Univariate Right Fractional Polynomial High Order Monotone Approximation

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Abstract
Let \( f \in C^r([-1,1]), \ r \geq 0 \) and let \( L^* \) be a linear right fractional differential operator such that \( L^*(f) \geq 0 \) throughout \([-1,0]\). We can find a sequence of polynomials \( Q_n \) of degree \( \leq n \) such that \( L^*(Q_n) \geq 0 \) over \([-1,0]\), furthermore \( f \) is approximated right fractionally and simultaneously by \( Q_n \) on \([-1,1]\). The degree of these restricted approximations is given via inequalities using a higher order modulus of smoothness for \( f^{(p)} \).

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1 Introduction

The topic of monotone approximation started in [6] has become a major trend in approximation theory. A typical problem in this subject is: given a positive integer \( k \), approximate a given function whose \( k \)th derivative is \( \geq 0 \) by polynomials having this property.

In [3] the authors replaced the \( k \)th derivative with a linear differential operator of order \( k \). We mention this motivating result.

Theorem 1 Let \( h,k,p \) be integers, \( 0 \leq h \leq k \leq p \) and let \( f \) be a real function, \( f^{(p)} \) continuous in \([-1,1]\) with modulus of continuity \( \omega_1(f^{(p)}, \cdot) \) there. Let \( a_j(x), j = h,h + 1, \ldots, k \) be real functions, defined and bounded on \([-1,1]\)
and assume \( a_h(x) \) is either \( \geq \) some number \( \alpha > 0 \) or \( \leq \) some number \( \beta < 0 \) throughout \([-1, 1]\). Consider the operator
\[
L = \sum_{j=h}^{k} a_j(x) \left[ \frac{d^j}{dx^j} \right]
\]
and suppose, throughout \([-1, 1]\),
\[
L(f) \geq 0. \tag{1}
\]
Then, for every integer \( n \geq 1 \), there is a real polynomial \( Q_n(x) \) of degree \( \leq n \) such that
\[
L(Q_n) \geq 0 \text{ throughout } [-1, 1]
\]
and
\[
\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq Cn^{k-p}\omega_{1}\left(\frac{f^{(p)}}{n}\right),
\]
where \( C \) is independent of \( n \) or \( f \).

We use also the notation \( I = [-1, 1] \).

We would like to mention

Theorem 2 (Gonska and Hinnemann [5]). Let \( r \geq 0 \) and \( s \geq 1 \). Then there exists a sequence \( Q_n = Q_n^{(r,s)} \) of linear polynomial operators mapping \( C^r(I) \) into \( P_n \) (space of polynomials of degree \( \leq n \)), such that for all \( f \in C^r(I) \), all \( |x| \leq 1 \) and all \( n \geq \max(4(r+1), r+s) \) we have
\[
\left| f^{(k)}(x) - (Q_n f)^{(k)}(x) \right| \leq C_{r,s} \omega_{s}\left(\frac{f^r}{n}\right), \quad 0 \leq k \leq r,
\]
where \( \Delta_n(x) = \sqrt[r-s]{1 + \frac{1}{n}} \) and \( C_{r,s} \) is a constant independent of \( f, x, \) and \( n \). Above \( \omega_{s} \) is the usual modulus of smoothness of order \( s \) with respect to the supremum norm.

Theorem 2 implies the useful

Corollary 3 ([2]). Let \( r \geq 0 \) and \( s \geq 1 \). Then there exists a sequence \( Q_n = Q_n^{(r,s)} \) of linear polynomial operators mapping \( C^r(I) \) into \( P_n \), such that for all \( f \in C^r(I) \) and all \( n \geq \max(4(r+1), r+s) \) we have
\[
\left\| f^{(k)}(x) - (Q_n f)^{(k)} \right\|_{\infty} \leq \frac{C_{r,s}}{n^{r-k}}\omega_{s}\left(\frac{f^r}{n}\right), \quad k = 0, 1, ..., r, \tag{3}
\]
where \( C_{r,s} \) is a constant independent of \( f \) and \( n \).

In [2] we proved the motivational
Theorem 4 Let \( h, v, r \) be integers, \( 0 \leq h \leq v \leq r \) and let \( f \in C^s (I) \), with \( f^{(r)} \) having modulus of smoothness \( \omega_s (f^{(r)}, \delta) \) there, \( s \geq 1 \). Let \( \alpha_j (x) \), \( j = h, h+1, ..., v \) be real functions, defined and bounded on \( I \) and suppose \( \alpha_h \) is either \( \geq \alpha > 0 \) or \( \leq \beta < 0 \) throughout \( I \). Take the operator
\[
L = \sum_{j=h}^{v} \alpha_j (x) \frac{d^j}{dx^j} \tag{4}
\]
and assume, throughout \( I \),
\[
L (f) \geq 0. \tag{5}
\]
Then for every integer \( n \geq \max (4 (r+1), r+s) \), there exists a real polynomial \( Q_n (x) \) of degree \( \leq n \) such that
\[
L (Q_n) \geq 0 \text{ throughout } I, \tag{6}
\]
and
\[
\| f^{(k)} - Q_n^{(k)} \|_\infty \leq \frac{C}{n^{r-k}} \omega_s \left( f^{(r)}, \frac{1}{n} \right), \quad 0 \leq k \leq h. \tag{7}
\]
Moreover, we get
\[
\| f^{(k)} - Q_n^{(k)} \|_\infty \leq \frac{C}{n^{r-k}} \omega_s \left( f^{(r)}, \frac{1}{n} \right), \quad h+1 \leq k \leq r, \tag{8}
\]
were \( C \) is a constant independent of \( f \) and \( n \).

In this article we extend Theorem 4 to the right fractional level. Indeed here \( L \) is replaced by \( L^* \), a linear right Caputo fractional differential operator. Now the monotonicity property is only true on the critical interval \([-1, 0]\). Simultaneous and right fractional convergence remains true on all of \( I \).

We are also inspired by [1]. We make

\textbf{Definition 5} ([4]) Let \( \alpha > 0 \) and \( [\alpha] = m \), \( ([\cdot] \text{ ceiling of the number}) \). Consider \( f \in C^m ([−1, 1]) \). We define the right Caputo fractional derivative of \( f \) of order \( \alpha \) as follows:
\[
(D^\alpha_{−} f) (x) = \frac{(-1)^m}{\Gamma (m-\alpha)} \int_{x}^{1} (t-x)^{m-\alpha-1} f^{(m)} (t) \, dt, \tag{9}
\]
for any \( x \in [−1, 1] \), where \( \Gamma \) is the gamma function.

We set
\[
D^\alpha_{−} f (x) = f (x),
D^m_{−} f (x) = (-1)^m f^{(m)} (x), \quad \forall \, x \in [−1, 1]. \tag{10}
\]
2 Main Result

We present

**Theorem 6** Let \( h, v, r \) be integers, \( h \) is even, \( 1 \leq h \leq v \leq r \) and let \( f \in C^r([-1,1]), \) with \( f^{(r)} \) having modulus of smoothness \( \omega_s(f^{(r)}, \delta) \) there, \( s \geq 1. \)

Let \( \alpha_j(x), j = h, h+1, ..., v \) be real functions, defined and bounded on \([-1,1]\) and suppose \( \alpha_h(x) \) is either \( \geq 0 \) or \( \leq 0 \) on \([-1,0]\). Let the real numbers \( 0 = 0 < h < 1 < 2 < ... < r < r. \) Here \( D^\alpha_1 \) stands for the right Caputo fractional derivative of \( f \) of order \( \alpha \) anchored at 1. Consider the linear right fractional differential operator

\[
L^*: = \sum_{j=h}^{k} \alpha_j(x) \left[ D^\alpha_{1-j} \right] \tag{11}
\]

and suppose, throughout \([-1,0]\),

\[
L^*(f) \geq 0. \tag{12}
\]

Then, for any \( n \in \mathbb{N} \) such that \( n \geq \max(4(r+1), r+s) \), there exists a real polynomial \( Q_n(x) \) of degree \( \leq n \) such that

\[
L^*(Q_n) \geq 0 \quad \text{throughout} \quad [-1,0], \tag{13}
\]

and

\[
\sup_{-1 \leq x \leq 1} \left| (D^\alpha_{1-j} f) (x) - (D^\alpha_{1-j} Q_n) (x) \right| \leq \frac{2^{j-\alpha_j} C_{r,s}}{\Gamma(j - \alpha_j + 1) n^{r-j} \omega_s \left( f^{(r)}, \frac{1}{n} \right)}, \tag{14}
\]

\( j = h+1, ..., r; \) \( C_{r,s} \) is a constant independent of \( f \) and \( n. \)

Set

\[
l_j := \sup_{x \in [-1,1]} \left| \alpha_h^{-1} (x) \alpha_j (x) \right|, \quad h \leq j \leq v. \tag{15}
\]

When \( j = 1, ..., h \) we derive

\[
\sup_{-1 \leq x \leq 1} \left| (D^\alpha_{1-j} f) (x) - (D^\alpha_{1-j} Q_n) (x) \right| \leq \frac{C_{r,s}}{n^{r-v} \omega_s \left( f^{(r)}, \frac{1}{n} \right)} \left( \sum_{\tau=h}^{v} \frac{2^{2\tau-\alpha_{\tau}}}{\Gamma(\tau - \alpha_{\tau} + 1)} \left( \sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda \Gamma(h - \alpha_j - \lambda + 1)} + \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \right) \right). \tag{16}
\]

Finally it holds

\[
\sup_{-1 \leq x \leq 1} \left| f(x) - Q_n(x) \right| \leq \frac{C_{r,s}}{n^{r-v} \omega_s \left( f^{(r)}, \frac{1}{n} \right)} \left[ \frac{1}{h!} \sum_{\tau=h}^{v} \frac{2^{2\tau-\alpha_{\tau}}}{\Gamma(\tau - \alpha_{\tau} + 1)} + 1 \right]. \tag{17}
\]
Proof. Here let \( Q_n \) as in Corollary 3.

Let \( \alpha_j > 0, j = 1, \ldots, r \), such that \( 0 < \alpha_1 < 1 < \alpha_2 < 2 < \alpha_3 < \ldots < \alpha_r < r \). That is \( \lfloor \alpha_j \rfloor = j, j = 1, \ldots, r \).

We consider the right Caputo fractional derivatives

\[
(D^\alpha_j f) (x) = \frac{(-1)^j}{\Gamma(j - \alpha_j)} \int_x^1 (t - x)^{j-\alpha_j-1} f^{(j)} (t) \, dt, \quad (18)
\]

and

\[
(D^\alpha_j Q_n) (x) = \frac{(-1)^j}{\Gamma(j - \alpha_j)} \int_x^1 (t-x)^{j-\alpha_j-1} Q_n^{(j)} (t) \, dt, \quad (19)
\]

\[
(D^\alpha_j Q_n) (x) = (-1)^j Q_n^{(j)} (x); \quad j = 1, \ldots, r.
\]

We notice that

\[
\left| (D^\alpha_j f) (x) - (D^\alpha_j Q_n) (x) \right| = \frac{1}{\Gamma(j - \alpha_j)} \left| \int_x^1 (t-x)^{j-\alpha_j-1} f^{(j)} (t) \, dt - \int_x^1 (t-x)^{j-\alpha_j-1} Q_n^{(j)} (t) \, dt \right| = (20)
\]

\[
\frac{1}{\Gamma(j - \alpha_j)} \left| \int_x^1 (t-x)^{j-\alpha_j-1} f^{(j)} (t) \, dt - \int_x^1 (t-x)^{j-\alpha_j-1} Q_n^{(j)} (t) \, dt \right| \leq \frac{1}{\Gamma(j - \alpha_j)} \int_x^1 (t-x)^{j-\alpha_j-1} |f^{(j)} (t) - Q_n^{(j)} (t)| \, dt \leq (3)
\]

\[
\frac{1}{\Gamma(j - \alpha_j)} \left( \int_x^1 (t-x)^{j-\alpha_j-1} \, dt \right) \frac{C_{r,s}}{n^{r-j}} \omega_s \left( f^{(r)} \frac{1}{n} \right) = (21)
\]

\[
\frac{1}{\Gamma(j - \alpha_j)} \frac{(1-x)^{j-\alpha_j}}{(j - \alpha_j)} \frac{C_{r,s}}{n^{r-j}} \omega_s \left( f^{(r)} \frac{1}{n} \right) \leq (22)
\]

\[
\frac{(1-x)^{j-\alpha_j}}{(j - \alpha_j + 1)} \frac{C_{r,s}}{n^{r-j}} \omega_s \left( f^{(r)} \frac{1}{n} \right) \leq (23)
\]

We proved for any \( x \in [-1, 1] \) that

\[
\left| (D^\alpha_j f) (x) - (D^\alpha_j Q_n) (x) \right| \leq \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \frac{C_{r,s}}{n^{r-j}} \omega_s \left( f^{(r)} \frac{1}{n} \right).
\]

Hence it holds

\[
\sup_{-1 \leq x \leq 1} \left| (D^\alpha_j f) (x) - (D^\alpha_j Q_n) (x) \right| \leq \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \frac{C_{r,s}}{n^{r-j}} \omega_s \left( f^{(r)} \frac{1}{n} \right),
\]
\(j = 0, 1, \ldots, r.\)

Above we set \(D_1^0 f(x) = f(x), \ \text{and } D_1^0 Q_n(x) = Q_n(x), \ \forall \ x \in [-1, 1], \) and \(\alpha_0 = 0, \) i.e. \([\alpha_0] = 0.\)

Set also

\[
\rho_n := C_{r,s} \omega_s \left( f^{(r)}, \frac{1}{n} \right) \left( \sum_{j=h}^{v} I_j \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} n^{j-r} \right). \tag{25}
\]

I. Suppose, throughout \([-1, 0], \) \(\alpha_h(x) \geq \alpha > 0.\) Let \(Q_n(x), x \in [-1, 1],\) be a real polynomial of degree \(n\) so that

\[
\max_{-1 \leq x \leq 1} \left| (D_1^{\alpha_j} f)(x) - (D_1^{\alpha_j} Q_n)(x) \right| \leq \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \frac{C_{r,s}}{n^{r-j}} \omega_s \left( f^{(r)}, \frac{1}{n} \right), \tag{26}
\]

\(j = 0, 1, \ldots, r.\)

When \(j = h + 1, \ldots, r,\) then

\[
\max_{-1 \leq x \leq 1} \left| (D_1^{\alpha_j} f)(x) - (D_1^{\alpha_j} Q_n)(x) \right| \leq \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \frac{C_{r,s}}{n^{r-j}} \omega_s \left( f^{(r)}, \frac{1}{n} \right), \tag{27}
\]

proving (14).

When \(j = 1, \ldots, h\) we get

\[
D_1^{\alpha_j} \left( \frac{x^h}{h!} \right) = \frac{(-1)^h}{\Gamma(j - \alpha_j)} \int_0^1 (t - x)^{j - \alpha_j - 1} \frac{t^{h-j}}{(h-j)!} dt \tag{28}
\]

(we see that \(t = t + 1 - 1, \) and \(-t + 1 - t - 1\))

\[
= \frac{(-1)^{j+h-j}}{(h-j)! \Gamma(j - \alpha_j)} \int_0^1 (-t)^{h-j} (t - x)^{j - \alpha_j - 1} dt = \frac{(-1)^h}{(h-j)! \Gamma(j - \alpha_j)} \int_0^1 (1-t)^{h-j} (t - x)^{j - \alpha_j - 1} dt = \]

(we see that \((1-t)^{h-j} = \sum_{\lambda=0}^{h-j} \left( \frac{h-j}{\lambda} \right) (1-t)^{h-j-\lambda} (-1)^\lambda\))

\[
= \frac{(-1)^h}{(h-j)! \Gamma(j - \alpha_j)} \sum_{\lambda=0}^{h-j} \left( \frac{h-j}{\lambda} \right) (-1)^\lambda \int_0^1 (1-t)^{(h-j-\lambda)-1} (t - x)^{(j-\alpha_j)-1} dt
\]

6
\[
D_1^{\alpha,j} \left( \frac{x^h}{h!} \right) = (-1)^h \sum_{\lambda=0}^{n-h} \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1) n^{r-j}} \omega_s \left( f^{(r)} \frac{1}{n} \right),
\]

for \( j = 1, \ldots, h \) and \( n \).
Hence for \( j = 1, \ldots, h \) we derived (16):

\[
\max_{-1 \leq x \leq 1} \left| (D_{1/2}^{\alpha_j} f) (x) - (D_{1/2}^{\alpha_j} Q_n) (x) \right| \leq \frac{C_{r,s}}{n^{r-v}} \omega_s \left( f^{(r)}, \frac{1}{n} \right). 
\]

From (26) when \( j = 0 \) we obtain

\[
\max_{-1 \leq x \leq 1} \left| f (x) + \rho_n \frac{2^h}{h!} - Q_n (x) \right| \leq \frac{C_{r,s}}{n^{r-v}} \omega_s \left( f^{(r)}, \frac{1}{n} \right). 
\]

And

\[
\max_{-1 \leq x \leq 1} \left| f (x) - Q_n (x) \right| \leq \rho_n + \frac{C_{r,s}}{n^{r-v}} \omega_s \left( f^{(r)}, \frac{1}{n} \right) = \frac{C_{r,s}}{h!} \omega_s \left( f^{(r)}, \frac{1}{n} \right) \left( \sum_{\tau=h}^{v} \frac{2^{\tau-\alpha_r}}{\Gamma (\tau - \alpha_r + 1) n^{\tau-r}} \right) \frac{1}{n} \leq \frac{C_{r,s}}{n^{r-v}} \omega_s \left( f^{(r)}, \frac{1}{n} \right) \left[ \frac{1}{h!} \sum_{\tau=h}^{v} \frac{2^{\tau-\alpha_r}}{\Gamma (\tau - \alpha_r + 1) n^{\tau-r}} + \frac{1}{n^r} \right] \leq \frac{C_{r,s}}{n^{r-v}} \omega_s \left( f^{(r)}, \frac{1}{n} \right) \left[ \frac{1}{h!} \sum_{\tau=h}^{v} \frac{2^{\tau-\alpha_r}}{\Gamma (\tau - \alpha_r + 1) n^{\tau-r}} + 1 \right],
\]

that is proving (17).

Also if \(-1 \leq x \leq 0\), then

\[
\alpha_{h-1} (x) L^* (Q_n (x)) = \alpha_{h-1} (x) L^* (f (x)) + \rho_n \frac{(1-x)^{h-\alpha_h}}{\Gamma (h - \alpha_h + 1)} + \left( \sum_{j=h}^{v} \alpha_{h-1} (x) \alpha_j (x) \left[ D_{1/2}^{\alpha_j} Q_n (x) - D_{1/2}^{\alpha_j} f (x) - \frac{\rho_n}{h!} D_{1/2}^{\alpha_j} x^h \right] \right) \geq (26)
\]

\[
\rho_n \frac{(1-x)^{h-\alpha_h}}{\Gamma (h - \alpha_h + 1)} - \left( \sum_{j=h}^{v} \frac{2^{j-\alpha_j}}{\Gamma (j - \alpha_j + 1) n^{\tau-j}} \omega_s \left( f^{(r)}, \frac{1}{n} \right) \right) = \rho_n \left[ \frac{(1-x)^{h-\alpha_h}}{\Gamma (h - \alpha_h + 1)} - 1 \right] = \frac{(1-x)^{h-\alpha_h}}{\Gamma (h - \alpha_h + 1)} - \rho_n = \rho_n \left[ \frac{(1-x)^{h-\alpha_h}}{\Gamma (h - \alpha_h + 1)} - 1 \right] = (40)
\]
\[
\rho_n \left[ \frac{(1-x)^{h-\alpha_h} - \Gamma (h-\alpha_h+1)}{\Gamma (h-\alpha_h+1)} \right] \geq \rho_n \left[ \frac{1 - \Gamma (h-\alpha_h+1)}{\Gamma (h-\alpha_h+1)} \right] \geq 0. \tag{41}
\]

**Explanation:** We know that \( \Gamma (1) = 1, \Gamma (2) = 1 \), and \( \Gamma \) is convex and positive on \((0, \infty)\). Here \( 0 \leq h - \alpha_h < 1 \) and \( 1 \leq h - \alpha_h + 1 < 2 \). Thus \( \Gamma (h-\alpha_h+1) \leq 1 \) and \( 1 - \Gamma (h-\alpha_h+1) \geq 0 \). Hence \( L^* (Q_n (x)) \geq 0, \ x \in [-1, 0] \).

II. Suppose on \([-1, 0]\) that \( \alpha_h (x) \leq \beta < 0 \). Let \( Q_n (x), \ x \in [-1, 1] \), be a real polynomial of degree \( \leq n \) so that

\[
\max_{-1 \leq x \leq 1} \left\| D_{i-}^{\alpha_i} \left( f(x) - \rho_n x^h \frac{\Gamma(i)}{\Gamma(h + 1)} \right) - (D_{i-}^{\alpha_i} Q_n) (x) \right\| \leq \frac{2j^{-\alpha_j}}{\Gamma (j - \alpha_j + 1)} \frac{C_{r,s}}{n^{r-j}} \omega_s \left( f^{(r)}, \frac{1}{n} \right), \tag{42}
\]

\( j = 0, 1, \ldots, r. \)

Similarly we obtain again inequalities of convergence, see (14), (16) and (17).

Also if \(-1 \leq x \leq 0\), then

\[
\alpha_h^{-1} (x) L^* (Q_n (x)) = \alpha_h^{-1} (x) L^* (f (x)) - \rho_n \frac{(1-x)^{h-\alpha_h}}{\Gamma (h-\alpha_h+1)} + \tag{43}
\]

\[
\sum_{j=h}^{r} \alpha_h^{-1} (x) \alpha_j (x) \left[ D_{i-}^{\alpha_i} Q_n (x) - D_{i-}^{\alpha_i} f (x) + \rho_n \frac{\Gamma(i)}{\Gamma(h + 1)} (D_{i-}^{\alpha_i} x^h) \right] \leq \tag{42}
\]

\[
-\rho_n \frac{(1-x)^{h-\alpha_h}}{\Gamma (h-\alpha_h+1)} + \sum_{j=h}^{r} \frac{2j^{-\alpha_j}}{\Gamma (j - \alpha_j + 1)} \frac{C_{r,s}}{n^{r-j}} \omega_s \left( f^{(r)}, \frac{1}{n} \right) = \tag{43}
\]

\[
\rho_n \left( 1 - \frac{(1-x)^{h-\alpha_h}}{\Gamma (h-\alpha_h+1)} \right) = \rho_n \left( \frac{\Gamma (h-\alpha_h+1) - (1-x)^{h-\alpha_h}}{\Gamma (h-\alpha_h+1)} \right) \leq \tag{44}
\]

and hence on \([-1, 0]\) again holds \( L^* (Q_n (x)) \geq 0. \)

**Remark 7** (to Theorem 6) Suppose that \( \alpha_j (x), j = h, h+1, \ldots, v \) are continuous functions on \([-1, 1]\), and we have on \([-1, 0]\) only \( L^* (f) > 0 \). Relax the condition \( \alpha_n (x) \) is either \( \geq \alpha > 0 \) or \( \leq \beta < 0 \) on \([-1, 0]\). Let \( Q_n \) be the polynomial of degree \( \leq n \) corresponding to \( f \) from (24).

Then \( D_{i-}^{\alpha_i} Q_n \) converges uniformly to \( D_{i-}^{\alpha_i} f \) at a higher rate given by inequality (24), in particular for \( 0 \leq j \leq h \). Moreover, because \( L^* (Q_n) \) converges uniformly to \( L^* (f) \) on \([-1, 1]\), \( L^* (Q_n) > 0 \) on \([-1, 0]\) for sufficiently large \( n \).
References


