

Right Fractional Monotone Approximation

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Abstract

Let $f \in C^p([-1, 1])$, $p \geq 0$ and let L be a linear right fractional differential operator such that $L(f) \geq 0$ throughout $[-1, 0]$. We can find a sequence of polynomials Q_n of degree $\leq n$ such that $L(Q_n) \geq 0$ over $[-1, 0]$, furthermore f is approximated uniformly by Q_n . The degree of this restricted approximations is given by an inequalities using the modulus of continuity of $f^{(p)}$.

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1 Introduction

The topic of monotone approximation started in [5] has become a major trend in approximation theory. A typical problem in this subject is: given a positive integer k , approximate a given function whose k th derivative is ≥ 0 by polynomials having this property.

In [2] the authors replaced the k th derivative with a linear differential operator of order k . We mention this motivating result.

Theorem 1 *Let h, k, p be integers, $0 \leq h \leq k \leq p$ and let f be a real function, $f^{(p)}$ continuous in $[-1, 1]$ with modulus of continuity $\omega_1(f^{(p)}, x)$ there. Let $a_j(x)$, $j = h, h + 1, \dots, k$ be real functions, defined and bounded on $[-1, 1]$ and assume $a_h(x)$ is either \geq some number $\alpha > 0$ or \leq some number $\beta < 0$ throughout $[-1, 1]$. Consider the operator*

$$L = \sum_{j=h}^k a_j(x) \left[\frac{d^j}{dx^j} \right] \quad (1)$$

and suppose, throughout $[-1, 1]$,

$$L(f) \geq 0. \quad (2)$$

Then, for every integer $n \geq 1$, there is a real polynomial $Q_n(x)$ of degree $\leq n$ such that

$$L(Q_n) \geq 0 \text{ throughout } [-1, 1] \quad (3)$$

and

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq C n^{k-p} \omega_1\left(f^{(p)}, \frac{1}{n}\right), \quad (4)$$

where C is independent of n or f .

In this article we extend Theorem 1 to the right fractional level. Now L is a linear right Caputo fractional differential operator. Here the monotonicity property is only true on the critical interval $[-1, 0]$. Quantitative uniform approximation remains true on all of $[-1, 1]$.

To the best of our knowledge this is the first time right fractional monotone Approximation is introduced.

We need and make

Definition 2 ([3]) Let $\alpha > 0$ and $[\alpha] = m$, ($[\cdot]$ ceiling of the number). Consider $f \in C^m([-1, 1])$. We define the right Caputo fractional derivative of f of order α as follows:

$$(D_{1-}^\alpha f)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^1 (t-x)^{m-\alpha-1} f^{(m)}(t) dt, \quad (5)$$

for any $x \in [-1, 1]$, where Γ is the gamma function.

We set

$$\begin{aligned} D_{1-}^0 f(x) &= f(x), \\ D_{1-}^m f(x) &= (-1)^m f^{(m)}(x), \quad \forall x \in [-1, 1]. \end{aligned} \quad (6)$$

2 Main Result

We present

Theorem 3 Let h, k, p be integers, h is even, $0 \leq h \leq k \leq p$ and let f be a real function, $f^{(p)}$ continuous in $[-1, 1]$ with modulus of continuity $\omega_1(f^{(p)}, \delta)$, $\delta > 0$, there. Let $\alpha_j(x)$, $j = h, h+1, \dots, k$ be real functions, defined and bounded on $[-1, 1]$ and assume for $x \in [-1, 0]$ that $\alpha_h(x)$ is either \geq some number $\alpha > 0$ or \leq some number $\beta < 0$. Let the real numbers $\alpha_0 = 0 < \alpha_1 < 1 < \alpha_2 < 2 < \dots < \alpha_p < p$. Here $D_{1-}^{\alpha_j} f$ stands for the right Caputo fractional derivative of

f of order α_j anchored at 1. Consider the linear right fractional differential operator

$$L := \sum_{j=h}^k \alpha_j(x) [D_{1-}^{\alpha_j}] \quad (7)$$

and suppose, throughout $[-1, 0]$,

$$L(f) \geq 0. \quad (8)$$

Then, for any $n \in \mathbb{N}$, there exists a real polynomial $Q_n(x)$ of degree $\leq n$ such that

$$L(Q_n) \geq 0 \text{ throughout } [-1, 0], \quad (9)$$

and

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq C n^{k-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right), \quad (10)$$

where C is independent of n or f .

Proof. Let $n \in \mathbb{N}$. By a theorem of Trigub [6, 7], given a real function g , with $g^{(p)}$ continuous in $[-1, 1]$, there exists a real polynomial $q_n(x)$ of degree $\leq n$ such that

$$\max_{-1 \leq x \leq 1} |g^{(j)}(x) - q_n^{(j)}(x)| \leq R_p n^{j-p} \omega_1 \left(g^{(p)}, \frac{1}{n} \right), \quad (11)$$

$j = 0, 1, \dots, p$, where R_p is independent of n or g .

Here $h, k, p \in \mathbb{Z}_+$, $0 \leq h \leq k \leq p$.

Let $\alpha_j > 0$, $j = 1, \dots, p$, such that $0 < \alpha_1 < 1 < \alpha_2 < 2 < \alpha_3 < 3 \dots < \dots < \alpha_p < p$. That is $[\alpha_j] = j$, $j = 1, \dots, p$.

We consider the right Caputo fractional derivatives

$$(D_{1-}^{\alpha_j} g)(x) = \frac{(-1)^j}{\Gamma(j - \alpha_j)} \int_x^1 (t - x)^{j - \alpha_j - 1} g^{(j)}(t) dt, \quad (12)$$

$$(D_{1-}^j g)(x) = (-1)^j g^{(j)}(x),$$

and

$$(D_{1-}^{\alpha_j} q_n)(x) = \frac{(-1)^j}{\Gamma(j - \alpha_j)} \int_x^1 (t - x)^{j - \alpha_j - 1} q_n^{(j)}(t) dt, \quad (13)$$

$$(D_{1-}^j q_n)(x) = (-1)^j q_n^{(j)}(x); \quad j = 1, \dots, p,$$

where Γ is the gamma function

$$\Gamma(v) = \int_0^\infty e^{-t} t^{v-1} dt, \quad v > 0. \quad (14)$$

We notice that

$$\begin{aligned} & |(D_{1-}^{\alpha_j} g)(x) - (D_{1-}^{\alpha_j} q_n)(x)| = \\ & \frac{1}{\Gamma(j - \alpha_j)} \left| \int_x^1 (t-x)^{j-\alpha_j-1} g^{(j)}(t) dt - \int_x^1 (t-x)^{j-\alpha_j-1} q_n^{(j)}(t) dt \right| = \end{aligned} \quad (15)$$

$$\begin{aligned} & \frac{1}{\Gamma(j - \alpha_j)} \left| \int_x^1 (t-x)^{j-\alpha_j-1} (g^{(j)}(t) - q_n^{(j)}(t)) dt \right| \leq \\ & \frac{1}{\Gamma(j - \alpha_j)} \int_x^1 (t-x)^{j-\alpha_j-1} |g^{(j)}(t) - q_n^{(j)}(t)| dt \stackrel{(11)}{\leq} \end{aligned} \quad (16)$$

$$\begin{aligned} & \frac{1}{\Gamma(j - \alpha_j)} \left(\int_x^1 (t-x)^{j-\alpha_j-1} dt \right) R_p n^{j-p} \omega_1 \left(g^{(p)}, \frac{1}{n} \right) = \\ & \frac{1}{\Gamma(j - \alpha_j)} \frac{(1-x)^{j-\alpha_j}}{(j - \alpha_j)} R_p n^{j-p} \omega_1 \left(g^{(p)}, \frac{1}{n} \right) \leq \end{aligned} \quad (17)$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} R_p n^{j-p} \omega_1 \left(g^{(p)}, \frac{1}{n} \right).$$

We proved that for any $x \in [-1, 1]$ we have

$$|(D_{1-}^{\alpha_j} g)(x) - (D_{1-}^{\alpha_j} q_n)(x)| \leq \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} R_p n^{j-p} \omega_1 \left(g^{(p)}, \frac{1}{n} \right). \quad (18)$$

Hence it holds

$$\max_{-1 \leq x \leq 1} |(D_{1-}^{\alpha_j} g)(x) - (D_{1-}^{\alpha_j} q_n)(x)| \leq \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} R_p n^{j-p} \omega_1 \left(g^{(p)}, \frac{1}{n} \right), \quad (19)$$

$j = 0, 1, \dots, p$.

Above we set $D_{1-}^0 g(x) = g(x)$, $D_{1-}^0 q_n(x) = q_n(x)$, $\forall x \in [-1, 1]$, and $\alpha_0 = 0$, i.e. $[\alpha_0] = 0$.

Put

$$s_j \equiv \sup_{-1 \leq x \leq 1} |\alpha_h^{-1}(x) \alpha_j(x)|, \quad j = h, \dots, k, \quad (20)$$

and

$$\eta_n := R_p \omega_1 \left(f^{(p)}, \frac{1}{n} \right) \left(\sum_{j=h}^k s_j \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} n^{j-p} \right). \quad (21)$$

I. Suppose, throughout $[-1, 0]$, $\alpha_h(x) \geq \alpha > 0$. Let $Q_n(x)$, $x \in [-1, 1]$, be a real polynomial of degree $\leq n$ so that

$$\begin{aligned} & \max_{-1 \leq x \leq 1} \left| D_{1-}^{\alpha_j} \left(f(x) + \eta_n (h!)^{-1} x^h \right) - (D_{1-}^{\alpha_j} Q_n)(x) \right| \stackrel{(19)}{\leq} \\ & \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} R_p n^{j-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right), \end{aligned} \quad (22)$$

$j = 0, 1, \dots, p$.

In particular ($j = 0$) holds

$$\max_{-1 \leq x \leq 1} \left| \left(f(x) + \eta_n (h!)^{-1} x^h \right) - Q_n(x) \right| \leq R_p n^{-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right), \quad (23)$$

and

$$\begin{aligned} \max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| &\leq \eta_n (h!)^{-1} + R_p n^{-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right) = \\ &(h!)^{-1} R_p \omega_1 \left(f^{(p)}, \frac{1}{n} \right) \left(\sum_{j=h}^k s_j \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} n^{j-p} \right) \\ &\quad + R_p n^{-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right) \leq \end{aligned} \quad (24)$$

$$R_p \omega_1 \left(f^{(p)}, \frac{1}{n} \right) n^{k-p} \left(1 + (h!)^{-1} \sum_{j=h}^k s_j \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \right). \quad (25)$$

That is

$$\begin{aligned} \max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| &\leq \\ &R_p \left(1 + (h!)^{-1} \sum_{j=h}^k s_j \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \right) n^{k-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right), \end{aligned} \quad (26)$$

proving (10).

Here

$$L = \sum_{j=h}^k \alpha_j(x) [D_{1-}^{\alpha_j}],$$

and suppose, throughout $[-1, 0]$, $Lf \geq 0$.

So over $-1 \leq x \leq 0$, we get

$$\alpha_h^{-1}(x) L(Q_n(x)) = \alpha_h^{-1}(x) L(f(x)) + \eta_n \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} + \quad (27)$$

$$\begin{aligned} &\sum_{j=h}^k \alpha_h^{-1}(x) \alpha_j(x) \left[D_{1-}^{\alpha_j} Q_n(x) - D_{1-}^{\alpha_j} f(x) - \frac{\eta_n}{h!} D_{1-}^{\alpha_j} x^h \right] \stackrel{(22)}{\geq} \\ &\eta_n \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} - \left(\sum_{j=h}^k s_j \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} n^{j-p} \right) R_p \omega_1 \left(f^{(p)}, \frac{1}{n} \right) = \\ &\eta_n \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} - \eta_n = \eta_n \left[\frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} - 1 \right] = \end{aligned} \quad (28)$$

$$\eta_n \left[\frac{(1-x)^{h-\alpha_h} - \Gamma(h-\alpha_h+1)}{\Gamma(h-\alpha_h+1)} \right] \geq \eta_n \left[\frac{1-\Gamma(h-\alpha_h+1)}{\Gamma(h-\alpha_h+1)} \right] \geq 0. \quad (29)$$

Explanation: We know $\Gamma(1) = 1$, $\Gamma(2) = 1$, and Γ is convex and positive on $(0, \infty)$. Here $0 \leq h - \alpha_h < 1$ and $1 \leq h - \alpha_h + 1 < 2$. Thus $\Gamma(h - \alpha_h + 1) \leq 1$ and

$$1 - \Gamma(h - \alpha_h + 1) \geq 0. \quad (30)$$

Hence

$$L(Q_n(x)) \geq 0, x \in [-1, 0]. \quad (31)$$

II. Suppose, throughout $[-1, 0]$, $\alpha_h(x) \leq \beta < 0$. In this case let $Q_n(x)$, $x \in [-1, 1]$, be a real polynomial of degree $\leq n$ such that

$$\begin{aligned} \max_{-1 \leq x \leq 1} \left| D_{1-}^{\alpha_j} \left(f(x) - \eta_n (h!)^{-1} x^h \right) - (D_{1-}^{\alpha_j} Q_n)(x) \right| \leq \quad (32) \\ \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} R_p n^{j-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right), \end{aligned}$$

$j = 0, 1, \dots, p$.

In particular holds ($j = 0$)

$$\max_{-1 \leq x \leq 1} \left| \left(f(x) - \eta_n (h!)^{-1} x^h \right) - Q_n(x) \right| \leq R_p n^{-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right), \quad (33)$$

and

$$\begin{aligned} \max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq \eta_n (h!)^{-1} + R_p n^{-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right) \stackrel{\text{(as before)}}{\leq} \\ R_p \omega_1 \left(f^{(p)}, \frac{1}{n} \right) n^{k-p} \left(1 + (h!)^{-1} \sum_{j=h}^k s_j \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \right). \quad (34) \end{aligned}$$

That is

$$\begin{aligned} \max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq \\ R_p \left(1 + (h!)^{-1} \sum_{j=h}^k s_j \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \right) n^{k-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right), \quad (35) \end{aligned}$$

reproving (10).

Again suppose, throughout $[-1, 0]$, $Lf \geq 0$.

Also if $-1 \leq x \leq 0$, then

$$\alpha_h^{-1}(x) L(Q_n(x)) = \alpha_h^{-1}(x) L(f(x)) - \eta_n \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} + \quad (36)$$

$$\begin{aligned}
& \sum_{j=h}^k \alpha_h^{-1}(x) \alpha_j(x) \left[D_{1-}^{\alpha_j} Q_n(x) - D_{1-}^{\alpha_j} f(x) + \frac{\eta_n}{h!} (D_{1-}^{\alpha_j} x^h) \right] \stackrel{(32)}{\leq} \\
& -\eta_n \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} + \left(\sum_{j=h}^k s_j \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} n^{j-p} \right) R_p \omega_1 \left(f^{(p)}, \frac{1}{n} \right) = \\
& \eta_n - \eta_n \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} = \eta_n \left(1 - \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} \right) = \\
& \eta_n \left(\frac{\Gamma(h-\alpha_h+1) - (1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} \right) \leq \eta_n \left(\frac{1 - (1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} \right) \leq 0, \quad (37)
\end{aligned}$$

and hence again

$$L(Q_n(x)) \geq 0, \quad \forall x \in [-1, 0]. \quad (38)$$

■

Remark 4 Based on [1], here we have that $D_{1-}^{\alpha_j} f$ are continuous functions, $j = 0, 1, \dots, p$. Suppose that $\alpha_h(x), \dots, \alpha_k(x)$ are continuous functions in $[-1, 1]$, and $L(f) \geq 0$ on $[-1, 0]$ is replaced by $L(f) > 0$ on $[-1, 0]$. Disregard the assumption made in the Theorem 3 on $\alpha_h(x)$. For $n \in \mathbb{N}$, let $Q_n(x)$ be $q_n(x)$ of (19) for $g = f$. Then $Q_n(x)$ converges to f at the Jackson rate [4, p. 18, Theorem VIII] and at the same time, since $L(Q_n)$ converges uniformly to $L(f)$ on $[-1, 1]$, $L(Q_n) > 0$ on $[-1, 0]$ for all n sufficiently large.

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