Spline left fractional monotone approximation involving left fractional differential operators

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Abstract

Let $f \in C^s([-1,1])$, s$\in \mathbb{N}$ and L^* be a linear left fractional differential operator such that $L^*(f) \geq 0$ on $[0,1]$. Then there exists a sequence Q_n, $n \in \mathbb{N}$ of polynomial splines with equally spaced knots of given fixed order such that $L^*(Q_n) \geq 0$ on $[0,1]$. Furthermore f is approximated with rates fractionally and simultaneously by Q_n in the uniform norm. This constrained fractional approximation on $[-1,1]$ is given via inequalities invoving a higher modulus of smoothness of $f^{(s)}$.

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1 Introduction

Let $[a,b] \subset \mathbb{R}$ and for $n \geq 1$ consider the partition Δ_n with points $x_{in} = a + i\left(\frac{b-a}{n}\right)$, $i = 0,1,...,n$. Hence $\overline{\Delta}_n \equiv \max_{1\leq i\leq n}(x_{in} - x_{i-1,n}) = \frac{b-a}{n}$.

Let $S_m(\Delta_n)$ be the space of polynomial splines of order $m > 0$ with simple knots at the points x_{in}, $i = 1,...,n-1$. Then there exists a linear operator $Q_n : Q_n \equiv Q_n(f)$, mapping $B[a,b]$: the space of bounded real valued functions f on $[a,b]$, into $S_m(\Delta_n)$ (see [4], p. 224, Theorem 6.18).

From the same reference [4], p. 227, Corollary 6.21, we get

Corollary 1 *Let $1 \leq \sigma \leq m$, $n \geq 1$. Then for all $f \in C^{\sigma-1}[a,b]$; $r = 0,...,\sigma-1$,*

$$\left\| f^{(r)} - Q_n^{(r)} \right\|_\infty \leq C_1 \left(\frac{b-a}{n}\right)^{\sigma-r-1} \omega_{m-\sigma+1}\left(f^{(b-1)}, \frac{b-a}{n}\right), \qquad (1)$$

where C_1 depends only on m, $C_1 = C_1(m)$.

By denoting $C_2 = C_1 \max_{0 \le r \le \sigma-1} (b-a)^{\sigma-r-1}$ we obtain

Lemma 2 *([1]) Let $1 \le \sigma \le m$, $n \ge 1$. Then for all $f \in C^{\sigma-1}[a,b]$; $r = 0, ..., \sigma-1$,*

$$\left\| f^{(r)} - Q_n^{(r)} \right\|_\infty \le \frac{C_2}{n^{\sigma-r-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{b-a}{n} \right), \tag{2}$$

where C_2 depends only on m, σ and $b-a$. Here $\omega_{m-\sigma+1}$ is the usual modulus of smoothness of order $m-\sigma+1$.

We are motivated by

Theorem 3 *([1]) Let h, k, σ, m be integers, $0 \le h \le k \le \sigma-1$, $\sigma \le m$ and let $f \in C^{\sigma-1}[a,b]$. Let $\alpha_j(x) \in B[a,b]$, $j = h, h+1, ..., k$ and suppose that $\alpha_h(x) \ge \alpha > 0$ or $\alpha_h(x) \le \beta < 0$ for all $x \in [a,b]$. Take the linear differential operator*

$$L = \sum_{j=h}^{k} \alpha_j(x) \left[\frac{d^j}{dx^j} \right] \tag{3}$$

and assume, throughout $[a,b]$,

$$L(f) \ge 0. \tag{4}$$

Then, for every integer $n \ge 1$, there is a polynomial spline function $Q_n(x)$ of order m with simple knots at $\left\{ a + i\left(\frac{b-a}{n}\right),\ i = 1, ..., n-1 \right\}$ such that $L(Q_n) \ge 0$ throughout $[a,b]$ and

$$\left\| f^{(r)} - Q_n^{(r)} \right\|_\infty \le \frac{C}{n^{\sigma-k-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{b-a}{n} \right), \quad 0 \le r \le h. \tag{5}$$

Moreover, we find

$$\left\| f^{(r)} - Q_n^{(r)} \right\|_\infty \le \frac{C}{n^{\sigma-r-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{b-a}{n} \right), \quad h+1 \le r \le \sigma-1, \tag{6}$$

where C is a constant independent of f and n. It depends only on m, σ, L, a, b.

Next we specialize on the case of $a = -1$, $b = 1$. That is working on $[-1,1]$. By Lemma 2 we get

Lemma 4 *Let $1 \le \sigma \le m$, $n \ge 1$. Then for all $f \in C^{\sigma-1}([-1,1])$; $j = 0, 1, ..., \sigma-1$,*

$$\left\| f^{(j)} - Q_n^{(j)} \right\|_\infty \le \frac{C_2}{n^{\sigma-j-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{2}{n} \right), \tag{7}$$

where $C_2 := C_2(m,\sigma) := C_1(m) 2^{\sigma-1}$.

Since

$$\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{2}{n}\right)\leq 2^{m-\sigma+1}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right) \tag{8}$$

(see [2], p. 45), we get

Lemma 5 *Let $1\leq\sigma\leq m$, $n\geq 1$. Then for all $f\in C^{\sigma-1}([-1,1])$; $j=0,1,...,\sigma-1$,*

$$\left\|f^{(j)}-Q_n^{(j)}\right\|_\infty\leq\frac{C_2^*}{n^{\sigma-j-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right), \tag{9}$$

where $C_2^*:=C_2^*(m,\sigma):=C_1(m)\,2^m$.

We use a lot in this article Lemma 5.

In this article we extend Theorem 3 over $[-1,1]$ to the fractional level. Indeed here L is replaced by L^*, a linear left Caputo fractional differential operator. Now the monotonicity property is only true on the critical interval $[0,1]$. Simultaneous fractional convergence remains true on all of $[-1,1]$.

We make

Definition 6 *([3], p. 50) Let $\alpha>0$ and $\lceil\alpha\rceil=m$, ($\lceil\cdot\rceil$ ceiling of the number). Consider $f\in C^m([-1,1])$. We define the left Caputo fractional derivative of f of order α as follows:*

$$\left(D_{*-1}^{\alpha}f\right)(x)=\frac{1}{\Gamma(m-\alpha)}\int_{-1}^{x}(x-t)^{m-\alpha-1}f^{(m)}(t)\,dt, \tag{10}$$

for any $x\in[-1,1]$, where Γ is the gamma function.

We set

$$D_{*-1}^{0}f(x)=f(x),$$
$$D_{*-1}^{m}f(x)=f^{(m)}(x),\ \ \forall\ x\in[-1,1]. \tag{11}$$

2 Main Result

Theorem 7 *Let h,k,σ,m be integers, $1\leq\sigma\leq m$, $n\in\mathbb{N}$, with $0\leq h\leq k\leq\sigma-2$ and let $f\in C^{\sigma-1}([-1,1])$, with $f^{(\sigma-1)}$ having modulus of smoothness $\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\delta\right)$ there, $\delta>0$. Let $\alpha_j(x)$, $j=h,h+1,...,k$ be real functions, defined and bounded on $[-1,1]$ and suppose $\alpha_h(x)$ is either $\geq\alpha>0$ or $\leq\beta<0$ on $[0,1]$. Let the real numbers $\alpha_0=0<\alpha_1\leq 1<\alpha_2\leq 2<...<\alpha_{\sigma-2}\leq\sigma-2$. Here $D_{*-1}^{\alpha_j}f$ stands for the left Caputo fractional derivative of f of order α_j anchored at -1. Consider the linear left fractional differential operator*

$$L^*:=\sum_{j=h}^{k}\alpha_j(x)\left[D_{*-1}^{\alpha_j}\right] \tag{12}$$

and suppose, throughout $[0,1]$, $L^*(f) \geq 0$.

Then, for every integer $n \geq 1$, *there exists a polynomial spline function* $Q_n(x)$ *of order* $m > 0$ *with simple knots at* $\left\{-1 + i\frac{2}{n},\ i = 1, ..., n-1\right\}$ *such that* $L^*(Q_n) \geq 0$ *throughout* $[0,1]$, *and*

$$\sup_{-1 \leq x \leq 1} \left| \left(D_{*-1}^{\alpha_j} f\right)(x) - \left(D_{*-1}^{\alpha_j} Q_n\right)(x) \right| \leq$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right), \tag{13}$$

$j = h+1, ..., \sigma-2$.

Set

$$l_j :\equiv \sup_{x \in [-1,1]} \left| \alpha_h^{-1}(x) \alpha_j(x) \right|, \quad h \leq j \leq k. \tag{14}$$

When $j = 1, ..., h$ *we derive*

$$\max_{-1 \leq x \leq 1} \left| \left(D_{*-1}^{\alpha_j} f\right)(x) - \left(D_{*-1}^{\alpha_j} Q_n\right)(x) \right| \leq \frac{C_2^*}{n^{\sigma-k-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right) \cdot$$

$$\left[\left(\sum_{\tau=h}^{k} l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1)} \right) \left(\sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda! \Gamma(h-\alpha_j-\lambda+1)} \right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \right]. \tag{15}$$

Finally it holds

$$\sup_{-1 \leq x \leq 1} \left| f(x) - Q_n(x) \right| \leq$$

$$\frac{C_2^*}{n^{\sigma-k-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right) \left[\frac{1}{h!} \sum_{\tau=h}^{k} l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1)} + 1 \right]. \tag{16}$$

Proof. Set $\alpha_0 = 0$, thus $\lceil \alpha_0 \rceil = 0$. We have $\lceil \alpha_j \rceil = j$, $j = 1, ..., \sigma-2$.
Let Q_n *as in Lemma* 5.
We notice that $(x \in [-1,1])$

$$\left| \left(D_{*-1}^{\alpha_j} f\right)(x) - \left(D_{*-1}^{\alpha_j} Q_n\right)(x) \right| =$$

$$\frac{1}{\Gamma(j-\alpha_j)} \left| \int_{-1}^{x} (x-t)^{j-\alpha_j-1} f^{(j)}(t)\, dt - \int_{-1}^{x} (x-t)^{j-\alpha_j-1} Q_n^{(j)}(t)\, dt \right| = \tag{17}$$

$$\frac{1}{\Gamma(j-\alpha_j)} \left| \int_{-1}^{x} (x-t)^{j-\alpha_j-1} \left(f^{(j)}(t) - Q_n^{(j)}(t)\right) dt \right| \leq$$

$$\frac{1}{\Gamma(j-\alpha_j)} \int_{-1}^{x} (x-t)^{j-\alpha_j-1} \left| f^{(j)}(t) - Q_n^{(j)}(t) \right| dt \overset{(9)}{\leq} \tag{18}$$

$$\frac{1}{\Gamma(j-\alpha_j)} \left(\int_{-1}^{x} (x-t)^{j-\alpha_j-1} dt \right) \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right) =$$

$$\frac{1}{\Gamma\left(j-\alpha_j\right)}\frac{(x+1)^{j-\alpha_j}}{\left(j-\alpha_j\right)}\frac{C_2^*}{n^{\sigma-j-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)= \tag{19}$$

$$\frac{(x+1)^{j-\alpha_j}}{\Gamma\left(j-\alpha_j+1\right)}\frac{C_2^*}{n^{\sigma-j-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)\leq$$

$$\frac{2^{j-\alpha_j}}{\Gamma\left(j-\alpha_j+1\right)}\frac{C_2^*}{n^{\sigma-j-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right). \tag{20}$$

Hence

$$\left\|D_{*-1}^{\alpha_j}f-D_{*-1}^{\alpha_j}Q_n\right\|_{\infty,[-1,1]}\leq\frac{2^{j-\alpha_j}}{\Gamma\left(j-\alpha_j+1\right)}\frac{C_2^*}{n^{\sigma-j-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right), \tag{21}$$

$j=0,1,...,\sigma-2$.

We set

$$\rho_n:=C_2^*\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)\left(\sum_{j=h}^{k}l_j\frac{2^{j-\alpha_j}}{\Gamma\left(j-\alpha_j+1\right)n^{\sigma-j-1}}\right). \tag{22}$$

I. Suppose, throughout $[0,1]$, $\alpha_h(x)\geq\alpha>0$. Let $Q_n(x)$, $x\in[-1,1]$, the polynomial spline of order $m>0$ with simple knots at the points x_{in}, $i=1,...,n-1$, on $[-1,1]$ ($x_{in}=-1+i\frac{2}{n}$, $i=0,1,...,n$, here $\overline{\Delta}_n=\frac{2}{n}$), so that

$$\max_{-1\leq x\leq1}\left|D_{*-1}^{\alpha_j}\left(f(x)+\rho_n\frac{x^h}{h!}\right)-\left(D_{*-1}^{\alpha_j}Q_n\right)(x)\right|\leq$$

$$\frac{2^{j-\alpha_j}}{\Gamma\left(j-\alpha_j+1\right)}\frac{C_2^*}{n^{\sigma-j-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right), \tag{23}$$

$j=0,1,...,\sigma-2$.

When $j=h+1,...,\sigma-2$, then

$$\max_{-1\leq x\leq1}\left|\left(D_{*-1}^{\alpha_j}f\right)(x)-\left(D_{*-1}^{\alpha_j}Q_n\right)(x)\right|\leq$$

$$\frac{2^{j-\alpha_j}}{\Gamma\left(j-\alpha_j+1\right)}\frac{C_2^*}{n^{\sigma-j-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right), \tag{24}$$

proving (13).

For $j=1,...,h$ we find that

$$D_{*-1}^{\alpha_j}\left(\frac{x^h}{h!}\right)=\sum_{\lambda=0}^{h-j}\frac{(-1)^{\lambda}(x+1)^{h-\alpha_j-\lambda}}{\lambda!\Gamma\left(h-\alpha_j-\lambda+1\right)}. \tag{25}$$

Therefore we get from (23)

$$\max_{-1\leq x\leq1}\left|\left(D_{*-1}^{\alpha_j}f\right)(x)+\rho_n\left(\sum_{\lambda=0}^{h-j}\frac{(-1)^{\lambda}(x+1)^{h-\alpha_j-\lambda}}{\lambda!\Gamma\left(h-\alpha_j-\lambda+1\right)}\right)-\left(D_{*-1}^{\alpha_j}Q_n\right)(x)\right|\leq \tag{26}$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_2^*}{n^{\sigma-j-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right),$$

$j=1,...,h$.

Therefore we get for $j=1,...,h$, that

$$\max_{-1\le x\le 1}\left|\left(D_{*-1}^{\alpha_j}f\right)(x)-\left(D_{*-1}^{\alpha_j}Q_n\right)(x)\right|\le \tag{27}$$

$$\rho_n\left(\sum_{\lambda=0}^{h-j}\frac{2^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)}\right)+\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_2^*}{n^{\sigma-j-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)=$$

$$C_2^*\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)\left(\sum_{\overline{j}=h}^{k}l_{\overline{j}}\frac{2^{\overline{j}-\alpha_{\overline{j}}}}{\Gamma\left(\overline{j}-\alpha_{\overline{j}}+1\right)n^{\sigma-\overline{j}-1}}\right)\cdot$$

$$\left(\sum_{\lambda=0}^{h-j}\frac{2^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)}\right)+\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_2^*}{n^{\sigma-j-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)=$$

$$C_2^*\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)\left[\left(\sum_{\overline{j}=h}^{k}l_{\overline{j}}\frac{2^{\overline{j}-\alpha_{\overline{j}}}}{\Gamma\left(\overline{j}-\alpha_{\overline{j}}+1\right)}\frac{1}{n^{\sigma-\overline{j}-1}}\right)\cdot\right. \tag{28}$$

$$\left.\left(\sum_{\lambda=0}^{h-j}\frac{2^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)}\right)+\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{1}{n^{\sigma-j-1}}\right]\le$$

$$C_2^*\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)\frac{1}{n^{\sigma-k-1}}\left[\left(\sum_{\overline{j}=h}^{k}l_{\overline{j}}\frac{2^{\overline{j}-\alpha_{\overline{j}}}}{\Gamma\left(\overline{j}-\alpha_{\overline{j}}+1\right)}\right)\cdot\right. \tag{29}$$

$$\left.\left(\sum_{\lambda=0}^{h-j}\frac{2^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)}\right)+\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\right].$$

Hence for $j=1,...,h$ we derived (15):

$$\max_{-1\le x\le 1}\left|\left(D_{*-1}^{\alpha_j}f\right)(x)-\left(D_{*-1}^{\alpha_j}Q_n\right)(x)\right|\le\frac{C_2^*}{n^{\sigma-k-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)\cdot$$

$$\left[\left(\sum_{\tau=h}^{k}l_\tau\frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1)}\right)\left(\sum_{\lambda=0}^{h-j}\frac{2^{h-\alpha_j-\lambda}}{\lambda!\Gamma(h-\alpha_j-\lambda+1)}\right)+\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\right]. \tag{30}$$

When $j=0$ from (23) we obtain

$$\max_{-1\le x\le 1}\left|f(x)+\rho_n\frac{x^h}{h!}-Q_n(x)\right|\le\frac{C_2^*}{n^{\sigma-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right). \tag{31}$$

And

$$\max_{-1\le x\le 1}\left|f(x)-Q_n(x)\right|\le\frac{\rho_n}{h!}+\frac{C_2^*}{n^{\sigma-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)= \tag{32}$$

$$\frac{C_2^*}{h!}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)\left(\sum_{\tau=h}^{k} l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1)\, n^{\sigma-\tau-1}}\right)$$

$$+\frac{C_2^*}{n^{\sigma-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)=$$

$$C_2^*\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)\left[\frac{1}{h!}\sum_{\tau=h}^{k} l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1)\, n^{\sigma-\tau-1}}+\frac{1}{n^{\sigma-1}}\right]\leq \tag{33}$$

$$\frac{C_2^*}{n^{\sigma-k-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)\left[\frac{1}{h!}\sum_{\tau=h}^{k} l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1)}+1\right].$$

Proving

$$\max_{-1\leq x\leq 1}\left|f(x)-Q_n(x)\right|\leq$$

$$\frac{C_2^*}{n^{\sigma-k-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)\left[\frac{1}{h!}\sum_{\tau=h}^{k} l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau-\alpha_\tau+1)}+1\right], \tag{34}$$

So that (16) is established.

Also if $0\leq x\leq 1$, then

$$\alpha_h^{-1}(x)\,L^*(Q_n(x))=\alpha_h^{-1}(x)\,L^*(f(x))+\rho_n\frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}+ \tag{35}$$

$$\sum_{j=h}^{k}\alpha_h^{-1}(x)\,\alpha_j(x)\left[D_{*-1}^{\alpha_j}Q_n(x)-D_{*-1}^{\alpha_j}f(x)-\frac{\rho_n}{h!}D_{*-1}^{\alpha_j}x^h\right]\overset{(23)}{\geq}$$

$$\rho_n\frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}-\left(\sum_{j=h}^{k} l_j\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_2^*}{n^{\sigma-j-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)\right)=$$

$$\rho_n\frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}-\rho_n=\rho_n\left[\frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}-1\right]=$$

$$\rho_n\left[\frac{(x+1)^{h-\alpha_h}-\Gamma(h-\alpha_h+1)}{\Gamma(h-\alpha_h+1)}\right]\geq\rho_n\left[\frac{1-\Gamma(h-\alpha_h+1)}{\Gamma(h-\alpha_h+1)}\right]\geq 0. \tag{36}$$

Explanation: We know that $\Gamma(1)=1$, $\Gamma(2)=1$, and Γ is convex and positive on $(0,\infty)$. Here $0\leq h-\alpha_h<1$ and $1\leq h-\alpha_h+1<2$. Thus $\Gamma(h-\alpha_h+1)\leq 1$ and $1-\Gamma(h-\alpha_h+1)\geq 0$. Hence $L^*(Q_n(x))\geq 0$, $x\in[0,1]$.

II. Suppose on $[0,1]$ that $\alpha_h(x)\leq\beta<0$. Let $Q_n(x)$, $x\in[-1,1]$, be the polynomial spline of order $m>0$, (as before), so that

$$\max_{-1\leq x\leq 1}\left|D_{*-1}^{\alpha_j}\left(f(x)-\rho_n\frac{x^h}{h!}\right)-\left(D_{*-1}^{\alpha_j}Q_n\right)(x)\right|\leq$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_2^*}{n^{\sigma-j-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right), \tag{37}$$

$j=0,1,...,\sigma-2$.

Similarly as before we obtain again inequalities of convergence (13), (15) and (16).

Also if $0\leq x\leq 1$, then

$$\alpha_h^{-1}(x)\,L^*(Q_n(x))=\alpha_h^{-1}(x)\,L^*(f(x))-\rho_n\frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}+ \tag{38}$$

$$\sum_{j=h}^{k}\alpha_h^{-1}(x)\,\alpha_j(x)\left[D_{*-1}^{\alpha_j}Q_n(x)-D_{*-1}^{\alpha_j}f(x)+\frac{\rho_n}{h!}\left(D_{*-1}^{\alpha_j}x^h\right)\right]\overset{(37)}{\leq}$$

$$-\rho_n\frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}+\sum_{j=h}^{k}l_j\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_2^*}{n^{\sigma-j-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)= \tag{39}$$

$$\rho_n\left(1-\frac{(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}\right)=\rho_n\left(\frac{\Gamma(h-\alpha_h+1)-(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}\right)\leq \tag{40}$$

$$\rho_n\left(\frac{1-(x+1)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}\right)\leq 0,$$

and hence again $L^*(Q_n(x))\geq 0$, $x\in[0,1]$. ■

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