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Spline right fractional monotone approximation involving right fractional differential operators

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Abstract

Let $f \in C^s([-1,1])$, s$\in \mathbb{N}$ and L^* be a linear right fractional differential operator such that $L^*(f) \geq 0$ on $[-1,0]$. Then there exists a sequence Q_n, $n \in \mathbb{N}$ of polynomial splines with equally spaced knots of given fixed order such that $L^*(Q_n) \geq 0$ on $[-1,0]$. Furthermore f is approximated with rates right fractionally and simultaneously by Q_n in the uniform norm. This constrained right fractional approximation on $[-1,1]$ is given via inequalities invoving a higher modulus of smoothness of $f^{(s)}$.

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1 Introduction

Let $[a,b] \subset \mathbb{R}$ and for $n \geq 1$ consider the partition Δ_n with points $x_{in} = a + i\left(\frac{b-a}{n}\right)$, $i = 0,1,...,n$. Hence $\overline{\Delta}_n \equiv \max_{1\leq i\leq n}(x_{in} - x_{i-1,n}) = \frac{b-a}{n}$.

Let $S_m(\Delta_n)$ be the space of polynomial splines of order $m > 0$ with simple knots at the points x_{in}, $i = 1,...,n-1$. Then there exists a linear operator $Q_n : Q_n \equiv Q_n(f)$, mapping $B[a,b]$: the space of bounded real valued functions f on $[a,b]$, into $S_m(\Delta_n)$ (see [4], p. 224, Theorem 6.18).

From the same reference [4], p. 227, Corollary 6.21, we get

Corollary 1 *Let $1 \leq \sigma \leq m$, $n \geq 1$. Then for all $f \in C^{\sigma-1}[a,b]$; $r = 0, ..., \sigma - 1$,*

$$\left\| f^{(r)} - Q_n^{(r)} \right\|_\infty \leq C_1 \left(\frac{b-a}{n} \right)^{\sigma-r-1} \omega_{m-\sigma+1} \left(f^{(b-1)}, \frac{b-a}{n} \right), \tag{1}$$

where C_1 depends only on m, $C_1 = C_1(m)$.

By denoting $C_2 = C_1 \max_{0 \leq r \leq \sigma-1} (b-a)^{\sigma-r-1}$ we obtain

Lemma 2 *([1]) Let $1 \leq \sigma \leq m$, $n \geq 1$. Then for all $f \in C^{\sigma-1}[a,b]$; $r = 0, ..., \sigma - 1$,*

$$\left\| f^{(r)} - Q_n^{(r)} \right\|_\infty \leq \frac{C_2}{n^{\sigma-r-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{b-a}{n} \right), \tag{2}$$

where C_2 depends only on m, σ and $b-a$. Here $\omega_{m-\sigma+1}$ is the usual modulus of smoothness of order $m - \sigma + 1$.

We are motivated by

Theorem 3 *([1]) Let h, k, σ, m be integers, $0 \leq h \leq k \leq \sigma - 1$, $\sigma \leq m$ and let $f \in C^{\sigma-1}[a,b]$. Let $\alpha_j(x) \in B[a,b]$, $j = h, h+1, ..., k$ and suppose that $\alpha_h(x) \geq \alpha > 0$ or $\alpha_h(x) \leq \beta < 0$ for all $x \in [a,b]$. Take the linear differential operator*

$$L = \sum_{j=h}^{k} \alpha_j(x) \left[\frac{d^j}{dx^j} \right] \tag{3}$$

and assume, throughout $[a,b]$,

$$L(f) \geq 0. \tag{4}$$

Then, for every integer $n \geq 1$, there is a polynomial spline function $Q_n(x)$ of order m with simple knots at $\left\{ a + i\left(\frac{b-a}{n}\right), \ i = 1, ..., n-1 \right\}$ such that $L(Q_n) \geq 0$ throughout $[a,b]$ and

$$\left\| f^{(r)} - Q_n^{(r)} \right\|_\infty \leq \frac{C}{n^{\sigma-k-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{b-a}{n} \right), \quad 0 \leq r \leq h. \tag{5}$$

Moreover, we find

$$\left\| f^{(r)} - Q_n^{(r)} \right\|_\infty \leq \frac{C}{n^{\sigma-r-1}} \omega_{m-\sigma+1} \left(f^{(\sigma-1)}, \frac{b-a}{n} \right), \quad h+1 \leq r \leq \sigma - 1, \tag{6}$$

where C is a constant independent of f and n. It depends only on m, σ, L, a, b.

Next we specialize on the case of $a = -1$, $b = 1$. That is working on $[-1, 1]$. By Lemma 2 we get

Lemma 4 *Let $1 \leq \sigma \leq m$, $n \geq 1$. Then for all $f \in C^{\sigma-1}([-1,1])$; $j = 0,1,...,\sigma-1$,*

$$\left\| f^{(j)} - Q_n^{(j)} \right\|_\infty \leq \frac{C_2}{n^{\sigma-j-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{2}{n} \right), \tag{7}$$

where $C_2 := C_2(m,\sigma) := C_1(m)\, 2^{\sigma-1}$.

Since

$$\omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{2}{n} \right) \leq 2^{m-\sigma+1} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n} \right) \tag{8}$$

(see [2], p. 45), we get

Lemma 5 *Let $1 \leq \sigma \leq m$, $n \geq 1$. Then for all $f \in C^{\sigma-1}([-1,1])$; $j = 0,1,...,\sigma-1$,*

$$\left\| f^{(j)} - Q_n^{(j)} \right\|_\infty \leq \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n} \right), \tag{9}$$

where $C_2^* := C_2^*(m,\sigma) := C_1(m)\, 2^m$.

We use a lot in this article Lemma 5.

In this article we extend Theorem 3 over $[-1,1]$ to the right fractional level. Indeed here L is replaced by L^*, a linear right Caputo fractional differential operator. Now the monotonicity property is only true on the critical interval $[-1,0]$. Simultaneous fractional convergence remains true on all of $[-1,1]$.

We make

Definition 6 *([3]) Let $\alpha > 0$ and $\lceil \alpha \rceil = m$, ($\lceil \cdot \rceil$ ceiling of the number). Consider $f \in C^m([-1,1])$. We define the right Caputo fractional derivative of f of order α as follows:*

$$\left(D_{1-}^{\alpha} f \right)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^1 (t-x)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{10}$$

for any $x \in [-1,1]$, where Γ is the gamma function.

We set

$$D_{1-}^{0} f(x) := f(x),$$

$$D_{1-}^{m} f(x) := (-1)^m f^{(m)}(x), \quad \forall\, x \in [-1,1]. \tag{11}$$

2 Main Result

Theorem 7 *Let h, k, σ, m be integers, $1 \leq \sigma \leq m$, $n \in \mathbb{N}$, h is even, with $0 \leq h \leq k \leq \sigma - 2$ and let $f \in C^{\sigma-1}([-1,1])$, with $f^{(\sigma-1)}$ having modulus of smoothness $\omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \delta\right)$ there, $\delta > 0$. Let $\alpha_j(x)$, $j = h, h+1, ..., k$ be real functions, defined and bounded on $[-1,1]$ and suppose $\alpha_h(x)$ is either $\geq \alpha > 0$ or $\leq \beta < 0$ on $[-1,0]$. Let the real numbers $\alpha_0 = 0 < \alpha_1 < 1 < \alpha_2 < 2 < ... < \alpha_{\sigma-2} < \sigma - 2$. Here $D_{1-}^{\alpha_j} f$ stands for the right Caputo fractional derivative of f of order α_j anchored at 1. Consider the linear right fractional differential operator*

$$L^* := \sum_{j=h}^{k} \alpha_j(x) \left[D_{1-}^{\alpha_j}\right] \tag{12}$$

and suppose, throughout $[-1,0]$, $L^*(f) \geq 0$.

Then, for every integer $n \geq 1$, there exists a polynomial spline function $Q_n(x)$ of order $m > 0$ with simple knots at $\left\{-1 + i\frac{2}{n},\ i = 1, ..., n-1\right\}$ such that $L^*(Q_n) \geq 0$ throughout $[-1,0]$, and

$$\sup_{-1 \leq x \leq 1} \left| \left(D_{1-}^{\alpha_j} f\right)(x) - \left(D_{1-}^{\alpha_j} Q_n\right)(x) \right| \leq$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right), \tag{13}$$

$j = h+1, ..., \sigma - 2$.

Set

$$l_j :\equiv \sup_{x \in [-1,1]} \left| \alpha_h^{-1}(x) \alpha_j(x) \right|, \quad h \leq j \leq k. \tag{14}$$

When $j = 1, ..., h$ we derive

$$\sup_{-1 \leq x \leq 1} \left| \left(D_{1-}^{\alpha_j} f\right)(x) - \left(D_{1-}^{\alpha_j} Q_n\right)(x) \right| \leq \frac{C_2^*}{n^{\sigma-k-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right) \cdot$$

$$\left[\left(\sum_{\tau=h}^{k} l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau - \alpha_\tau + 1)}\right)\left(\sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda! \Gamma(h - \alpha_j - \lambda + 1)}\right) + \frac{2^{j-\alpha_j}}{\Gamma(j - \alpha_j + 1)}\right]. \tag{15}$$

Finally it holds

$$\sup_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq$$

$$\frac{C_2^*}{n^{\sigma-k-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right) \left[\frac{1}{h!} \sum_{\tau=h}^{k} l_\tau \frac{2^{\tau-\alpha_\tau}}{\Gamma(\tau - \alpha_\tau + 1)} + 1\right]. \tag{16}$$

Proof. Set $\alpha_0 = 0$, thus $\lceil \alpha_0 \rceil = 0$. We have $\lceil \alpha_j \rceil = j$, $j = 1, ..., \sigma - 2$.
Let Q_n *as in Lemma* 5.

We notice that ($x \in [-1,1]$)

$$\left|\left(D_{1-}^{\alpha_j} f\right)(x) - \left(D_{1-}^{\alpha_j} Q_n\right)(x)\right| =$$

$$\frac{1}{\Gamma(j-\alpha_j)}\left|\int_x^1 (t-x)^{j-\alpha_j-1} f^{(j)}(t)\,dt - \int_x^1 (t-x)^{j-\alpha_j-1} Q_n^{(j)}(t)\,dt\right| = \quad (17)$$

$$\frac{1}{\Gamma(j-\alpha_j)}\left|\int_x^1 (t-x)^{j-\alpha_j-1}\left(f^{(j)}(t) - Q_n^{(j)}(t)\right)dt\right| \leq$$

$$\frac{1}{\Gamma(j-\alpha_j)}\int_x^1 (t-x)^{j-\alpha_j-1}\left|f^{(j)}(t) - Q_n^{(j)}(t)\right| dt \overset{(9)}{\leq} \quad (18)$$

$$\frac{1}{\Gamma(j-\alpha_j)}\left(\int_x^1 (t-x)^{j-\alpha_j-1}\,dt\right)\frac{C_2^*}{n^{\sigma-j-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right) =$$

$$\frac{1}{\Gamma(j-\alpha_j)}\frac{(1-x)^{j-\alpha_j}}{(j-\alpha_j)}\frac{C_2^*}{n^{\sigma-j-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right) = \quad (19)$$

$$\frac{(1-x)^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_2^*}{n^{\sigma-j-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right) \leq$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_2^*}{n^{\sigma-j-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right). \quad (20)$$

Hence

$$\left\|D_{1-}^{\alpha_j} f - D_{1-}^{\alpha_j} Q_n\right\|_{\infty,[-1,1]} \leq \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_2^*}{n^{\sigma-j-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right), \quad (21)$$

$j = 0, 1, ..., \sigma - 2$.

We set

$$\rho_n := C_2^*\omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right)\left(\sum_{j=h}^{k} l_j \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)\,n^{\sigma-j-1}}\right). \quad (22)$$

I. Suppose, throughout $[-1,0]$, $\alpha_h(x) \geq \alpha > 0$. Let $Q_n(x)$, $x \in [-1,1]$, the polynomial spline of order $m > 0$ with simple knots at the points x_{in}, $i = 1, ..., n-1$, on $[-1,1]$ ($x_{in} = -1 + i\frac{2}{n}$, $i = 0, 1, ..., n$, here $\overline{\Delta}_n = \frac{2}{n}$), so that

$$\max_{-1\leq x\leq 1}\left|D_{1-}^{\alpha_j}\left(f(x) + \rho_n \frac{x^h}{h!}\right) - \left(D_{1-}^{\alpha_j} Q_n\right)(x)\right| \leq$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{C_2^*}{n^{\sigma-j-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right), \quad (23)$$

$j = 0, 1, ..., \sigma - 2$.

When $j = h+1, ..., \sigma - 2$, then

$$\max_{-1 \le x \le 1} \left| \left(D_{1-}^{\alpha_j} f \right)(x) - \left(D_{1-}^{\alpha_j} Q_n \right)(x) \right| \le$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n} \right), \tag{24}$$

proving (13).

For $j = 1, ..., h$ we find that

$$D_{1-}^{\alpha_j}\left(\frac{x^h}{h!} \right) = (-1)^h \sum_{\lambda=0}^{h-j} \frac{(-1)^\lambda (1-x)^{h-\alpha_j-\lambda}}{\lambda! \Gamma(h-\alpha_j-\lambda+1)}. \tag{25}$$

Therefore we get from (23)

$$\max_{-1 \le x \le 1} \left| \left(D_{1-}^{\alpha_j} f \right)(x) + \rho_n \left((-1)^h \sum_{\lambda=0}^{h-j} \frac{(-1)^\lambda (1-x)^{h-\alpha_j-\lambda}}{\lambda! \Gamma(h-\alpha_j-\lambda+1)} \right) - \left(D_{1-}^{\alpha_j} Q_n \right)(x) \right| \le \tag{26}$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n} \right),$$

$j = 1, ..., h$.

Therefore we get for $j = 1, ..., h$, that

$$\max_{-1 \le x \le 1} \left| \left(D_{1-}^{\alpha_j} f \right)(x) - \left(D_{1-}^{\alpha_j} Q_n \right)(x) \right| \le \tag{27}$$

$$\rho_n \left(\sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda! \Gamma(h-\alpha_j-\lambda+1)} \right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n} \right) =$$

$$C_2^* \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n} \right) \left(\sum_{\bar{j}=h}^{k} l_{\bar{j}} \frac{2^{\bar{j}-\alpha_{\bar{j}}}}{\Gamma\left(\bar{j}-\alpha_{\bar{j}}+1\right) n^{\sigma-\bar{j}-1}} \right) \cdot$$

$$\left(\sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda! \Gamma(h-\alpha_j-\lambda+1)} \right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n} \right) =$$

$$C_2^* \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n} \right) \left[\left(\sum_{\bar{j}=h}^{k} l_{\bar{j}} \frac{2^{\bar{j}-\alpha_{\bar{j}}}}{\Gamma\left(\bar{j}-\alpha_{\bar{j}}+1\right)} \frac{1}{n^{\sigma-\bar{j}-1}} \right) \cdot \right. \tag{28}$$

$$\left. \left(\sum_{\lambda=0}^{h-j} \frac{2^{h-\alpha_j-\lambda}}{\lambda! \Gamma(h-\alpha_j-\lambda+1)} \right) + \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{1}{n^{\sigma-j-1}} \right] \le$$

$$C_2^* \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n} \right) \frac{1}{n^{\sigma-k-1}} \left[\left(\sum_{\bar{j}=h}^{k} l_{\bar{j}} \frac{2^{\bar{j}-\alpha_{\bar{j}}}}{\Gamma\left(\bar{j}-\alpha_{\bar{j}}+1\right)} \right) \cdot \right. \tag{29}$$

$$\left(\sum_{\lambda=0}^{h-j}\frac{2^{h-\alpha_j-\lambda}}{\lambda!\Gamma\left(h-\alpha_j-\lambda+1\right)}\right)+\frac{2^{j-\alpha_j}}{\Gamma\left(j-\alpha_j+1\right)}\Bigg].$$

Hence for $j=1,...,h$ we derived (15):

$$\max_{-1\leq x\leq 1}\left|\left(D_{1-}^{\alpha_j}f\right)(x)-\left(D_{1-}^{\alpha_j}Q_n\right)(x)\right|\leq\frac{C_2^*}{n^{\sigma-k-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)\cdot$$

$$\left[\left(\sum_{\tau=h}^{k}l_\tau\frac{2^{\tau-\alpha_\tau}}{\Gamma\left(\tau-\alpha_\tau+1\right)}\right)\left(\sum_{\lambda=0}^{h-j}\frac{2^{h-\alpha_j-\lambda}}{\lambda!\Gamma\left(h-\alpha_j-\lambda+1\right)}\right)+\frac{2^{j-\alpha_j}}{\Gamma\left(j-\alpha_j+1\right)}\right].\tag{30}$$

When $j=0$ from (23) we obtain

$$\max_{-1\leq x\leq 1}\left|f(x)+\rho_n\frac{x^h}{h!}-Q_n(x)\right|\leq\frac{C_2^*}{n^{\sigma-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right).\tag{31}$$

And

$$\max_{-1\leq x\leq 1}\left|f(x)-Q_n(x)\right|\leq\frac{\rho_n}{h!}+\frac{C_2^*}{n^{\sigma-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)=\tag{32}$$

$$\frac{C_2^*}{h!}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)\left(\sum_{\tau=h}^{k}l_\tau\frac{2^{\tau-\alpha_\tau}}{\Gamma\left(\tau-\alpha_\tau+1\right)n^{\sigma-\tau-1}}\right)$$

$$+\frac{C_2^*}{n^{\sigma-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)=$$

$$C_2^*\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)\left[\frac{1}{h!}\sum_{\tau=h}^{k}l_\tau\frac{2^{\tau-\alpha_\tau}}{\Gamma\left(\tau-\alpha_\tau+1\right)n^{\sigma-\tau-1}}+\frac{1}{n^{\sigma-1}}\right]\leq\tag{33}$$

$$\frac{C_2^*}{n^{\sigma-k-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)\left[\frac{1}{h!}\sum_{\tau=h}^{k}l_\tau\frac{2^{\tau-\alpha_\tau}}{\Gamma\left(\tau-\alpha_\tau+1\right)}+1\right].$$

Proving

$$\max_{-1\leq x\leq 1}\left|f(x)-Q_n(x)\right|\leq$$

$$\frac{C_2^*}{n^{\sigma-k-1}}\omega_{m-\sigma+1}\left(f^{(\sigma-1)},\frac{1}{n}\right)\left[\frac{1}{h!}\sum_{\tau=h}^{k}l_\tau\frac{2^{\tau-\alpha_\tau}}{\Gamma\left(\tau-\alpha_\tau+1\right)}+1\right],\tag{34}$$

So that (16) is established.

Also if $-1\leq x\leq 0$, then

$$\alpha_h^{-1}(x)L^*\left(Q_n(x)\right)=\alpha_h^{-1}(x)L^*\left(f(x)\right)+\rho_n\frac{(1-x)^{h-\alpha_h}}{\Gamma\left(h-\alpha_h+1\right)}+\tag{35}$$

$$\sum_{j=h}^{k}\alpha_h^{-1}(x)\alpha_j(x)\left[D_{1-}^{\alpha_j}Q_n(x)-D_{1-}^{\alpha_j}f(x)-\frac{\rho_n}{h!}D_{1-}^{\alpha_j}x^h\right]\overset{(23)}{\geq}$$

$$\rho_n \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} - \left(\sum_{j=h}^{k} l_j \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right) \right) =$$

$$\rho_n \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} - \rho_n = \rho_n \left[\frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} - 1 \right] =$$

$$\rho_n \left[\frac{(1-x)^{h-\alpha_h} - \Gamma(h-\alpha_h+1)}{\Gamma(h-\alpha_h+1)} \right] \geq \rho_n \left[\frac{1-\Gamma(h-\alpha_h+1)}{\Gamma(h-\alpha_h+1)} \right] \geq 0. \tag{36}$$

Explanation: We know that $\Gamma(1) = 1$, $\Gamma(2) = 1$, and Γ is convex and positive on $(0, \infty)$. Here $0 \leq h-\alpha_h < 1$ and $1 \leq h-\alpha_h+1 < 2$. Thus $\Gamma(h-\alpha_h+1) \leq 1$ and $1-\Gamma(h-\alpha_h+1) \geq 0$. Hence $L^*(Q_n(x)) \geq 0$, $x \in [-1, 0]$.

II. Suppose on $[-1, 0]$ that $\alpha_h(x) \leq \beta < 0$. Let $Q_n(x)$, $x \in [-1, 1]$, be the polynomial spline of order $m > 0$, (as before), so that

$$\max_{-1 \leq x \leq 1} \left| D_{1-}^{\alpha_j} \left(f(x) - \rho_n \frac{x^h}{h!} \right) - \left(D_{1-}^{\alpha_j} Q_n \right)(x) \right| \leq$$

$$\frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right), \tag{37}$$

$j = 0, 1, ..., \sigma - 2$.

Similarly as before we obtain again inequalities of convergence (13), (15) and (16).

Also if $-1 \leq x \leq 0$, then

$$\alpha_h^{-1}(x) L^*(Q_n(x)) = \alpha_h^{-1}(x) L^*(f(x)) - \rho_n \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} + \tag{38}$$

$$\sum_{j=h}^{k} \alpha_h^{-1}(x) \alpha_j(x) \left[D_{1-}^{\alpha_j} Q_n(x) - D_{1-}^{\alpha_j} f(x) + \frac{\rho_n}{h!} \left(D_{1-}^{\alpha_j} x^h \right) \right] \overset{(37)}{\leq}$$

$$-\rho_n \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} + \sum_{j=h}^{k} l_j \frac{2^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)} \frac{C_2^*}{n^{\sigma-j-1}} \omega_{m-\sigma+1}\left(f^{(\sigma-1)}, \frac{1}{n}\right) = \tag{39}$$

$$\rho_n \left(1 - \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} \right) = \rho_n \left(\frac{\Gamma(h-\alpha_h+1) - (1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} \right) \leq \tag{40}$$

$$\rho_n \left(\frac{1-(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} \right) \leq 0,$$

and hence again $L^*(Q_n(x)) \geq 0$, $x \in [-1, 0]$. ■

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