Received 17/03/14

Bivariate Left Fractional Polynomial Monotone Approximation

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Let $f \in C^{r,p}\left([0,1]^2\right)$, $r,p \in \mathbb{N}$, and let L^* be a linear left fractional mixed partial differential operator such that $L^*(f) \geq 0$, for all (x,y) in a critical region of $[0,1]^2$ that depends on L^*. Then there exists a sequence of two-dimensional polynomials $Q_{\overline{m_1},\overline{m_2}}(x,y)$ with $L^*\left(Q_{\overline{m_1},\overline{m_2}}(x,y)\right) \geq 0$ there, where $\overline{m_1},\overline{m_2} \in \mathbb{N}$ such that $\overline{m_1} > r$, $\overline{m_2} > p$, so that f is approximated left fractionally simultaneously and uniformly by $Q_{\overline{m_1},\overline{m_2}}$ on $[0,1]^2$. This restricted left fractional approximation is accomplished quantitatively by the use of a suitable integer partial derivatives two-dimensional first modulus of continuity.

2010 AMS Mathematics Subject Classification : 26A33, 41A10, 41A17, 41A25, 41A28, 41A29, 41A63.

Keywords and Phrases: Monotone Approximation, Caputo fractional derivative, fractional linear differential operator, two-dimensional modulus of continuity.

1 Introduction

The topic of monotone approximation started in [5] has become a major trend in approximation theory. A typical problem in this subject is: given a positive integer k, approximate a given function whose kth derivative is ≥ 0 by polynomials having this property.

In [2] the authors replaced the kth derivative with a linear differential operator of order k. We mention this motivating result.

Theorem 1 *Let h,k,p be integers, $0 \leq h \leq k \leq p$ and let f be a real function, $f^{(p)}$ continuous in $[-1,1]$ with modulus of continuity $\omega\left(f^{(p)},x\right)$ there. Let*

$a_j(x)$, $j = h, h+1, ..., k$ be real functions, defined and bounded on $[-1,1]$ and assume $a_h(x)$ is either $\geq$ some number $\alpha > 0$ or $\leq$ some number $\beta < 0$ throughout $[-1,1]$. Consider the operator

$$L = \sum_{j=h}^{k} a_j(x) \left[\frac{d^j}{dx^j} \right]$$

and suppose, throughout $[-1,1]$,

$$L(f) \geq 0. \tag{1}$$

Then, for every integer $n \geq 1$, there is a real polynomial $Q_n(x)$ of degree $\leq n$ such that

$$L(Q_n) \geq 0 \text{ throughout } [-1,1]$$

and

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq C n^{k-p} \omega \left(f^{(p)}, \frac{1}{n} \right),$$

where C is independent of n or f.

We need

Definition 2 *(D.D. Stancu [6]). Let $f \in C\left([0,1]^2\right)$, $[0,1]^2 = [0,1] \times [0,1]$, where (x_1, y_1), $(x_2, y_2) \in [0,1]^2$ and $\delta_1, \delta_2 \geq 0$. The first modulus of continuity of f is defined as follows:*

$$\omega_1(f, \delta_1, \delta_2) = \sup_{\substack{|x_1 - x_2| \leq \delta_1 \\ |y_1 - y_2| \leq \delta_2}} |f(x_1, y_1) - f(x_2, y_2)|.$$

Definition 3 *Let f be a real-valued function defined on $[0,1]^2$ and let m, n be two positive integers. Let $B_{m,n}$ be the Bernstein (polynomial) operator of order (m,n) given by*

$$B_{m,n}(f; x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} f\left(\frac{i}{m}, \frac{j}{n}\right) \cdot \binom{m}{i} \cdot \binom{n}{j} \cdot x^i \cdot (1-x)^{m-i} \cdot y^j \cdot (1-y)^{n-j}. \tag{2}$$

For integers $r, s \geq 0$, we denote by $f^{(r,s)}$ the differential operator of order (r,s), given by

$$f^{(r,s)}(x,y) = \frac{\partial^{r+s} f(x,y)}{\partial x^r \partial y^s}.$$

We use

Theorem 4 *(I. Badea, C. Badea [3]). It holds that*

$$\left\| f^{(k,l)} - \left(B_{m,n}f\right)^{(k,l)} \right\|_{\infty} \leq t\left(k,l\right) \cdot$$

$$\omega_1 \left(f^{(k,l)}; \frac{1}{\sqrt{m-k}}, \frac{1}{\sqrt{n-l}} \right) + \max \left\{ \frac{k\left(k-1\right)}{m}, \frac{l\left(l-1\right)}{n} \right\} \cdot \left\| f^{(k,l)} \right\|_{\infty}, \quad (3)$$

where $m > k \geq 0$, $n > l \geq 0$ *are integers,* f *is a real-valued function on* $[0,1]^2$ *such that* $f^{(k,l)}$ *is continuous, and* t *is a positive real-valued function on* $\mathbb{Z}_+ = \{0,1,2,...\}$. *Here* $\|\cdot\|_{\infty}$ *is the supremum norm on* $[0,1]^2$.

Denote $C^{r,p}\left([0,1]^2\right) := \{f : [0,1]^2 \to \mathbb{R}; f^{(k,l)}$ is continuous for $0 \leq k \leq r$, $0 \leq l \leq p\}$.

In [1] the author proved the following main motivational result.

Theorem 5 *Let* h_1, h_2, v_1, v_2, r, p *be integers,* $0 \leq h_1 \leq v_1 \leq r$, $0 \leq h_2 \leq v_2 \leq p$ *and let* $f \in C^{r,p}\left([0,1]^2\right)$. *Let* $\alpha_{i,j}\left(x,y\right)$, $i = h_1, h_1{+}1, ..., v_1$; $j = h_2, h_2{+}1, ..., v_2$ *be real-valued functions, defined and bounded in* $[0,1]^2$ *and assume* $\alpha_{h_1 h_2}$ *is either* $\geq \alpha > 0$ *or* $\leq \beta < 0$ *throughout* $[0,1]^2$. *Consider the operator*

$$L = \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{ij}\left(x,y\right) \frac{\partial^{i+j}}{\partial x^i \partial y^j} \quad (4)$$

and suppose that throughout $[0,1]^2$,

$$L\left(f\right) \geq 0.$$

Then for integers m, n *with* $m > r$, $n > p$, *there exists a polynomial* $Q_{m,n}\left(x,y\right)$ *of degree* (m,n) *such that* $L\left(Q_{m,n}\left(x,y\right)\right) \geq 0$ *throughout* $[0,1]^2$ *and*

$$\left\| f^{(k,l)} - Q_{m,n}^{(k,l)} \right\|_{\infty} \leq \frac{P_{m,n}\left(L,f\right)}{\left(h_1-k\right)!\left(h_2-l\right)!} + M_{m,n}^{k,l}\left(f\right), \quad (5)$$

all $(0,0) \leq (k,l) \leq (h_1,h_2)$. *Furhermore w get*

$$\left\| f^{(k,l)} - Q_{m,n}^{(k,l)} \right\|_{\infty} \leq M_{m,n}^{k,l}\left(f\right), \quad (6)$$

for all $(h_1+1, h_2+1) \leq (k,l) \leq (r,p)$. *Also (6) is true whenever* $0 \leq k \leq h_1$, $h_2+1 \leq l \leq p$ *or* $h_1+1 \leq k \leq r$, $0 \leq l \leq h_2$. *Here*

$$M_{m,n}^{k,l} \equiv M_{m,n}^{k,l}\left(f\right) \equiv t\left(k,l\right) \cdot$$

$$\omega_1 \left(f^{(k,l)}; \frac{1}{\sqrt{m-k}}, \frac{1}{\sqrt{n-l}} \right) + \max \left\{ \frac{k\left(k-1\right)}{m}, \frac{l\left(l-1\right)}{n} \right\} \cdot \left\| f^{(k,l)} \right\|_{\infty} \quad (7)$$

and

$$P_{m,n} \equiv P_{m,n}(L,f) \equiv \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} l_{ij} \cdot M_{m,n}^{i,j}, \tag{8}$$

where t *is a positive real-valued function on* $\mathbb{Z}_+^2$ *and*

$$l_{ij} \equiv \sup_{(x,y)\in[0,1]^2} \left|\alpha_{h_1 h_2}^{-1}(x,y) \cdot \alpha_{ij}(x,y)\right| < \infty. \tag{9}$$

In this article we extend Theorem 5 to the fractional level. Indeed here L is replaced by L^*, a linear left Caputo fractional mixed partial differential operator. Now the monotonicity property is only true on a critical region of $[0,1]^2$ that depends on L^* parameters. Simultaneous fractional convergence remains true on all of $[0,1]^2$.

We need

Definition 6 *Let* $\alpha_1, \alpha_2 > 0$; $\alpha = (\alpha_1, \alpha_2)$, $f \in C\left([0,1]^2\right)$ *and let* $x = (x_1, x_2)$, $t = (t_1, t_2) \in [0,1]^2$. *We define the left mixed Riemann-Liouville fractional two dimensional integral of order* α *(see also [4]):*

$$\left(I_{0+}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^{x_1} \int_0^{x_2} (x_1 - t_1)^{\alpha_1 - 1} (x_2 - t_2)^{\alpha_2 - 1} f(t_1, t_2)\, dt_1 dt_2, \tag{10}$$

with $x_1, x_2 > 0$.

Notice here $I_{0+}^{\alpha}(|f|) < \infty$.

Definition 7 *Let* $\alpha_1, \alpha_2 > 0$ *with* $\lceil \alpha_1 \rceil = m_1$, $\lceil \alpha_2 \rceil = m_2$, *(*$\lceil \cdot \rceil$ *ceiling of the number). Let here* $f \in C^{m_1, m_2}\left([0,1]^2\right)$. *We consider the left (Caputo type) fractional partial derivative:*

$$D_{*0}^{(\alpha_1,\alpha_2)} f(x) := \frac{1}{\Gamma(m_1 - \alpha_1)\Gamma(m_2 - \alpha_2)} \cdot$$

$$\int_0^{x_1} \int_0^{x_2} (x_1 - t_1)^{m_1 - \alpha_1 - 1} (x_2 - t_2)^{m_2 - \alpha_2 - 1} \frac{\partial^{m_1 + m_2} f(t_1, t_2)}{\partial t_1^{m_1} \partial t_2^{m_2}} dt_1 dt_2, \tag{11}$$

$\forall\ x = (x_1, x_2) \in [0,1]^2$, *where* Γ *is the gamma function*

$$\Gamma(\nu) = \int_0^{\infty} e^{-t} t^{\nu - 1} dt, \quad \nu > 0. \tag{12}$$

We set

$$D_{*0}^{(0,0)} f(x) := f(x), \quad \forall\ x \in [0,1]^2; \tag{13}$$

$$D_{*0}^{(m_1,m_2)} f(x) := \frac{\partial^{m_1+m_2} f(x)}{\partial x_1^{m_1} \partial x_2^{m_2}}, \ \ \forall\ x \in [0,1]^2. \tag{14}$$

Definition 8 *We also set*

$$D_{*0}^{(0,\alpha_2)} f(x) := \frac{1}{\Gamma(m_2-\alpha_2)} \int_0^{x_2} (x_2-t_2)^{m_2-\alpha_2-1} \frac{\partial^{m_2} f(x_1,t_2)}{\partial t_2^{m_2}} dt_2, \tag{15}$$

$$D_{*0}^{(\alpha_1,0)} f(x) := \frac{1}{\Gamma(m_1-\alpha_1)} \int_0^{x_1} (x_1-t_1)^{m_1-\alpha_1-1} \frac{\partial^{m_1} f(t_1,x_2)}{\partial t_1^{m_1}} dt_1, \tag{16}$$

and

$$D_{*0}^{(m,\alpha_2)} f(x) := \frac{1}{\Gamma(m_2-\alpha_2)} \int_0^{x_2} (x_2-t_2)^{m_2-\alpha_2-1} \frac{\partial^{m_1+m_2} f(x_1,t_2)}{\partial x_1^{m_1} \partial t_2^{m_2}} dt_2, \tag{17}$$

$$D_{*0}^{(\alpha_1,m_2)} f(x) := \frac{1}{\Gamma(m_1-\alpha_1)} \int_0^{x_1} (x_1-t_1)^{m_1-\alpha_1-1} \frac{\partial^{m_1+m_2} f(t_1,x_2)}{\partial t_1^{m_1} \partial x_2^{m_2}} dt_1. \tag{18}$$

2 Main Result

We present our main result

Theorem 9 *Let h_1, h_2, v_1, v_2, r, p be integers, $0 \le h_1 \le v_1 \le r$, $0 \le h_2 \le v_2 \le p$ and let $f \in C^{r,p}\left([0,1]^2\right)$. Let $\alpha_{ij}(x,y)$, $i = h_1, h_1+1, ..., v_1$; $j = h_2, h_2+1, ..., v_2$ be real valued functions, defined and bounded in $[0,1]^2$ and assume $\alpha_{h_1h_2}$ is either $\ge \alpha > 0$ or $\le \beta < 0$ throughout $[0,1]^2$. Let $0 \le \alpha_{1h_1} \le h_1 \le \alpha_{11} \le h_1+1 < \alpha_{12} \le h_1+2 < \alpha_{13} \le h_1+3 < ... < \alpha_{1v_1} \le v_1 < ... < \alpha_{1r} \le r$, with $\lceil \alpha_{1h_1} \rceil = h_1$; $0 \le \alpha_{2h_2} \le h_2 < \alpha_{21} \le h_2+1 < \alpha_{22} \le h_2+2 < \alpha_{23} \le h_2+3 < ... < \alpha_{2v_2} \le v_2 < ... < \alpha_{2p} \le p$, with $\lceil \alpha_{2h_2} \rceil = h_2$. Consider the left fractional difefrential bivariate operator*

$$L^* := \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{ij}(x,y) D_{*0}^{(\alpha_{1i},\alpha_{2j})}. \tag{19}$$

Let integers $\overline{m_1}, \overline{m_2}$ with $\overline{m_1} > r$, $\overline{m_2} > p$. Set

$$l_{ij} := \sup_{(x,y)\in[0,1]^2} \left| \alpha_{h_1h_2}^{-1}(x,y) \cdot \alpha_{ij}(x,y) \right| < \infty.$$

Also set ($\lceil \alpha_{1i} \rceil = i$, $\lceil \alpha_{2j} \rceil = j$, $\lceil \cdot \rceil$ ceiling of number)

$$M_{\overline{m_1},\overline{m_2}}^{i,j} := M_{\overline{m_1},\overline{m_2}}^{i,j}(f) := \tag{20}$$

$$\frac{1}{\Gamma(i-\alpha_{1i}+1)\Gamma(j-\alpha_{2j}+1)} \left\{ t(i,j)\,\omega_1\left(f^{(i,j)}; \frac{1}{\sqrt{\overline{m_1}-i}}, \frac{1}{\sqrt{\overline{m_2}-j}}\right) + \max\left\{ \frac{i(i-1)}{\overline{m_1}}, \frac{j(j-1)}{\overline{m_2}} \right\} \cdot \left\| f^{(i,j)} \right\|_\infty \right\},$$

$i = h_1, ..., v_1;\ j = h_2, ..., v_2$.

Here t *is a positive real-valued function on* $\mathbb{Z}_+^2$, $\|\cdot\|_\infty$ *is the supremum norm on* $[0,1]^2$*. Call*

$$P_{\overline{m_1},\overline{m_2}} := P_{\overline{m_1},\overline{m_2}}(f) = \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} l_{ij} \cdot M_{\overline{m_1},\overline{m_2}}^{i,j}. \tag{21}$$

Then there exists a polynomial $Q_{\overline{m_1},\overline{m_2}}(x,y)$ *of degree* $(\overline{m_1},\overline{m_2})$ *on* $[0,1]^2$ *such that*

$$\left\| D_{*0}^{(\alpha_{1k},\alpha_{2l})}(f) - D_{*0}^{(\alpha_{1k},\alpha_{2l})}\left(Q_{\overline{m_1},\overline{m_2}}\right) \right\|_\infty \leq$$

$$\frac{\Gamma(h_1-k+1)\,\Gamma(h_2-l+1)\,P_{\overline{m_1},\overline{m_2}}}{\Gamma(h_1-\alpha_{1k}+1)\,\Gamma(h_2-\alpha_{2l}+1)\,(h_1-k)!\,(h_2-l)!} + M_{\overline{m_1},\overline{m_2}}^{k,l}, \tag{22}$$

for $(0,0) \leq (k,l) \leq (h_1,h_2)$.

If $(h_1+1, h_2+1) \leq (k,l) \leq (r,p)$*, or* $0 \leq k \leq h_1$*,* $h_2+1 \leq l \leq p$*, or* $h_1+1 \leq k \leq r$*,* $0 \leq l \leq h_2$*, we get*

$$\left\| D_{*0}^{(\alpha_{1k},\alpha_{2l})}(f) - D_{*0}^{(\alpha_{1k},\alpha_{2l})}\left(Q_{\overline{m_1},\overline{m_2}}\right) \right\|_\infty \leq M_{\overline{m_1},\overline{m_2}}^{k,l}. \tag{23}$$

By assuming $L^*(f(1,1)) \geq 0$*, we get* $L^*\left(Q_{\overline{m_1},\overline{m_2}}(1,1)\right) \geq 0$.

Let $1 > x, y > 0$*, with* $\alpha_{1h_1} \neq h_1$ *and* $\alpha_{2h_2} \neq h_2$*, such that*

$$x \geq \left(\Gamma\left(h_1-\alpha_{1h_1}+1\right)\right)^{\frac{1}{\left(h_1-\alpha_{1h_1}\right)}}, \tag{24}$$

$$y \geq \left(\Gamma\left(h_2-\alpha_{2h_2}+1\right)\right)^{\frac{1}{\left(h_2-\alpha_{2h_2}\right)}},$$

and

$$L^*(f(x,y)) \geq 0.$$

Then

$$L^*\left(Q_{\overline{m_1},\overline{m_2}}(x,y)\right) \geq 0.$$

To prove Theorem 9 it takes some preparation. We need

Definition 10 *Let* f *be a real-valued function defined on* $[0,1]^2$ *and let* $\overline{m_1}, \overline{m_2} \in \mathbb{N}$*. Let* $B_{\overline{m_1},\overline{m_2}}$ *be the Bernstein (polynomial) operator of order* $(\overline{m_1},\overline{m_2})$ *given by*

$$B_{\overline{m_1},\overline{m_2}}(f; x_1, x_2) :=$$

$$\sum_{i_1=0}^{\overline{m_1}} \sum_{i_2=0}^{\overline{m_2}} f\left(\frac{i_1}{\overline{m_1}}, \frac{i_2}{\overline{m_2}}\right) \binom{\overline{m_1}}{i_1} \binom{\overline{m_2}}{i_2} x_1^{i_1} (1-x_1)^{\overline{m_1}-i_1} x_2^{i_2} (1-x_2)^{\overline{m_2}-i_2}. \tag{25}$$

We need the following simultaneous approximation result.

Theorem 11 *(I. Badea, C. Badea [3]). It holds that*

$$\left\| f^{(k_1,k_2)} - \left(B_{\overline{m_1},\overline{m_2}} f\right)^{(k_1,k_2)} \right\|_\infty \leq t\left(k_1,k_2\right) \omega_1 \left(f^{(k_1,k_2)}; \frac{1}{\sqrt{\overline{m_1}-k_1}}, \frac{1}{\sqrt{\overline{m_2}-k_2}} \right) \tag{26}$$
$$+ \max\left\{ \frac{k_1\left(k_1-1\right)}{\overline{m_1}}, \frac{k_2\left(k_2-1\right)}{\overline{m_2}} \right\} \cdot \left\| f^{(k_1,k_2)} \right\|_\infty ,$$

where $\overline{m_1} > k_1 \geq 0$, $\overline{m_2} > k_2 \geq 0$ are integers, f is a real-valued function on $[0,1]^2$, such that $f^{(k_1,k_2)}$ is continuous, and t is a positive real-valued function on $\mathbb{Z}_+^2$. Here $\|\cdot\|_\infty$ is the supremum norm on $[0,1]^2$.

Remark 12 *We assume that $\overline{m_1} > m_1 = \lceil \alpha_1 \rceil$, $\overline{m_2} > m_2 = \lceil \alpha_2 \rceil$, where $\alpha_1, \alpha_2 > 0$.*

We consider also

$$D_{*0}^{(\alpha_1,\alpha_2)} \left(B_{\overline{m_1},\overline{m_2}} f\right)\left(x_1,x_2\right) = \frac{1}{\Gamma\left(m_1-\alpha_1\right)\Gamma\left(m_2-\alpha_2\right)} \cdot \tag{27}$$
$$\int_0^{x_1}\int_0^{x_2} \left(x_1-t_1\right)^{m_1-\alpha_1-1}\left(x_2-t_2\right)^{m_2-\alpha_2-1} \frac{\partial^{m_1+m_2}\left(B_{\overline{m_1},\overline{m_2}} f\right)\left(t_1,t_2\right)}{\partial t_1^{m_1}\partial t_2^{m_2}} dt_1 dt_2,$$

$\forall\ \left(x_1,x_2\right) \in [0,1]^2$.

Proposition 13 *Let $\alpha_1, \alpha_2 > 0$ with $\lceil \alpha_1 \rceil = m_1$, $\lceil \alpha_2 \rceil = m_2$, $f \in C^{m_1,m_2}\left([0,1]^2\right)$, where $\overline{m_1}, \overline{m_2} \in \mathbb{N} : \overline{m_1} > m_1$, $\overline{m_2} > m_2$. Then*

$$\left\| D_{*0}^{(\alpha_1,\alpha_2)} f - D_{*0}^{(\alpha_1,\alpha_2)} \left(B_{\overline{m_1},\overline{m_2}} f\right) \right\|_\infty \leq \frac{1}{\Gamma\left(m_1-\alpha_1+1\right)\Gamma\left(m_2-\alpha_2+1\right)} \cdot$$
$$\left\{ t\left(m_1,m_2\right)\omega_1\left(f^{(m_1,m_2)}; \frac{1}{\sqrt{\overline{m_1}-m_1}}, \frac{1}{\sqrt{\overline{m_2}-m_2}} \right) + \right.$$
$$\left. \max\left\{ \frac{m_1\left(m_1-1\right)}{\overline{m_1}}, \frac{m_2\left(m_2-1\right)}{\overline{m_2}} \right\} \cdot \left\| f^{(m_1,m_2)} \right\|_\infty \right\}, \tag{28}$$

Proof. We observe the following

$$\left| D_{*0}^{(\alpha_1,\alpha_2)} f\left(x_1,x_2\right) - D_{*0}^{(\alpha_1,\alpha_2)} \left(B_{\overline{m_1},\overline{m_2}} f\right)\left(x_1,x_2\right) \right| = \frac{1}{\Gamma\left(m_1-\alpha_1\right)\Gamma\left(m_2-\alpha_2\right)} \cdot \tag{29}$$
$$\left| \int_0^{x_1}\int_0^{x_2} \left(x_1-t_1\right)^{m_1-\alpha_1-1}\left(x_2-t_2\right)^{m_2-\alpha_2-1} \cdot \right.$$
$$\left. \left(\frac{\partial^{m_1+m_2} f\left(t_1,t_2\right)}{\partial t_1^{m_1}\partial t_2^{m_2}} - \frac{\partial^{m_1+m_2}\left(B_{\overline{m_1},\overline{m_2}} f\right)\left(t_1,t_2\right)}{\partial t_1^{m_1}\partial t_2^{m_2}} \right) dt_1 dt_2 \right|$$
$$\leq \frac{1}{\Gamma\left(m_1-\alpha_1\right)\Gamma\left(m_2-\alpha_2\right)} \int_0^{x_1}\int_0^{x_2} \left(x_1-t_1\right)^{m_1-\alpha_1-1}\left(x_2-t_2\right)^{m_2-\alpha_2-1} \cdot \tag{30}$$

$$\left|\frac{\partial^{m_1+m_2} f\left(t_1,t_2\right)}{\partial t_1^{m_1}\partial t_2^{m_2}}-\frac{\partial^{m_1+m_2}\left(B_{\overline{m_1},\overline{m_2}}f\right)\left(t_1,t_2\right)}{\partial t_1^{m_1}\partial t_2^{m_2}}\right|dt_1dt_2\overset{(26)}{\leq}$$

$$\left\{t\left(m_1,m_2\right)\omega_1\left(f^{(m_1,m_2)};\frac{1}{\sqrt{\overline{m_1}-m_1}},\frac{1}{\sqrt{\overline{m_2}-m_2}}\right)+\right.$$

$$\left.\max\left\{\frac{m_1\left(m_1-1\right)}{\overline{m_1}},\frac{m_2\left(m_2-1\right)}{\overline{m_2}}\right\}\cdot\left\|f^{(m_1,m_2)}\right\|_\infty\right\}$$

$$\cdot\frac{1}{\Gamma\left(m_1-\alpha_1\right)\Gamma\left(m_2-\alpha_2\right)}\int_0^{x_1}\int_0^{x_2}\left(x_1-t_1\right)^{m_1-\alpha_1-1}\left(x_2-t_2\right)^{m_2-\alpha_2-1}dt_1dt_2= \tag{31}$$

$$\frac{x_1^{m_1-\alpha_1}x_2^{m_2-\alpha_2}}{\Gamma\left(m_1-\alpha_1+1\right)\Gamma\left(m_2-\alpha_2+1\right)}\cdot$$

$$\left\{t\left(m_1,m_2\right)\omega_1\left(f^{(m_1,m_2)};\frac{1}{\sqrt{\overline{m_1}-m_1}},\frac{1}{\sqrt{\overline{m_2}-m_2}}\right)\right.$$

$$\left.+\max\left\{\frac{m_1\left(m_1-1\right)}{\overline{m_1}},\frac{m_2\left(m_2-1\right)}{\overline{m_2}}\right\}\left\|f^{(m_1,m_2)}\right\|_\infty\right\}, \tag{32}$$

$\forall\ (x_1,x_2)\in[0,1]^2$. ■

Proof. of Theorem 9. Here we use a lot Proposition 13.

Case (i). Assume that throughout $[0,1]^2$, $\alpha_{h_1h_2}\geq\alpha>0$.

Call

$$Q_{\overline{m_1},\overline{m_2}}(x,y):=B_{\overline{m_1},\overline{m_2}}(f;x,y)+P_{\overline{m_1},\overline{m_2}}\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}. \tag{33}$$

Then by (28) we get

$$\left\|D_{*0}^{(\alpha_{1k},\alpha_{2l})}\left(f+P_{\overline{m_1},\overline{m_2}}\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}\right)-D_{*0}^{(\alpha_{1k},\alpha_{2l})}\left(Q_{\overline{m_1},\overline{m_2}}\right)\right\|_\infty\leq M_{\overline{m_1},\overline{m_2}}^{k,l}, \tag{34}$$

all $0\leq k\leq r$, $0\leq l\leq p$. When $(0,0)\leq(k,l)\leq(h_1,h_2)$, inequality (34) becomes

$$\left\|D_{*0}^{(\alpha_{1k},\alpha_{2l})}(f)+P_{\overline{m_1},\overline{m_2}}\frac{\Gamma\left(h_1-k+1\right)\Gamma\left(h_2-l+1\right)x^{h_1-\alpha_{1k}}y^{h_2-\alpha_{2l}}}{\Gamma\left(h_1-\alpha_{1k}+1\right)\Gamma\left(h_2-\alpha_{2l}+1\right)\left(h_1-k\right)!\left(h_2-l\right)!}\right.$$

$$\left.-D_{*0}^{(\alpha_{1k},\alpha_{2l})}\left(Q_{\overline{m_1},\overline{m_2}}\right)\right\|_\infty\leq M_{\overline{m_1},\overline{m_2}}^{k,l}. \tag{35}$$

We prove (35) as follows:

In (34) we need to calculate $((0,0)\leq(k,l)\leq(h_1,h_2))$

$$D_{*0}^{(\alpha_{1k},\alpha_{2l})}\left(\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}\right)=\frac{1}{\Gamma\left(k-\alpha_{1k}\right)\Gamma\left(l-\alpha_{2l}\right)}\cdot \tag{36}$$

$$\int_0^x\int_0^y\left(x-t_1\right)^{k-\alpha_{1k}-1}\left(y-t_2\right)^{l-\alpha_{2l}-1}\frac{t_1^{h_1-k}}{\left(h_1-k\right)!}\frac{t_2^{h_2-l}}{\left(h_2-l\right)!}dt_1dt_2=$$

$$\left(\frac{1}{\Gamma(k-\alpha_{1k})}\int_0^x (x-t_1)^{k-\alpha_{1k}-1}\frac{t_1^{h_1-k}}{(h_1-k)!}dt_1\right)\cdot$$
$$\left(\frac{1}{\Gamma(l-\alpha_{2l})}\int_0^y (y-t_2)^{l-\alpha_{2l}-1}\frac{t_2^{h_2-l}}{(h_2-l)!}dt_2\right)=$$

$$\left(\frac{1}{(h_1-k)!\Gamma(k-\alpha_{1k})}\int_0^x (x-t_1)^{k-\alpha_{1k}-1}(t_1-0)^{(h_1-k+1)-1}dt_1\right)\cdot \tag{37}$$
$$\left(\frac{1}{(h_2-l)!\Gamma(l-\alpha_{2l})}\int_0^y (y-t_2)^{l-\alpha_{2l}-1}(t_2-0)^{(h_2-l+1)-1}dt_2\right)=$$

$$\left(\frac{1}{(h_1-k)!\Gamma(k-\alpha_{1k})}\frac{\Gamma(k-\alpha_{1k})\Gamma(h_1-k+1)}{\Gamma(h_1-\alpha_{1k}+1)}x^{h_1-\alpha_{1k}}\right)\cdot \tag{38}$$
$$\left(\frac{1}{(h_2-l)!\Gamma(l-\alpha_{2l})}\frac{\Gamma(l-\alpha_{2l})\Gamma(h_2-l+1)}{\Gamma(h_2-\alpha_{2l}+1)}y^{h_2-\alpha_{2l}}\right)=$$

$$\left(\frac{\Gamma(h_1-k+1)}{(h_1-k)!\Gamma(h_1-\alpha_{1k}+1)}\right)\left(\frac{\Gamma(h_2-l+1)}{(h_2-l)!\Gamma(h_2-\alpha_{2l}+1)}\right)x^{h_1-\alpha_{1k}}y^{h_2-\alpha_{2l}}. \tag{39}$$

So when $(0,0)\le(k,l)\le(h_1,h_2)$ we get

$$D_{*0}^{(\alpha_{1k},\alpha_{2l})}\left(\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}\right)=\frac{\Gamma(h_1-k+1)\Gamma(h_2-l+1)x^{h_1-\alpha_{1k}}y^{h_2-\alpha_{2l}}}{\Gamma(h_1-\alpha_{1k}+1)\Gamma(h_2-\alpha_{2l}+1)(h_1-k)!(h_2-l)!}. \tag{40}$$

Hence we plug in (40) into (34) to get (35).

Using (35) and triangle inequality we obtain $((0,0)\le(k,l)\le(h_1,h_2))$ that

$$\left\|D_{*0}^{(\alpha_{1k},\alpha_{2l})}(f)-D_{*0}^{(\alpha_{1k},\alpha_{2l})}(Q_{\overline{m_1},\overline{m_2}})\right\|_\infty\le$$
$$\frac{\Gamma(h_1-k+1)\Gamma(h_2-l+1)P_{\overline{m_1},\overline{m_2}}}{\Gamma(h_1-\alpha_{1k}+1)\Gamma(h_2-\alpha_{2l}+1)(h_1-k)!(h_2-l)!}+M^{k,l}_{\overline{m_1},\overline{m_2}}, \tag{41}$$

proving (22).

Next if $(h_1+1,h_2+1)\le(k,l)\le(r,p)$, or $0\le k\le h_1$, $h_2+1\le l\le p$, or $h_1+1\le k\le r$, $0\le l\le h_2$, we get by (34) that

$$\left\|D_{*0}^{(\alpha_{1k},\alpha_{2l})}(f)-D_{*0}^{(\alpha_{1k},\alpha_{2l})}(Q_{\overline{m_1},\overline{m_2}})\right\|_\infty\le M^{k,l}_{\overline{m_1},\overline{m_2}}, \tag{42}$$

proving (23).

Furthermore, if (x,y) in critical region, see (24), we get

$$\alpha_{h_1h_2}^{-1}(x,y)\,L^*\left(Q_{\overline{m_1},\overline{m_2}}(x,y)\right)=\alpha_{h_1h_2}^{-1}(x,y)\,L^*\left(f(x,y)\right)+ \tag{43}$$

$$P_{\overline{m_1},\overline{m_2}}\frac{x^{h_1-\alpha_{1h_1}}y^{h_2-\alpha_{2h_2}}}{\Gamma\left(h_1-\alpha_{1h_1}+1\right)\Gamma\left(h_2-\alpha_{2h_2}+1\right)}+\sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2}\alpha_{h_1h_2}^{-1}(x,y)\,\alpha_{ij}(x,y)\cdot$$

$$D_{*0}^{(\alpha_{1i},\alpha_{2j})}\left[Q_{\overline{m_1},\overline{m_2}}(x,y)-f(x,y)-P_{\overline{m_1},\overline{m_2}}\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}\right]\overset{(34)}{\geq}$$

$$P_{\overline{m_1},\overline{m_2}}\frac{x^{h_1-\alpha_{1h_1}}y^{h_2-\alpha_{2h_2}}}{\Gamma\left(h_1-\alpha_{1h_1}+1\right)\Gamma\left(h_2-\alpha_{2h_2}+1\right)}-\sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2}l_{ij}M_{\overline{m_1},\overline{m_2}}^{i,j}=$$

$$P_{\overline{m_1},\overline{m_2}}\frac{x^{h_1-\alpha_{1h_1}}y^{h_2-\alpha_{2h_2}}}{\Gamma\left(h_1-\alpha_{1h_1}+1\right)\Gamma\left(h_2-\alpha_{2h_2}+1\right)}-P_{\overline{m_1},\overline{m_2}}= \tag{44}$$

$$P_{\overline{m_1},\overline{m_2}}\left[\frac{x^{h_1-\alpha_{1h_1}}y^{h_2-\alpha_{2h_2}}}{\Gamma\left(h_1-\alpha_{1h_1}+1\right)\Gamma\left(h_2-\alpha_{2h_2}+1\right)}-1\right]=$$

$$P_{\overline{m_1},\overline{m_2}}\left[\frac{x^{h_1-\alpha_{1h_1}}y^{h_2-\alpha_{2h_2}}-\Gamma\left(h_1-\alpha_{1h_1}+1\right)\Gamma\left(h_2-\alpha_{2h_2}+1\right)}{\Gamma\left(h_1-\alpha_{1h_1}+1\right)\Gamma\left(h_2-\alpha_{2h_2}+1\right)}\right]=:(*)\,. \tag{45}$$

We know $\Gamma(1)=1$, $\Gamma(2)=1$, and Γ is convex and positive on $(0,\infty)$. Here $0\leq h_1-\alpha_{1h_1}<1$ and $0\leq h_2-\alpha_{2h_2}<1$, hence $1\leq h_1-\alpha_{1h_1}+1<2$, $1\leq h_2-\alpha_{2h_2}+1<2$. Thus $0<\Gamma\left(h_1-\alpha_{1h_1}+1\right),\Gamma\left(h_2-\alpha_{2h_2}+1\right)\leq 1$, and $1-\Gamma\left(h_1-\alpha_{1h_1}+1\right)\Gamma\left(h_2-\alpha_{2h_2}+1\right)\geq 0$. Clearly acting as in (43)-(45), when $L^*\left(f(1,1)\right)\geq 0$, we get $L^*\left(Q_{\overline{m_1},\overline{m_2}}(1,1)\right)\geq 0$.

Based on the above comments about Gamma function we get $(*)\geq 0$. That is $L^*\left(Q_{\overline{m_1},\overline{m_2}}(x,y)\right)\geq 0$, over the critical region of (24).

Case (ii). Assume that throughout $[0,1]^2$, $\alpha_{h_1h_2}\leq\beta<0$. Consider

$$Q_{\overline{m_1},\overline{m_2}}^{-}(x,y):\equiv B_{\overline{m_1},\overline{m_2}}(f;x,y)-P_{\overline{m_1},\overline{m_2}}\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}.$$

Then by (28) we get

$$\left\|D_{*0}^{(\alpha_{1k},\alpha_{2l})}\left(f-P_{\overline{m_1},\overline{m_2}}\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}\right)-D_{*0}^{(\alpha_{1k},\alpha_{2l})}\left(Q_{\overline{m_1},\overline{m_2}}^{-}(x,y)\right)\right\|_\infty\leq M_{\overline{m_1},\overline{m_2}}^{k,l}, \tag{46}$$

all $0\leq k\leq r$, $0\leq l\leq p$.

When $(0,0)\leq(k,l)\leq(h_1,h_2)$ (46) becomes

$$\Bigg\|D_{*0}^{(\alpha_{1k},\alpha_{2l})}(f)-P_{\overline{m_1},\overline{m_2}}\frac{\Gamma(h_1-k+1)\,\Gamma(h_2-l+1)\,x^{h_1-\alpha_{1k}}y^{h_2-\alpha_{2l}}}{\Gamma(h_1-\alpha_{1k}+1)\,\Gamma(h_2-\alpha_{2l}+1)\,(h_1-k)!\,(h_2-l)!}$$

$$-D_{*0}^{(\alpha_{1k},\alpha_{2l})}\left(Q_{\overline{m_1},\overline{m_2}}^{-}\right)\Bigg\|_\infty\leq M_{\overline{m_1},\overline{m_2}}^{k,l}. \tag{47}$$

Using (47) and triangle inequality we obtain for $(0,0) \leq (k,l) \leq (h_1,h_2)$ that

$$\left\| D_{*0}^{(\alpha_{1k},\alpha_{2l})}(f) - D_{*0}^{(\alpha_{1k},\alpha_{2l})}\left(Q^{-}_{\overline{m_1},\overline{m_2}}\right)\right\|_{\infty} \leq \frac{\Gamma(h_1-k+1)\,\Gamma(h_2-l+1)\,P_{\overline{m_1},\overline{m_2}}}{\Gamma(h_1-\alpha_{1k}+1)\,\Gamma(h_2-\alpha_{2l}+1)\,(h_1-k)!\,(h_2-l)!} + M^{k,l}_{\overline{m_1},\overline{m_2}}. \tag{48}$$

Next if $(h_1+1,h_2+1) \leq (k,l) \leq (r,p)$, or $0 \leq k \leq h_1$, $h_2+1 \leq l \leq p$, or $h_1+1 \leq k \leq r$, $0 \leq l \leq h_2$, we get by (46) that

$$\left\| D_{*0}^{(\alpha_{1k},\alpha_{2l})}(f) - D_{*0}^{(\alpha_{1k},\alpha_{2l})}\left(Q^{-}_{\overline{m_1},\overline{m_2}}\right)\right\|_{\infty} \leq M^{k,l}_{\overline{m_1},\overline{m_2}}. \tag{49}$$

We proved again (22) and (23). Furthermore, if (x,y) in critical region, see (24), we obtain

$$\alpha^{-1}_{h_1h_2}(x,y)\,L^{*}\left(Q^{-}_{\overline{m_1},\overline{m_2}}(x,y)\right) = \alpha^{-1}_{h_1h_2}(x,y)\,L^{*}(f(x,y)) - P_{\overline{m_1},\overline{m_2}}\frac{x^{h_1-\alpha_{1h_1}}y^{h_2-\alpha_{2h_2}}}{\Gamma(h_1-\alpha_{1h_1}+1)\,\Gamma(h_2-\alpha_{2h_2}+1)} + \sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2}\alpha^{-1}_{h_1h_2}(x,y)\,\alpha_{ij}(x,y)\cdot \tag{50}$$

$$D_{*0}^{(\alpha_{1i},\alpha_{2j})}\left[Q^{-}_{\overline{m_1},\overline{m_2}}(x,y) - f(x,y) + P_{\overline{m_1},\overline{m_2}}\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}\right] \overset{(46)}{\leq} -P_{\overline{m_1},\overline{m_2}}\frac{x^{h_1-\alpha_{1h_1}}y^{h_2-\alpha_{2h_2}}}{\Gamma(h_1-\alpha_{1h_1}+1)\,\Gamma(h_2-\alpha_{2h_2}+1)} + \sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2} l_{ij}M^{i,j}_{\overline{m_1},\overline{m_2}} = \tag{51}$$

$$P_{\overline{m_1},\overline{m_2}}\left(1 - \frac{x^{h_1-\alpha_{1h_1}}y^{h_2-\alpha_{2h_2}}}{\Gamma(h_1-\alpha_{1h_1}+1)\,\Gamma(h_2-\alpha_{2h_2}+1)}\right) = \tag{52}$$

$$P_{\overline{m_1},\overline{m_2}}\left(\frac{\Gamma(h_1-\alpha_{1h_1}+1)\,\Gamma(h_2-\alpha_{2h_2}+1) - x^{h_1-\alpha_{1h_1}}y^{h_2-\alpha_{2h_2}}}{\Gamma(h_1-\alpha_{1h_1}+1)\,\Gamma(h_2-\alpha_{2h_2}+1)}\right) =: (**). \tag{53}$$

We know $\Gamma(1) = 1$, $\Gamma(2) = 1$, and Γ is convex and positive on $(0,\infty)$. Here $0 \leq h_1 - \alpha_{1h_1} < 1$ and $0 \leq h_2 - \alpha_{2h_2} < 1$, hence $1 \leq h_1 - \alpha_{1h_1} + 1 < 2$, $1 \leq h_2 - \alpha_{2h_2} + 1 < 2$. Thus $0 < \Gamma(h_1 - \alpha_{1h_1} + 1), \Gamma(h_2 - \alpha_{2h_2} + 1) \leq 1$, and $1 - \Gamma(h_1 - \alpha_{1h_1} + 1)\,\Gamma(h_2 - \alpha_{2h_2} + 1) \geq 0$. Clearly acting as in (50)-(53), when $L^{*}(f(1,1)) \geq 0$, we get $L^{*}\left(Q^{-}_{\overline{m_1},\overline{m_2}}(1,1)\right) \geq 0$.

Based on the above comments about Gamma function we get $(**) \leq 0$. That is $L^{*}\left(Q^{-}_{\overline{m_1},\overline{m_2}}(x,y)\right) \geq 0$, over the critical region of (24). ■

References

[1] G.A. Anastassiou, *Bivariate Monotone Approximation*, Proc. of AMS, vol. 112, No. 4 (1991), 959-963.

[2] G.A. Anastassiou, O. Shisha, *Monotone approximation with linear differential operators*, J. Approx. Theory 44 (1985), 391-393.

[3] I. Badea, C. Badea, *On the order of simultaneously approximation of bivariate functions by Bernstein operators*, Anal. Numér. Théor. Approx. 16 (1987), 11-17.

[4] T. Mamatov, S. Samko, *Mixed fractional integration operators in mixed weighted Hölder spaces*, Fractional Calculus and Applied Analysis, Vol. 13, No. 3 (2010), 245-259.

[5] O. Shisha, *Monotone approximation*, Pacific J. Math. 15 (1965), 667-671.

[6] D.D. Stancu, *Studii Si Cercetari Stiintifice*, XI (2) (1960), 221-233.