Univariate Left Fractional Polynomial High Order Monotone Approximation

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Abstract

Let $f \in C^r([-1,1])$, $r \geq 0$ and let L^* be a linear left fractional differential operator such that $L^*(f) \geq 0$ throughout $[0,1]$. We can find a sequence of polynomials Q_n of degree $\leq n$ such that $L^*(Q_n) \geq 0$ over $[0,1]$, furthermore f is approximated left fractionally and simultaneously by Q_n on $[-1,1]$. The degree of these restricted approximations is given via inequalities using a higher order modulus of smoothness for $f^{(r)}$.

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1 Introduction

The topic of monotone approximation started in [6] has become a major trend in approximation theory. A typical problem in this subject is: given a positive integer k, approximate a given function whose kth derivative is ≥ 0 by polynomials having this property.

In [3] the authors replaced the kth derivative with a linear differential operator of order k. We mention this motivating result.

Theorem 1 *Let h, k, p be integers, $0 \leq h \leq k \leq p$ and let f be a real function, $f^{(p)}$ continuous in $[-1,1]$ with modulus of continuity $\omega_1\left(f^{(p)}, x\right)$ there. Let $a_j(x)$, $j = h, h+1, ..., k$ be real functions, defined and bounded on $[-1,1]$ and assume $a_h(x)$ is either $\geq$ some number $\alpha > 0$ or $\leq$ some number $\beta < 0$*

throughout $[-1,1]$. *Consider the operator*

$$L=\sum_{j=h}^{k}a_j\left(x\right)\left[\frac{d^j}{dx^j}\right]$$

and suppose, throughout $[-1,1]$,

$$L\left(f\right)\geq 0. \tag{1}$$

Then, for every integer $n\geq 1$, *there is a real polynomial* $Q_n\left(x\right)$ *of degree* $\leq n$ *such that*

$$L\left(Q_n\right)\geq 0 \text{ throughout } [-1,1]$$

and

$$\max_{-1\leq x\leq 1}\left|f\left(x\right)-Q_n\left(x\right)\right|\leq Cn^{k-p}\omega_1\left(f^{(p)},\frac{1}{n}\right),$$

where C *is independent of* n *or* f.

We use also the notation $I=[-1,1]$.

We would like to mention

Theorem 2 *(Gonska and Hinnemann [4]). Let* $r\geq 0$ *and* $s\geq 1$. *Then there exists a sequence* $Q_n=Q_n^{(r,s)}$ *of linear polynomial operators mapping* $C^r\left(I\right)$ *into* P_n *(space of polynomials of degree* $\leq n$*), such that for all* $f\in C^r\left(I\right)$, *all* $|x|\leq 1$ *and all* $n\geq\max\left(4\left(r+1\right),r+s\right)$ *we have*

$$\left|f^{(k)}\left(x\right)-\left(Q_nf\right)^{(k)}\left(x\right)\right|\leq M_{r,s}\left(\Delta_n\left(x\right)\right)^{r-k}\omega_s\left(f^{(r)},\Delta_n\left(x\right)\right),\quad 0\leq k\leq r, \tag{2}$$

where $\Delta_n\left(x\right)=\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}$, *and* $M_{r,s}$ *is a constant independent of* f, x, *and* n. *Above* ω_s *is the usual modulus of smoothnees of order* s *with respect to the supremum norm.*

Theorem 2 implies the useful

Corollary 3 *([2]) Let* $r\geq 0$ *and* $s\geq 1$. *Then there exists a sequence* $Q_n=Q_n^{(r,s)}$ *of linear polynomial operators mapping* $C^r\left(I\right)$ *into* P_n, *such that for all* $f\in C^r\left(I\right)$ *and all* $n\geq\max\left(4\left(r+1\right),r+s\right)$ *we have*

$$\left\|f^{(k)}-\left(Q_nf\right)^{(k)}\right\|_\infty\leq\frac{C_{r,s}}{n^{r-k}}\omega_s\left(f^{(r)},\frac{1}{n}\right),\quad k=0,1,...,r, \tag{3}$$

where $C_{r,s}$ *is a constant independent of* f *and* n.

In [2] we proved the motivational

Theorem 4 *Let h, v, r be integers, $0 \leq h \leq v \leq r$ and let $f \in C^r(I)$, with $f^{(r)}$ having modulus of smoothness $\omega_s\left(f^{(r)}, \delta\right)$ there, $s \geq 1$. Let $\alpha_j(x)$, $j = h, h+1, ..., v$ be real functions, defined and bounded on I and suppose α_h is either $\geq \alpha > 0$ or $\leq \beta < 0$ throughout I. Take the operator*

$$L = \sum_{j=h}^{v} \alpha_j(x) \left[\frac{d^j}{dx^j}\right] \tag{4}$$

and assume, throughout I,

$$L(f) \geq 0. \tag{5}$$

Then for every integer $n \geq \max(4(r+1), r+s)$, there exists a real polynomial $Q_n(x)$ of degree $\leq n$ such that

$$L(Q_n) \geq 0 \text{ throughout } I, \tag{6}$$

and

$$\left\|f^{(k)} - Q_n^{(k)}\right\|_\infty \leq \frac{C}{n^{r-v}} \omega_s\left(f^{(r)}, \frac{1}{n}\right), \quad 0 \leq k \leq h. \tag{7}$$

Moreover, we get

$$\left\|f^{(k)} - Q_n^{(k)}\right\|_\infty \leq \frac{C}{n^{r-k}} \omega_s\left(f^{(r)}, \frac{1}{n}\right), \quad h+1 \leq k \leq r, \tag{8}$$

were C is a constant independent of f and n.

In this article we extend Theorem 4 to the fractional level. Indeed here L is replaced by L^*, a linear left Caputo fractional differential operator. Now the monotonicity property is only true on the critical interval $[0,1]$. Simultaneous and fractional convergence remains true on all of I.

We are also inspired by [1].

We make

Definition 5 *([5], p. 50) Let $\alpha > 0$ and $\lceil \alpha \rceil = m$, ($\lceil \cdot \rceil$ ceiling of the number). Consider $f \in C^m([-1,1])$. We define the left Caputo fractional derivative of f of order α as follows:*

$$\left(D_{*-1}^{\alpha} f\right)(x) = \frac{1}{\Gamma(m-\alpha)} \int_{-1}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{9}$$

for any $x \in [-1,1]$, where Γ is the gamma function.

We set

$$D_{*-1}^{0} f(x) = f(x),$$
$$D_{*-1}^{m} f(x) = f^{(m)}(x), \ \ \forall\, x \in [-1,1]. \tag{10}$$

2 Main Result

We present

Theorem 6 *Let h, v, r be integers, $1 \leq h \leq v \leq r$ and let $f \in C^r([-1,1])$, with $f^{(r)}$ having modulus of smoothness $\omega_s\left(f^{(r)}, \delta\right)$ there, $s \geq 1$. Let $\alpha_j(x)$, $j = h, h+1, ..., v$ be real functions, defined and bounded on $[-1,1]$ and suppose $\alpha_h(x)$ is either $\geq \alpha > 0$ or $\leq \beta < 0$ on $[0,1]$. Let the real numbers $\alpha_0 = 0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < ... < \alpha_r \leq r$. Here $D^{\alpha_j}_{*-1} f$ stands for the left Caputo fractional derivative of f of order α_j anchored at -1. Consider the linear left fractional differential operator*

$$L^* := \sum_{j=h}^{k} \alpha_j(x) \left[D^{\alpha_j}_{*-1}\right] \tag{11}$$

and suppose, throughout $[0,1]$,

$$L^*(f) \geq 0. \tag{12}$$

Then, for any $n \in \mathbb{N}$ such that $n \geq \max(4(r+1), r+s)$, there exists a real polynomial $Q_n(x)$ of degree $\leq n$ such that

$$L^*(Q_n) \geq 0 \quad \text{throughout} \quad [0,1], \tag{13}$$

and

$$\sup_{-1\leq x\leq 1} \left|\left(D^{\alpha_j}_{*-1} f\right)(x) - \left(D^{\alpha_j}_{*-1} Q_n\right)(x)\right| \leq \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_{r,s}}{n^{r-j}} \omega_s\left(f^{(r)}, \frac{1}{n}\right), \tag{14}$$

$j = h+1, ..., r$; $C_{r,s}$ is a constant independent of f and n.

Set

$$l_j :\equiv \sup_{x\in[-1,1]} \left|\alpha_h^{-1}(x) \alpha_j(x)\right|, \quad h \leq j \leq v. \tag{15}$$

When $j = 1, ..., h$ we derive

$$\sup_{-1\leq x\leq 1} \left|\left(D^{\alpha_j}_{*-1} f\right)(x) - \left(D^{\alpha_j}_{*-1} Q_n\right)(x)\right| \leq \frac{C_{r,s}}{n^{r-v}} \omega_s\left(f^{(r)}, \frac{1}{n}\right) \cdot \left[\left(\sum_{\tau=h}^{v} l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1)}\right)\left(\sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)}\right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\right]. \tag{16}$$

Finally it holds

$$\sup_{-1\leq x\leq 1} |f(x) - Q_n(x)| \leq \frac{C_{r,s}}{n^{r-v}} \omega_s\left(f^{(r)}, \frac{1}{n}\right) \left[\frac{1}{h!} \sum_{\tau=h}^{v} l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1)} + 1\right]. \tag{17}$$

Proof. Here let Q_n as in Corollary 3.

Let $\alpha_j > 0$, $j = 1, ..., r$, such that $0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < \alpha_3 \leq 3... < ... < \alpha_r \leq r$. That is $\lceil \alpha_j \rceil = j$, $j = 1, ..., r$.

We consider the left Caputo fractional derivatives

$$\left(D_{*-1}^{\alpha_j} f\right)(x) = \frac{1}{\Gamma(j-\alpha_j)} \int_{-1}^{x} (x-t)^{j-\alpha_j-1} f^{(j)}(t)\, dt, \tag{18}$$

and

$$\left(D_{*-1}^{j} f\right)(x) = f^{(j)}(x),$$

and

$$\left(D_{*-1}^{\alpha_j} Q_n\right)(x) = \frac{1}{\Gamma(j-\alpha_j)} \int_{-1}^{x} (x-t)^{j-\alpha_j-1} Q_n^{(j)}(t)\, dt, \tag{19}$$

$$\left(D_{*-1}^{j} Q_n\right)(x) = Q_n^{(j)}(x);\ j = 1, ..., r.$$

We notice that

$$\left|\left(D_{*-1}^{\alpha_j} f\right)(x) - \left(D_{*-1}^{\alpha_j} Q_n\right)(x)\right| =$$

$$\frac{1}{\Gamma(j-\alpha_j)} \left| \int_{-1}^{x} (x-t)^{j-\alpha_j-1} f^{(j)}(t)\, dt - \int_{-1}^{x} (x-t)^{j-\alpha_j-1} Q_n^{(j)}(t)\, dt \right| = \tag{20}$$

$$\frac{1}{\Gamma(j-\alpha_j)} \left| \int_{-1}^{x} (x-t)^{j-\alpha_j-1} \left(f^{(j)}(t) - Q_n^{(j)}(t)\right) dt \right| \leq$$

$$\frac{1}{\Gamma(j-\alpha_j)} \int_{-1}^{x} (x-t)^{j-\alpha_j-1} \left| f^{(j)}(t) - Q_n^{(j)}(t) \right| dt \overset{(3)}{\leq} \tag{21}$$

$$\frac{1}{\Gamma(j-\alpha_j)} \left(\int_{-1}^{x} (x-t)^{j-\alpha_j-1} dt \right) \frac{C_{r,s}}{n^{r-j}} \omega_s\left(f^{(r)}, \frac{1}{n}\right) =$$

$$\frac{1}{\Gamma(j-\alpha_j)} \frac{(x+1)^{j-\alpha_j}}{(j-\alpha_j)} \frac{C_{r,s}}{n^{r-j}} \omega_s\left(f^{(r)}, \frac{1}{n}\right) = \tag{22}$$

$$\frac{(x+1)^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_{r,s}}{n^{r-j}} \omega_s\left(f^{(r)}, \frac{1}{n}\right) \leq$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_{r,s}}{n^{r-j}} \omega_s\left(f^{(r)}, \frac{1}{n}\right).$$

We proved for any $x \in [-1, 1]$ that

$$\left|\left(D_{*-1}^{\alpha_j} f\right)(x) - \left(D_{*-1}^{\alpha_j} Q_n\right)(x)\right| \leq \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_{r,s}}{n^{r-j}} \omega_s\left(f^{(r)}, \frac{1}{n}\right). \tag{23}$$

Hence it holds

$$\sup_{-1\leq x\leq 1} \left|\left(D_{*-1}^{\alpha_j} f\right)(x) - \left(D_{*-1}^{\alpha_j} Q_n\right)(x)\right| \leq \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_{r,s}}{n^{r-j}} \omega_s\left(f^{(r)}, \frac{1}{n}\right), \tag{24}$$

$j = 0, 1, ..., r$.

Above we set $D^0_{*-1} f(x) = f(x)$, $D^0_{*-1} Q_n(x) = Q_n(x)$, $\forall\ x \in [-1, 1]$, and $\alpha_0 = 0$, i.e. $\lceil \alpha_0 \rceil = 0$.

Set also

$$\rho_n := C_{r,s} \omega_s \left(f^{(r)}, \frac{1}{n} \right) \left(\sum_{j=h}^{v} l_j \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} n^{j-r} \right). \tag{25}$$

I. Suppose, throughout $[0, 1]$, $\alpha_h(x) \geq \alpha > 0$. Let $Q_n(x)$, $x \in [-1, 1]$, be a real polynomial of degree $\leq n$ so that

$$\max_{-1 \leq x \leq 1} \left| D^{\alpha_j}_{*-1} \left(f(x) + \rho_n \frac{x^h}{h!} \right) - \left(D^{\alpha_j}_{*-1} Q_n \right)(x) \right| \leq$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \frac{C_{r,s}}{n^{r-j}} \omega_s \left(f^{(r)}, \frac{1}{n} \right), \tag{26}$$

$j = 0, 1, ..., r$.

When $j = h + 1, ..., r$, then

$$\max_{-1 \leq x \leq 1} \left| \left(D^{\alpha_j}_{*-1} f \right)(x) - \left(D^{\alpha_j}_{*-1} Q_n \right)(x) \right| \leq$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \frac{C_{r,s}}{n^{r-j}} \omega_s \left(f^{(r)}, \frac{1}{n} \right), \tag{27}$$

proving (14).

For $j = 1, ..., h$ we get

$$D^{\alpha_j}_{*-1} \left(\frac{x^h}{h!} \right) = \frac{1}{\Gamma(j - \alpha_j)} \int_{-1}^{x} (x - t)^{j - \alpha_j - 1} \frac{t^{h-j}}{(h - j)!} dt \tag{28}$$

(we see that $t = t + 1 - 1$, and
$t^{h-j} = ((t + 1) - 1)^{h-j} = \sum_{\lambda=0}^{h-j} \binom{h-j}{\lambda} (t + 1)^{h-j-\lambda} (-1)^{\lambda}$)

$$= \frac{1}{(h - j)! \Gamma(j - \alpha_j)} \sum_{\lambda=0}^{h-j} (-1)^{\lambda} \binom{h-j}{\lambda} \int_{-1}^{x} (x - t)^{j - \alpha_j - 1} (t + 1)^{h-j-\lambda+1-1} dt$$

$$= \frac{1}{(h - j)! \Gamma(j - \alpha_j)} \sum_{\lambda=0}^{h-j} (-1)^{\lambda} \frac{(h - j)!}{\lambda! (h - j - \lambda)!} \cdot$$

$$\frac{\Gamma(j - \alpha_j) \Gamma(h - j - \lambda + 1)}{\Gamma(h - \alpha_j - \lambda + 1)} (x + 1)^{h - \alpha_j - \lambda}$$

$$= \sum_{\lambda=0}^{h-j} \frac{(-1)^{\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)} (x+1)^{h-\alpha_j-\lambda}. \tag{29}$$

Hence for $j = 1, ..., h$ we found that

$$D_{*-1}^{\alpha_j}\left(\frac{x^h}{h!}\right) = \sum_{\lambda=0}^{h-j} \frac{(-1)^{\lambda}(x+1)^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)}. \tag{30}$$

Therefore we get from (26) that

$$\max_{-1\leq x\leq 1}\left|\left(D_{*-1}^{\alpha_j}f\right)(x) + \rho_n\left(\sum_{\lambda=0}^{h-j} \frac{(-1)^{\lambda}(x+1)^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)}\right) - \left(D_{*-1}^{\alpha_j}Q_n\right)(x)\right| \leq \tag{31}$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_{r,s}}{n^{r-j}}\omega_s\left(f^{(r)},\frac{1}{n}\right),$$

$j = 1, ..., h$.

Therefore we get for $j = 1, ..., h$, that

$$\max_{-1\leq x\leq 1}\left|\left(D_{*-1}^{\alpha_j}f\right)(x) - \left(D_{*-1}^{\alpha_j}Q_n\right)(x)\right| \leq$$

$$\rho_n\left(\sum_{\lambda=0}^{h-j}\frac{2^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)}\right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_{r,s}}{n^{r-j}}\omega_s\left(f^{(r)},\frac{1}{n}\right) = \tag{32}$$

$$C_{r,s}\omega_s\left(f^{(r)},\frac{1}{n}\right)\left(\sum_{\bar{j}=h}^{k} l_{\bar{j}}\frac{2^{\bar{j}-\alpha_{\bar{j}}}}{\Gamma\left(\bar{j}-\alpha_{\bar{j}}+1\right)}n^{\bar{j}-r}\right)\cdot$$

$$\left(\sum_{\lambda=0}^{h-j}\frac{2^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)}\right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_{r,s}}{n^{r-j}}\omega_s\left(f^{(r)},\frac{1}{n}\right) =$$

$$C_{r,s}\omega_s\left(f^{(r)},\frac{1}{n}\right)\left[\left(\sum_{\bar{j}=h}^{k} l_{\bar{j}}\frac{2^{\bar{j}-\alpha_{\bar{j}}}}{\Gamma\left(\bar{j}-\alpha_{\bar{j}}+1\right)}\frac{1}{n^{r-\bar{j}}}\right)\cdot\right. \tag{33}$$

$$\left.\left(\sum_{\lambda=0}^{h-j}\frac{2^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)}\right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{1}{n^{r-j}}\right] \leq$$

$$C_{r,s}\omega_s\left(f^{(r)},\frac{1}{n}\right)\frac{1}{n^{r-v}}\left[\left(\sum_{\bar{j}=h}^{v} l_{\bar{j}}\frac{2^{\bar{j}-\alpha_{\bar{j}}}}{\Gamma\left(\bar{j}-\alpha_{\bar{j}}+1\right)}\right)\cdot\right. \tag{34}$$

$$\left.\left(\sum_{\lambda=0}^{h-j}\frac{2^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)}\right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\right].$$

Hence for $j = 1, ..., h$ we derived (16):

$$\max_{-1\le x\le 1}\left|\left(D_{*-1}^{\alpha_j}f\right)(x)-\left(D_{*-1}^{\alpha_j}Q_n\right)(x)\right|\le \frac{C_{r,s}}{n^{r-v}}\omega_s\left(f^{(r)},\frac{1}{n}\right)\cdot$$

$$\left[\left(\sum_{\tau=h}^{v} l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1)}\right)\left(\sum_{\lambda=0}^{h-j}\frac{2^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)}\right)+\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\right]. \tag{35}$$

From (26) when $j = 0$ we obtain

$$\max_{-1\le x\le 1}\left|f(x)+\rho_n\frac{x^h}{h!}-Q_n(x)\right|\le \frac{C_{r,s}}{n^r}\omega_s\left(f^{(r)},\frac{1}{n}\right). \tag{36}$$

And

$$\max_{-1\le x\le 1}|f(x)-Q_n(x)|\le \frac{\rho_n}{h!}+\frac{C_{r,s}}{n^r}\omega_s\left(f^{(r)},\frac{1}{n}\right)= \tag{37}$$

$$\frac{C_{r,s}}{h!}\omega_s\left(f^{(r)},\frac{1}{n}\right)\left(\sum_{\tau=h}^{v} l_\tau\frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1)}n^{\tau-r}\right)$$

$$+\frac{C_{r,s}}{n^r}\omega_s\left(f^{(r)},\frac{1}{n}\right)=$$

$$C_{r,s}\omega_s\left(f^{(r)},\frac{1}{n}\right)\left[\frac{1}{h!}\sum_{\tau=h}^{v} l_\tau\frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1)n^{r-\tau}}+\frac{1}{n^r}\right]\le$$

$$\frac{C_{r,s}}{n^{r-v}}\omega_s\left(f^{(r)},\frac{1}{n}\right)\left[\frac{1}{h!}\sum_{\tau=h}^{k} l_\tau\frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1)}+1\right], \tag{38}$$

that is proving (17).

Also if $0 \le x \le 1$, then

$$\alpha_h^{-1}(x)L^*(Q_n(x))=\alpha_h^{-1}(x)L^*(f(x))+\rho_n\frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}+ \tag{39}$$

$$\sum_{j=h}^{v}\alpha_h^{-1}(x)\alpha_j(x)\left[D_{*-1}^{\alpha_j}Q_n(x)-D_{*-1}^{\alpha_j}f(x)-\frac{\rho_n}{h!}D_{*-1}^{\alpha_j}x^h\right]\overset{(26)}{\geq}$$

$$\rho_n\frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}-\left(\sum_{j=h}^{v} l_j\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_{r,s}}{n^{r-j}}\omega_s\left(f^{(r)},\frac{1}{n}\right)\right)=$$

$$\rho_n\frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}-\rho_n=\rho_n\left[\frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}-1\right]= \tag{40}$$

$$\rho_n\left[\frac{(x+1)^{h-\alpha_h}-\Gamma(h-\alpha_h+1)}{\Gamma(h-\alpha_h+1)}\right]\ge\rho_n\left[\frac{1-\Gamma(h-\alpha_h+1)}{\Gamma(h-\alpha_h+1)}\right]\ge 0. \tag{41}$$

Explanation: We know that $\Gamma(1)=1$, $\Gamma(2)=1$, and Γ is convex and positive on $(0,\infty)$. Here $0\leq h-\alpha_h<1$ and $1\leq h-\alpha_h+1<2$. Thus $\Gamma(h-\alpha_h+1)\leq 1$ and $1-\Gamma(h-\alpha_h+1)\geq 0$. Hence $L^*(Q_n(x))\geq 0$, $x\in[0,1]$.

II. Suppose on $[0,1]$ that $\alpha_h(x)\leq\beta<0$. Let $Q_n(x)$, $x\in[-1,1]$, be a real polynomial of degree $\leq n$ so that

$$\max_{-1\leq x\leq 1}\left|D_{*-1}^{\alpha_j}\left(f(x)-\rho_n\frac{x^h}{h!}\right)-\left(D_{*-1}^{\alpha_j}Q_n\right)(x)\right|\leq$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_{r,s}}{n^{r-j}}\omega_s\left(f^{(r)},\frac{1}{n}\right), \tag{42}$$

$j=0,1,...,r.$

Similarly we obtain again inequalities of convergence, see (14), (16) and (17).

Also if $0\leq x\leq 1$, then

$$\alpha_h^{-1}(x)L^*(Q_n(x))=\alpha_h^{-1}(x)L^*(f(x))-\rho_n\frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}+ \tag{43}$$

$$\sum_{j=h}^{v}\alpha_h^{-1}(x)\alpha_j(x)\left[D_{*-1}^{\alpha_j}Q_n(x)-D_{*-1}^{\alpha_j}f(x)+\frac{\rho_n}{h!}\left(D_{*-1}^{\alpha_j}x^h\right)\right]\overset{(42)}{\leq}$$

$$-\rho_n\frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}+\sum_{j=h}^{v}l_j\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_{r,s}}{n^{r-j}}\omega_s\left(f^{(r)},\frac{1}{n}\right)=$$

$$\rho_n\left(1-\frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}\right)=\rho_n\left(\frac{\Gamma(h-\alpha_h+1)-(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}\right)\leq \tag{44}$$

$$\rho_n\left(\frac{1-(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}\right)\leq 0, \tag{45}$$

and hence on $[0,1]$ again holds $L^*(Q_n(x))\geq 0$. ∎

Remark 7 *(to Theorem 6) Suppose that $\alpha_j(x)$, $j=h,h+1,...,v$ are continuous functions on $[-1,1]$, and we have on $[0,1]$ only $L^*(f)>0$. Relax the condition $\alpha_h(x)$ is either $\geq\alpha>0$ or $\leq\beta<0$ on $[0,1]$. Let Q_n be the polynomial of degree $\leq n$ coresponding to f from (24).*

Then $D_{*-1}^{\alpha_j}Q_n$ converges uniformly to $D_{*-1}^{\alpha_j}f$ at a higher rate given by inequality (24), in particular for $0\leq j\leq h$. Moreover, because $L^*(Q_n)$ converges uniformly to $L^*(f)$ on $[-1,1]$, $L^*(Q_n)>0$ on $[0,1]$ for sufficiently large n.

References

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