dt, \tag{13}$$

$$\left(D_{*-1}^{j} q_n\right)(x) = q_n^{(j)}(x);\ j = 1, ..., p,$$

where Γ is the gamma function

$$\Gamma(v) = \int_0^{\infty} e^{-t} t^{v-1} dt, \quad v > 0. \tag{14}$$

We notice that

$$\left|\left(D_{*-1}^{\alpha_j}g\right)(x)-\left(D_{*-1}^{\alpha_j}q_n\right)(x)\right|=$$

$$\frac{1}{\Gamma(j-\alpha_j)}\left|\int_{-1}^{x}(x-t)^{j-\alpha_j-1}g^{(j)}(t)\,dt-\int_{-1}^{x}(x-t)^{j-\alpha_j-1}q_n^{(j)}(t)\,dt\right|= \quad (15)$$

$$\frac{1}{\Gamma(j-\alpha_j)}\left|\int_{-1}^{x}(x-t)^{j-\alpha_j-1}\left(g^{(j)}(t)-q_n^{(j)}(t)\right)dt\right|\le$$

$$\frac{1}{\Gamma(j-\alpha_j)}\int_{-1}^{x}(x-t)^{j-\alpha_j-1}\left|g^{(j)}(t)-q_n^{(j)}(t)\right|dt\overset{(11)}{\le} \quad (16)$$

$$\frac{1}{\Gamma(j-\alpha_j)}\left(\int_{-1}^{x}(x-t)^{j-\alpha_j-1}dt\right)R_pn^{j-p}\omega_1\left(g^{(p)},\frac{1}{n}\right)=$$

$$\frac{1}{\Gamma(j-\alpha_j)}\frac{(x+1)^{j-\alpha_j}}{(j-\alpha_j)}R_pn^{j-p}\omega_1\left(g^{(p)},\frac{1}{n}\right)\le \quad (17)$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}R_pn^{j-p}\omega_1\left(g^{(p)},\frac{1}{n}\right).$$

We proved that for any $x\in[-1,1]$ we have

$$\left|\left(D_{*-1}^{\alpha_j}g\right)(x)-\left(D_{*-1}^{\alpha_j}q_n\right)(x)\right|\le\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}R_pn^{j-p}\omega_1\left(g^{(p)},\frac{1}{n}\right). \quad (18)$$

Hence it holds

$$\max_{-1\le x\le 1}\left|\left(D_{*-1}^{\alpha_j}g\right)(x)-\left(D_{*-1}^{\alpha_j}q_n\right)(x)\right|\le\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}R_pn^{j-p}\omega_1\left(g^{(p)},\frac{1}{n}\right), \quad (19)$$

$j=0,1,...,p$.

Above we set $D_{*-1}^{0}g(x)=g(x)$, $D_{*-1}^{0}q_n(x)=q_n(x)$, $\forall\ x\in[-1,1]$, and $\alpha_0=0$, i.e. $\lceil\alpha_0\rceil=0$.

Put

$$s_j\equiv\sup_{-1\le x\le 1}\left|\alpha_h^{-1}(x)\,\alpha_j(x)\right|,\ \ j=h,...,k, \quad (20)$$

and

$$\eta_n:=R_p\omega_1\left(f^{(p)},\frac{1}{n}\right)\left(\sum_{j=h}^{k}s_j\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}n^{j-p}\right). \quad (21)$$

I. Suppose, throughout $[0,1]$, $\alpha_h(x)\ge\alpha>0$. Let $Q_n(x)$, $x\in[-1,1]$, be a real polynomial of degree $\le n$ so that

$$\max_{-1\le x\le 1}\left|D_{*-1}^{\alpha_j}\left(f(x)+\eta_n(h!)^{-1}x^h\right)-\left(D_{*-1}^{\alpha_j}Q_n\right)(x)\right|\overset{(19)}{\le}$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}R_pn^{j-p}\omega_1\left(f^{(p)},\frac{1}{n}\right), \quad (22)$$

$j=0,1,...,p.$

In particular $(j=0)$ holds

$$\max_{-1\leq x\leq 1}\left|\left(f(x)+\eta_n(h!)^{-1}x^h\right)-Q_n(x)\right|\leq R_p n^{-p}\omega_1\left(f^{(p)},\frac{1}{n}\right), \tag{23}$$

and

$$\max_{-1\leq x\leq 1}|f(x)-Q_n(x)|\leq \eta_n(h!)^{-1}+R_p n^{-p}\omega_1\left(f^{(p)},\frac{1}{n}\right)=$$

$$(h!)^{-1}R_p\omega_1\left(f^{(p)},\frac{1}{n}\right)\left(\sum_{j=h}^{k}s_j\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}n^{j-p}\right)$$

$$+R_p n^{-p}\omega_1\left(f^{(p)},\frac{1}{n}\right)\leq \tag{24}$$

$$R_p\omega_1\left(f^{(p)},\frac{1}{n}\right)n^{k-p}\left(1+(h!)^{-1}\sum_{j=h}^{k}s_j\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\right). \tag{25}$$

That is

$$\max_{-1\leq x\leq 1}|f(x)-Q_n(x)|\leq$$

$$R_p\left(1+(h!)^{-1}\sum_{j=h}^{k}s_j\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\right)n^{k-p}\omega_1\left(f^{(p)},\frac{1}{n}\right), \tag{26}$$

proving (10).

Here

$$L=\sum_{j=h}^{k}\alpha_j(x)\left[D_{*-1}^{\alpha_j}\right],$$

and suppose, throughout $[0,1]$, $Lf\geq 0$.

So over $0\leq x\leq 1$, we get

$$\alpha_h^{-1}(x)L(Q_n(x))=\alpha_h^{-1}(x)L(f(x))+\eta_n\frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}+ \tag{27}$$

$$\sum_{j=h}^{k}\alpha_h^{-1}(x)\alpha_j(x)\left[D_{*-1}^{\alpha_j}Q_n(x)-D_{*-1}^{\alpha_j}f(x)-\frac{\eta_n}{h!}D_{*-1}^{\alpha_j}x^h\right]\overset{(22)}{\geq}$$

$$\eta_n\frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}-\left(\sum_{j=h}^{k}s_j\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}n^{j-p}\right)R_p\omega_1\left(f^{(p)},\frac{1}{n}\right)=$$

$$\eta_n\frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}-\eta_n=\eta_n\left[\frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}-1\right]= \tag{28}$$

$$\eta_n\left[\frac{(x+1)^{h-\alpha_h}-\Gamma(h-\alpha_h+1)}{\Gamma(h-\alpha_h+1)}\right]\geq\eta_n\left[\frac{1-\Gamma(h-\alpha_h+1)}{\Gamma(h-\alpha_h+1)}\right]\geq 0. \tag{29}$$

Explanation: We know $\Gamma(1)=1$, $\Gamma(2)=1$, and Γ is convex and positive on $(0,\infty)$. Here $0\leq h-\alpha_h<1$ and $1\leq h-\alpha_h+1<2$. Thus $\Gamma(h-\alpha_h+1)\leq 1$ and

$$1-\Gamma(h-\alpha_h+1)\geq 0. \tag{30}$$

Hence

$$L(Q_n(x))\geq 0, x\in[0,1]. \tag{31}$$

II. Suppose, throughout $[0,1]$, $\alpha_h(x)\leq\beta<0$. In this case let $Q_n(x)$, $x\in[-1,1]$, be a real polynomial of degree $\leq n$ such that

$$\max_{-1\leq x\leq 1}\left|D_{*-1}^{\alpha_j}\left(f(x)-\eta_n(h!)^{-1}x^h\right)-\left(D_{*-1}^{\alpha_j}Q_n\right)(x)\right|\leq \tag{32}$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}R_p n^{j-p}\omega_1\left(f^{(p)},\frac{1}{n}\right),$$

$j=0,1,...,p$.

In particular holds ($j=0$)

$$\max_{-1\leq x\leq 1}\left|\left(f(x)-\eta_n(h!)^{-1}x^h\right)-Q_n(x)\right|\leq R_p n^{-p}\omega_1\left(f^{(p)},\frac{1}{n}\right), \tag{33}$$

and

$$\max_{-1\leq x\leq 1}|f(x)-Q_n(x)|\leq\eta_n(h!)^{-1}+R_p n^{-p}\omega_1\left(f^{(p)},\frac{1}{n}\right)\overset{\text{(as before)}}{\leq}$$

$$R_p\omega_1\left(f^{(p)},\frac{1}{n}\right)n^{k-p}\left(1+(h!)^{-1}\sum_{j=h}^{k}s_j\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\right). \tag{34}$$

That is

$$\max_{-1\leq x\leq 1}|f(x)-Q_n(x)|\leq$$

$$R_p\left(1+(h!)^{-1}\sum_{j=h}^{k}s_j\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\right)n^{k-p}\omega_1\left(f^{(p)},\frac{1}{n}\right), \tag{35}$$

reproving (10).

Again suppose, throughout $[0,1]$, $Lf\geq 0$.

Also if $0\leq x\leq 1$, then

$$\alpha_h^{-1}(x)L(Q_n(x))=\alpha_h^{-1}(x)L(f(x))-\eta_n\frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}+ \tag{36}$$

$$\sum_{j=h}^{k} \alpha_h^{-1}(x)\,\alpha_j(x)\left[D_{*-1}^{\alpha_j} Q_n(x) - D_{*-1}^{\alpha_j} f(x) + \frac{\eta_n}{h!}\left(D_{*-1}^{\alpha_j} x^h\right)\right] \overset{(32)}{\leq}$$

$$-\eta_n \frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} + \left(\sum_{j=h}^{k} s_j \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} n^{j-p}\right) R_p \omega_1\left(f^{(p)}, \frac{1}{n}\right) =$$

$$\eta_n - \eta_n \frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} = \eta_n \left(1 - \frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}\right) =$$

$$\eta_n \left(\frac{\Gamma(h-\alpha_h+1) - (x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}\right) \leq \eta_n \left(\frac{1-(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}\right) \leq 0, \quad (37)$$

and hence again

$$L(Q_n(x)) \geq 0, \quad \forall\, x \in [0,1]. \quad (38)$$

■

Remark 4 *Based on [1], here we have that* $D_{*-1}^{\alpha_j} f$ *are continuous functions,* $j = 0, 1, ..., p$. *Suppose that* $\alpha_h(x), ..., \alpha_k(x)$ *are continuous functions in* $[-1, 1]$, *and* $L(f) \geq 0$ *on* $[0, 1]$ *is replaced by* $L(f) > 0$ *on* $[0, 1]$. *Disregard the assumption made in the Theorem 3 on* $\alpha_h(x)$. *For* $n \in \mathbb{N}$, *let* $Q_n(x)$ *be* $q_n(x)$ *of (19) for* $g = f$. *Then* $Q_n(x)$ *converges to* f *at the Jackson rate [4, p. 18, Theorem VIII] and at the same time, since* $L(Q_n)$ *converges uniformly to* $L(f)$ *on* $[-1, 1]$, $L(Q_n) > 0$ *on* $[0, 1]$ *for all* n *sufficiently large.*

References

[1] G.A. Anastassiou, *Fractional Korovkin Theorey*, Chaos, Solitons and Fractals, 42 (2009), 2080-2094.

[2] G.A. Anastassiou, O. Shisha, *Monotone approximation with linear differential operators*, J. Approx. Theory 44 (1985), 391-393.

[3] K. Diethelm, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics, Vol. 2004, 1st edition, Springer, New York, Heidelberg, 2010.

[4] D. Jackson, *The Theory of Approximation*, Amer. Math. Soc. Colloq., Vol. XI, New York, 1930.

[5] O. Shisha, *Monotone approximation*, Pacific J. Math. 15 (1965), 667-671.

[6] S.A. Teljakovskii, *Two theorems on the approximation of functions by algebraic polynomials*, Mat. Sb. 70 (112) (1966), 252-265 [Russian]; Amer. Math. Soc. Trans. 77 (2) (1968), 163-178.

[7] R.M. Trigub, *Approximation of functions by polynomials with integer coefficients*, Izv. Akad. Nauk SSSR Ser. Mat. 26 (1962), 261-280 [Russian].