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Univariate Right Fractional Polynomial High Order Monotone Approximation

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Abstract

Let $f \in C^r([-1,1])$, $r \geq 0$ and let L^* be a linear right fractional differential operator such that $L^*(f) \geq 0$ throughout $[-1,0]$. We can find a sequence of polynomials Q_n of degree $\leq n$ such that $L^*(Q_n) \geq 0$ over $[-1,0]$, furthermore f is approximated right fractionally and simultaneously by Q_n on $[-1,1]$. The degree of these restricted approximations is given via inequalities using a higher order modulus of smoothness for $f^{(r)}$.

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1 Introduction

The topic of monotone approximation started in [6] has become a major trend in approximation theory. A typical problem in this subject is: given a positive integer k, approximate a given function whose kth derivative is ≥ 0 by polynomials having this property.

In [3] the authors replaced the kth derivative with a linear differential operator of order k. We mention this motivating result.

Theorem 1 *Let h, k, p be integers, $0 \leq h \leq k \leq p$ and let f be a real function, $f^{(p)}$ continuous in $[-1,1]$ with modulus of continuity $\omega_1\left(f^{(p)}, x\right)$ there. Let $a_j(x)$, $j = h, h+1, ..., k$ be real functions, defined and bounded on $[-1,1]$*

and assume $a_h(x)$ is either $\geq$ some number $\alpha > 0$ or $\leq$ some number $\beta < 0$ throughout $[-1,1]$. Consider the operator

$$L = \sum_{j=h}^{k} a_j(x) \left[\frac{d^j}{dx^j} \right]$$

and suppose, throughout $[-1,1]$,

$$L(f) \geq 0. \tag{1}$$

Then, for every integer $n \geq 1$, there is a real polynomial $Q_n(x)$ of degree $\leq n$ such that

$$L(Q_n) \geq 0 \text{ throughout } [-1,1]$$

and

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq C n^{k-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right),$$

where C is independent of n or f.

We use also the notation $I = [-1,1]$.

We would like to mention

Theorem 2 *(Gonska and Hinnemann [5]). Let $r \geq 0$ and $s \geq 1$. Then there exists a sequence $Q_n = Q_n^{(r,s)}$ of linear polynomial operators mapping $C^r(I)$ into P_n (space of polynomials of degree $\leq n$), such that for all $f \in C^r(I)$, all $|x| \leq 1$ and all $n \geq \max(4(r+1), r+s)$ we have*

$$\left| f^{(k)}(x) - (Q_n f)^{(k)}(x) \right| \leq M_{r,s} (\Delta_n(x))^{r-k} \omega_s \left(f^{(r)}, \Delta_n(x) \right), \quad 0 \leq k \leq r, \tag{2}$$

where $\Delta_n(x) = \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}$, and $M_{r,s}$ is a constant independent of f, x, and n. Above ω_s is the usual modulus of smoothnees of order s with respect to the supremum norm.

Theorem 2 implies the useful

Corollary 3 *([2]) Let $r \geq 0$ and $s \geq 1$. Then there exists a sequence $Q_n = Q_n^{(r,s)}$ of linear polynomial operators mapping $C^r(I)$ into P_n, such that for all $f \in C^r(I)$ and all $n \geq \max(4(r+1), r+s)$ we have*

$$\left\| f^{(k)} - (Q_n f)^{(k)} \right\|_\infty \leq \frac{C_{r,s}}{n^{r-k}} \omega_s \left(f^{(r)}, \frac{1}{n} \right), \quad k = 0, 1, ..., r, \tag{3}$$

where $C_{r,s}$ is a constant independent of f and n.

In [2] we proved the motivational

Theorem 4 *Let h, v, r be integers, $0 \leq h \leq v \leq r$ and let $f \in C^r(I)$, with $f^{(r)}$ having modulus of smoothness $\omega_s\left(f^{(r)}, \delta\right)$ there, $s \geq 1$. Let $\alpha_j(x)$, $j = h, h+1, ..., v$ be real functions, defined and bounded on I and suppose α_h is either $\geq \alpha > 0$ or $\leq \beta < 0$ throughout I. Take the operator*

$$L = \sum_{j=h}^{v} \alpha_j(x) \left[\frac{d^j}{dx^j}\right] \tag{4}$$

and assume, throughout I,

$$L(f) \geq 0. \tag{5}$$

Then for every integer $n \geq \max(4(r+1), r+s)$, there exists a real polynomial $Q_n(x)$ of degree $\leq n$ such that

$$L(Q_n) \geq 0 \text{ throughout } I, \tag{6}$$

and

$$\left\| f^{(k)} - Q_n^{(k)} \right\|_\infty \leq \frac{C}{n^{r-v}} \omega_s\left(f^{(r)}, \frac{1}{n}\right), \quad 0 \leq k \leq h. \tag{7}$$

Moreover, we get

$$\left\| f^{(k)} - Q_n^{(k)} \right\|_\infty \leq \frac{C}{n^{r-k}} \omega_s\left(f^{(r)}, \frac{1}{n}\right), \quad h+1 \leq k \leq r, \tag{8}$$

were C is a constant independent of f and n.

In this article we extend Theorem 4 to the right fractional level. Indeed here L is replaced by L^*, a linear right Caputo fractional differential operator. Now the monotonicity property is only true on the critical interval $[-1, 0]$. Simultaneous and right fractional convergence remains true on all of I.

We are also inspired by [1].

We make

Definition 5 *([4]) Let $\alpha > 0$ and $\lceil \alpha \rceil = m$, ($\lceil \cdot \rceil$ ceiling of the number). Consider $f \in C^m([-1, 1])$. We define the right Caputo fractional derivative of f of order α as follows:*

$$\left(D_{1-}^{\alpha} f\right)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^1 (t-x)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{9}$$

for any $x \in [-1, 1]$, where Γ is the gamma function.

We set

$$D_{1-}^{0} f(x) = f(x),$$
$$D_{1-}^{m} f(x) = (-1)^m f^{(m)}(x), \ \ \forall\, x \in [-1, 1]. \tag{10}$$

2 Main Result

We present

Theorem 6 *Let h, v, r be integers, h is even, $1 \leq h \leq v \leq r$ and let $f \in C^r([-1,1])$, with $f^{(r)}$ having modulus of smoothness $\omega_s\left(f^{(r)}, \delta\right)$ there, $s \geq 1$. Let $\alpha_j(x)$, $j = h, h+1, ..., v$ be real functions, defined and bounded on $[-1,1]$ and suppose $\alpha_h(x)$ is either $\geq \alpha > 0$ or $\leq \beta < 0$ on $[-1,0]$. Let the real numbers $\alpha_0 = 0 < \alpha_1 < 1 < \alpha_2 < 2 < ... < \alpha_r < r$. Here $D_{1-}^{\alpha_j} f$ stands for the right Caputo fractional derivative of f of order α_j anchored at 1. Consider the linear right fractional differential operator*

$$L^* := \sum_{j=h}^{k} \alpha_j(x) \left[D_{1-}^{\alpha_j}\right] \tag{11}$$

and suppose, throughout $[-1,0]$,

$$L^*(f) \geq 0. \tag{12}$$

Then, for any $n \in \mathbb{N}$ such that $n \geq \max(4(r+1), r+s)$, there exists a real polynomial $Q_n(x)$ of degree $\leq n$ such that

$$L^*(Q_n) \geq 0 \quad \text{throughout} \quad [-1,0], \tag{13}$$

and

$$\sup_{-1 \leq x \leq 1} \left|\left(D_{1-}^{\alpha_j} f\right)(x) - \left(D_{1-}^{\alpha_j} Q_n\right)(x)\right| \leq \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_{r,s}}{n^{r-j}} \omega_s\left(f^{(r)}, \frac{1}{n}\right), \tag{14}$$

$j = h+1, ..., r$; $C_{r,s}$ is a constant independent of f and n.

Set

$$l_j :\equiv \sup_{x \in [-1,1]} \left|\alpha_h^{-1}(x) \alpha_j(x)\right|, \quad h \leq j \leq v. \tag{15}$$

When $j = 1, ..., h$ we derive

$$\sup_{-1 \leq x \leq 1} \left|\left(D_{1-}^{\alpha_j} f\right)(x) - \left(D_{1-}^{\alpha_j} Q_n\right)(x)\right| \leq \frac{C_{r,s}}{n^{r-v}} \omega_s\left(f^{(r)}, \frac{1}{n}\right) \cdot \left[\left(\sum_{\tau=h}^{v} l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1)}\right)\left(\sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda! \Gamma(h-\alpha_j-\lambda+1)}\right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\right]. \tag{16}$$

Finally it holds

$$\sup_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq \frac{C_{r,s}}{n^{r-v}} \omega_s\left(f^{(r)}, \frac{1}{n}\right) \left[\frac{1}{h!} \sum_{\tau=h}^{v} l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1)} + 1\right]. \tag{17}$$

Proof. Here let Q_n as in Corollary 3.

Let $\alpha_j > 0$, $j = 1, ..., r$, such that $0 < \alpha_1 < 1 < \alpha_2 < 2 < \alpha_3 < 3... < ... < \alpha_r < r$. That is $\lceil \alpha_j \rceil = j$, $j = 1, ..., r$.

We consider the right Caputo fractional derivatives

$$\left(D_{1-}^{\alpha_j} f\right)(x) = \frac{(-1)^j}{\Gamma(j - \alpha_j)} \int_x^1 (t - x)^{j-\alpha_j-1} f^{(j)}(t)\, dt, \tag{18}$$

$$\left(D_{1-}^{j} f\right)(x) = (-1)^j f^{(j)}(x),$$

and

$$\left(D_{1-}^{\alpha_j} Q_n\right)(x) = \frac{(-1)^j}{\Gamma(j - \alpha_j)} \int_x^1 (t\text{-}x)^{j-\alpha_j-1} Q_n^{(j)}(t)\, dt, \tag{19}$$

$$\left(D_{1-}^{j} Q_n\right)(x) = (-1)^j Q_n^{(j)}(x);\ j = 1, ..., r.$$

We notice that

$$\left|\left(D_{1-}^{\alpha_j} f\right)(x) - \left(D_{1-}^{\alpha_j} Q_n\right)(x)\right| =$$

$$\frac{1}{\Gamma(j - \alpha_j)} \left| \int_x^1 (t - x)^{j-\alpha_j-1} f^{(j)}(t)\, dt - \int_x^1 (t - x)^{j-\alpha_j-1} Q_n^{(j)}(t)\, dt \right| = \tag{20}$$

$$\frac{1}{\Gamma(j - \alpha_j)} \left| \int_x^1 (t - x)^{j-\alpha_j-1} \left(f^{(j)}(t) - Q_n^{(j)}(t)\right) dt \right| \leq$$

$$\frac{1}{\Gamma(j - \alpha_j)} \int_x^1 (t - x)^{j-\alpha_j-1} \left|f^{(j)}(t) - Q_n^{(j)}(t)\right| dt \overset{(3)}{\leq} \tag{21}$$

$$\frac{1}{\Gamma(j - \alpha_j)} \left(\int_x^1 (t - x)^{j-\alpha_j-1} dt\right) \frac{C_{r,s}}{n^{r-j}} \omega_s\left(f^{(r)}, \frac{1}{n}\right) =$$

$$\frac{1}{\Gamma(j - \alpha_j)} \frac{(1 - x)^{j-\alpha_j}}{(j - \alpha_j)} \frac{C_{r,s}}{n^{r-j}} \omega_s\left(f^{(r)}, \frac{1}{n}\right) = \tag{22}$$

$$\frac{(1 - x)^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \frac{C_{r,s}}{n^{r-j}} \omega_s\left(f^{(r)}, \frac{1}{n}\right) \leq$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \frac{C_{r,s}}{n^{r-j}} \omega_s\left(f^{(r)}, \frac{1}{n}\right).$$

We proved for any $x \in [-1, 1]$ that

$$\left|\left(D_{1-}^{\alpha_j} f\right)(x) - \left(D_{1-}^{\alpha_j} Q_n\right)(x)\right| \leq \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \frac{C_{r,s}}{n^{r-j}} \omega_s\left(f^{(r)}, \frac{1}{n}\right). \tag{23}$$

Hence it holds

$$\sup_{-1 \leq x \leq 1} \left|\left(D_{1-}^{\alpha_j} f\right)(x) - \left(D_{1-}^{\alpha_j} Q_n\right)(x)\right| \leq \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \frac{C_{r,s}}{n^{r-j}} \omega_s\left(f^{(r)}, \frac{1}{n}\right), \tag{24}$$

$j=0,1,...,r.$

Above we set $D_{1-}^{0}f(x)=f(x)$, $D_{1-}^{0}Q_n(x)=Q_n(x)$, $\forall\ x\in[-1,1]$, and $\alpha_0=0$, i.e. $\lceil\alpha_0\rceil=0$.

Set also

$$\rho_n:=C_{r,s}\omega_s\left(f^{(r)},\frac{1}{n}\right)\left(\sum_{j=h}^{v}l_j\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}n^{j-r}\right). \tag{25}$$

I. Suppose, throughout $[-1,0]$, $\alpha_h(x)\geq\alpha>0$. Let $Q_n(x)$, $x\in[-1,1]$, be a real polynomial of degree $\leq n$ so that

$$\max_{-1\leq x\leq 1}\left|D_{1-}^{\alpha_j}\left(f(x)+\rho_n\frac{x^h}{h!}\right)-\left(D_{1-}^{\alpha_j}Q_n\right)(x)\right|\leq$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_{r,s}}{n^{r-j}}\omega_s\left(f^{(r)},\frac{1}{n}\right), \tag{26}$$

$j=0,1,...,r.$

When $j=h+1,...,r$, then

$$\max_{-1\leq x\leq 1}\left|\left(D_{1-}^{\alpha_j}f\right)(x)-\left(D_{1-}^{\alpha_j}Q_n\right)(x)\right|\leq$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_{r,s}}{n^{r-j}}\omega_s\left(f^{(r)},\frac{1}{n}\right), \tag{27}$$

proving (14).

When $j=1,...,h$ we get

$$D_{1-}^{\alpha_j}\left(\frac{x^h}{h!}\right)=\frac{(-1)^h}{\Gamma(j-\alpha_j)}\int_x^1(t-x)^{j-\alpha_j-1}\frac{t^{h-j}}{(h-j)!}dt \tag{28}$$

(we see that $t=t+1-1$, and $-t+1-t-1$)

$$=\frac{(-1)^{j+h-j}}{(h-j)!\Gamma(j-\alpha_j)}\int_x^1(-t)^{h-j}(t-x)^{j-\alpha_j-1}dt=$$

$$\frac{(-1)^h}{(h-j)!\Gamma(j-\alpha_j)}\int_x^1(1-t-1)^{h-j}(t-x)^{j-\alpha_j-1}dt=$$

(we see that $((1-t)-1)^{h-j}=\sum_{\lambda=0}^{h-j}\binom{h-j}{\lambda}(1-t)^{h-j-\lambda}(-1)^{\lambda}$)

$$=\frac{(-1)^h}{(h-j)!\Gamma(j-\alpha_j)}\sum_{\lambda=0}^{h-j}\binom{h-j}{\lambda}(-1)^{\lambda}\int_x^1(1-t)^{(h-j-\lambda+1)-1}(t-x)^{(j-\alpha_j)-1}dt$$

$$= \frac{(-1)^h}{(h-j)!\Gamma(j-\alpha_j)} \sum_{\lambda=0}^{h-j} \frac{(h-j)!\,(-1)^\lambda}{\lambda!\,(h-j-\lambda)!} \cdot$$

$$\frac{\Gamma(h-j-\lambda+1)\,\Gamma(j-\alpha_j)}{\Gamma(h-\alpha_j-\lambda+1)} (1-x)^{h-\alpha_j-\lambda}$$

$$= (-1)^h \sum_{\lambda=0}^{h-j} \frac{(-1)^\lambda (1-x)^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)}. \tag{29}$$

Hence for $j = 1, ..., h$ we found that

$$D_{1-}^{\alpha_j} \left(\frac{x^h}{h!} \right) = (-1)^h \sum_{\lambda=0}^{h-j} \frac{(-1)^\lambda (1-x)^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)}. \tag{30}$$

Therefore we get from (26) that

$$\max_{-1\leq x\leq 1} \left| \left(D_{1-}^{\alpha_j} f\right)(x) + \rho_n \left((-1)^h \sum_{\lambda=0}^{h-j} \frac{(-1)^\lambda (1-x)^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)} \right) - \left(D_{1-}^{\alpha_j} Q_n\right)(x) \right| \leq \tag{31}$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_{r,s}}{n^{r-j}} \omega_s \left(f^{(r)}, \frac{1}{n} \right),$$

$j = 1, ..., h.$

Therefore we get for $j = 1, ..., h$, that

$$\max_{-1\leq x\leq 1} \left| \left(D_{1-}^{\alpha_j} f\right)(x) - \left(D_{1-}^{\alpha_j} Q_n\right)(x) \right| \leq$$

$$\rho_n \left(\sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)} \right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_{r,s}}{n^{r-j}} \omega_s \left(f^{(r)}, \frac{1}{n} \right) = \tag{32}$$

$$C_{r,s} \omega_s \left(f^{(r)}, \frac{1}{n} \right) \left(\sum_{\overline{j}=h}^{v} l_{\overline{j}} \frac{2^{\overline{j}-\alpha_{\overline{j}}}}{\Gamma\left(\overline{j}-\alpha_{\overline{j}}+1\right)} n^{\overline{j}-r} \right) \cdot$$

$$\left(\sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)} \right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_{r,s}}{n^{r-j}} \omega_s \left(f^{(r)}, \frac{1}{n} \right) =$$

$$C_{r,s} \omega_s \left(f^{(r)}, \frac{1}{n} \right) \left[\left(\sum_{\overline{j}=h}^{v} l_{\overline{j}} \frac{2^{\overline{j}-\alpha_{\overline{j}}}}{\Gamma\left(\overline{j}-\alpha_{\overline{j}}+1\right)} \frac{1}{n^{r-\overline{j}}} \right) \cdot \right. \tag{33}$$

$$\left. \left(\sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)} \right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{1}{n^{r-j}} \right] \leq$$

$$C_{r,s} \omega_s \left(f^{(r)}, \frac{1}{n} \right) \frac{1}{n^{r-v}} \left[\left(\sum_{\overline{j}=h}^{v} l_{\overline{j}} \frac{2^{\overline{j}-\alpha_{\overline{j}}}}{\Gamma\left(\overline{j}-\alpha_{\overline{j}}+1\right)} \right) \cdot \right. \tag{34}$$

$$\left(\sum_{\lambda=0}^{h-j}\frac{2^{h-\alpha_j-\lambda}}{\lambda!\Gamma\left(h-\alpha_j-\lambda+1\right)}\right)+\frac{2^{j-\alpha_j}}{\Gamma\left(j-\alpha_j+1\right)}\Bigg].$$

Hence for $j=1,...,h$ we derived (16):

$$\max_{-1\le x\le 1}\left|\left(D_{1-}^{\alpha_j}f\right)(x)-\left(D_{1-}^{\alpha_j}Q_n\right)(x)\right|\le\frac{C_{r,s}}{n^{r-v}}\omega_s\left(f^{(r)},\frac{1}{n}\right)\cdot$$

$$\left[\left(\sum_{\tau=h}^{v}l_\tau\frac{2^{\tau-\alpha_\tau}}{\Gamma\left(\tau-\alpha_\tau+1\right)}\right)\left(\sum_{\lambda=0}^{h-j}\frac{2^{h-\alpha_j-\lambda}}{\lambda!\Gamma\left(h-\alpha_j-\lambda+1\right)}\right)+\frac{2^{j-\alpha_j}}{\Gamma\left(j-\alpha_j+1\right)}\right]. \tag{35}$$

From (26) when $j=0$ we obtain

$$\max_{-1\le x\le 1}\left|f(x)+\rho_n\frac{x^h}{h!}-Q_n(x)\right|\le\frac{C_{r,s}}{n^r}\omega_s\left(f^{(r)},\frac{1}{n}\right). \tag{36}$$

And

$$\max_{-1\le x\le 1}\left|f(x)-Q_n(x)\right|\le\frac{\rho_n}{h!}+\frac{C_{r,s}}{n^r}\omega_s\left(f^{(r)},\frac{1}{n}\right)= \tag{37}$$

$$\frac{C_{r,s}}{h!}\omega_s\left(f^{(r)},\frac{1}{n}\right)\left(\sum_{\tau=h}^{v}l_\tau\frac{2^{\tau-\alpha_\tau}}{\Gamma\left(\tau-\alpha_\tau+1\right)}n^{\tau-r}\right)$$

$$+\frac{C_{r,s}}{n^r}\omega_s\left(f^{(r)},\frac{1}{n}\right)=$$

$$C_{r,s}\omega_s\left(f^{(r)},\frac{1}{n}\right)\left[\frac{1}{h!}\sum_{\tau=h}^{v}l_\tau\frac{2^{\tau-\alpha_\tau}}{\Gamma\left(\tau-\alpha_\tau+1\right)n^{r-\tau}}+\frac{1}{n^r}\right]\le$$

$$\frac{C_{r,s}}{n^{r-v}}\omega_s\left(f^{(r)},\frac{1}{n}\right)\left[\frac{1}{h!}\sum_{\tau=h}^{v}l_\tau\frac{2^{\tau-\alpha_\tau}}{\Gamma\left(\tau-\alpha_\tau+1\right)}+1\right], \tag{38}$$

that is proving (17).

Also if $-1\le x\le 0$, then

$$\alpha_h^{-1}(x)L^*\left(Q_n(x)\right)=\alpha_h^{-1}(x)L^*\left(f(x)\right)+\rho_n\frac{(1-x)^{h-\alpha_h}}{\Gamma\left(h-\alpha_h+1\right)}+ \tag{39}$$

$$\sum_{j=h}^{v}\alpha_h^{-1}(x)\alpha_j(x)\left[D_{1-}^{\alpha_j}Q_n(x)-D_{1-}^{\alpha_j}f(x)-\frac{\rho_n}{h!}D_{1-}^{\alpha_j}x^h\right]\overset{(26)}{\ge}$$

$$\rho_n\frac{(1-x)^{h-\alpha_h}}{\Gamma\left(h-\alpha_h+1\right)}-\left(\sum_{j=h}^{v}l_j\frac{2^{j-\alpha_j}}{\Gamma\left(j-\alpha_j+1\right)}\frac{C_{r,s}}{n^{r-j}}\omega_s\left(f^{(r)},\frac{1}{n}\right)\right)=$$

$$\rho_n\frac{(1-x)^{h-\alpha_h}}{\Gamma\left(h-\alpha_h+1\right)}-\rho_n=\rho_n\left[\frac{(1-x)^{h-\alpha_h}}{\Gamma\left(h-\alpha_h+1\right)}-1\right]= \tag{40}$$

$$\rho_n\left[\frac{(1-x)^{h-\alpha_h}-\Gamma(h-\alpha_h+1)}{\Gamma(h-\alpha_h+1)}\right]\geq\rho_n\left[\frac{1-\Gamma(h-\alpha_h+1)}{\Gamma(h-\alpha_h+1)}\right]\geq 0. \tag{41}$$

Explanation: We know that $\Gamma(1)=1$, $\Gamma(2)=1$, and Γ is convex and positive on $(0,\infty)$. Here $0\leq h-\alpha_h<1$ and $1\leq h-\alpha_h+1<2$. Thus $\Gamma(h-\alpha_h+1)\leq 1$ and $1-\Gamma(h-\alpha_h+1)\geq 0$. Hence $L^*(Q_n(x))\geq 0$, $x\in[-1,0]$.

II. Suppose on $[-1,0]$ that $\alpha_h(x)\leq\beta<0$. Let $Q_n(x)$, $x\in[-1,1]$, be a real polynomial of degree $\leq n$ so that

$$\max_{-1\leq x\leq 1}\left|D_{1-}^{\alpha_j}\left(f(x)-\rho_n\frac{x^h}{h!}\right)-\left(D_{1-}^{\alpha_j}Q_n\right)(x)\right|\leq$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_{r,s}}{n^{r-j}}\omega_s\left(f^{(r)},\frac{1}{n}\right), \tag{42}$$

$j=0,1,...,r$.

Similarly we obtain again inequalities of convergence, see (14), (16) and (17).

Also if $-1\leq x\leq 0$, then

$$\alpha_h^{-1}(x)L^*(Q_n(x))=\alpha_h^{-1}(x)L^*(f(x))-\rho_n\frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}+ \tag{43}$$

$$\sum_{j=h}^{v}\alpha_h^{-1}(x)\alpha_j(x)\left[D_{1-}^{\alpha_j}Q_n(x)-D_{1-}^{\alpha_j}f(x)+\frac{\rho_n}{h!}\left(D_{1-}^{\alpha_j}x^h\right)\right]\overset{(42)}{\leq}$$

$$-\rho_n\frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}+\sum_{j=h}^{v}l_j\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_{r,s}}{n^{r-j}}\omega_s\left(f^{(r)},\frac{1}{n}\right)=$$

$$\rho_n\left(1-\frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}\right)=\rho_n\left(\frac{\Gamma(h-\alpha_h+1)-(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}\right)\leq \tag{44}$$

$$\rho_n\left(\frac{1-(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}\right)\leq 0, \tag{45}$$

and hence on $[-1,0]$ again holds $L^*(Q_n(x))\geq 0$. ■

Remark 7 *(to Theorem 6) Suppose that $\alpha_j(x)$, $j=h,h+1,...,v$ are continuous functions on $[-1,1]$, and we have on $[-1,0]$ only $L^*(f)>0$. Relax the condition $\alpha_h(x)$ is either $\geq\alpha>0$ or $\leq\beta<0$ on $[-1,0]$. Let Q_n be the polynomial of degree $\leq n$ coresponding to f from (24).*

Then $D_{1-}^{\alpha_j}Q_n$ converges uniformly to $D_{1-}^{\alpha_j}f$ at a higher rate given by inequality (24), in particular for $0\leq j\leq h$. Moreover, because $L^*(Q_n)$ converges uniformly to $L^*(f)$ on $[-1,1]$, $L^*(Q_n)>0$ on $[-1,0]$ for sufficiently large n.

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