

Bivariate Right Fractional Polynomial Monotone Approximation

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Abstract

Let $f \in C^{r,p}([0, 1]^2)$, $r, p \in \mathbb{N}$, and let \bar{L} be a linear right fractional mixed partial differential operator such that $\bar{L}(f) \geq 0$, for all (x, y) in a critical region of $[0, 1]^2$ that depends on \bar{L} . Then there exists a sequence of two-dimensional polynomials $Q_{\bar{m}_1, \bar{m}_2}(x, y)$ with $\bar{L}(Q_{\bar{m}_1, \bar{m}_2}(x, y)) \geq 0$ there, where $\bar{m}_1, \bar{m}_2 \in \mathbb{N}$ such that $\bar{m}_1 > r$, $\bar{m}_2 > p$, so that f is approximated right fractionally simultaneously and uniformly by $Q_{\bar{m}_1, \bar{m}_2}$ on $[0, 1]^2$. This restricted right fractional approximation is accomplished quantitatively by the use of a suitable integer partial derivatives two-dimensional first modulus of continuity.

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1 Introduction

The topic of monotone approximation started in [5] has become a major trend in approximation theory. A typical problem in this subject is: given a positive integer k , approximate a given function whose k th derivative is ≥ 0 by polynomials having this property.

In [2] the authors replaced the k th derivative with a linear differential operator of order k . We mention this motivating result.

Theorem 1 *Let h, k, p be integers, $0 \leq h \leq k \leq p$ and let f be a real function, $f^{(p)}$ continuous in $[-1, 1]$ with modulus of continuity $\omega(f^{(p)}, x)$ there. Let*

$a_j(x)$, $j = h, h+1, \dots, k$ be real functions, defined and bounded on $[-1, 1]$ and assume $a_h(x)$ is either \geq some number $\alpha > 0$ or \leq some number $\beta < 0$ throughout $[-1, 1]$. Consider the operator

$$L = \sum_{j=h}^k a_j(x) \left[\frac{d^j}{dx^j} \right]$$

and suppose, throughout $[-1, 1]$,

$$L(f) \geq 0. \tag{1}$$

Then, for every integer $n \geq 1$, there is a real polynomial $Q_n(x)$ of degree $\leq n$ such that

$$L(Q_n) \geq 0 \text{ throughout } [-1, 1]$$

and

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq C n^{k-p} \omega\left(f^{(p)}, \frac{1}{n}\right),$$

where C is independent of n or f .

We need

Definition 2 (D.D. Stancu [6]). Let $f \in C([0, 1]^2)$, $[0, 1]^2 = [0, 1] \times [0, 1]$, where $(x_1, y_1), (x_2, y_2) \in [0, 1]^2$ and $\delta_1, \delta_2 \geq 0$. The first modulus of continuity of f is defined as follows:

$$\omega_1(f, \delta_1, \delta_2) = \sup_{\substack{|x_1 - x_2| \leq \delta_1 \\ |y_1 - y_2| \leq \delta_2}} |f(x_1, y_1) - f(x_2, y_2)|.$$

Definition 3 Let f be a real-valued function defined on $[0, 1]^2$ and let m, n be two positive integers. Let $B_{m,n}$ be the Bernstein (polynomial) operator of order (m, n) given by

$$B_{m,n}(f; x, y) = \sum_{i=0}^m \sum_{j=0}^n f\left(\frac{i}{m}, \frac{j}{n}\right) \cdot \binom{m}{i} \cdot \binom{n}{j} \cdot x^i \cdot (1-x)^{m-i} \cdot y^j \cdot (1-y)^{n-j}. \tag{2}$$

For integers $r, s \geq 0$, we denote by $f^{(r,s)}$ the differential operator of order (r, s) , given by

$$f^{(r,s)}(x, y) = \frac{\partial^{r+s} f(x, y)}{\partial x^r \partial y^s}.$$

We use

Theorem 4 (I. Badea, C. Badea [3]). *It holds that*

$$\left\| f^{(k,l)} - (B_{m,n}f)^{(k,l)} \right\|_{\infty} \leq t(k,l).$$

$$\omega_1 \left(f^{(k,l)}; \frac{1}{\sqrt{m-k}}, \frac{1}{\sqrt{n-l}} \right) + \max \left\{ \frac{k(k-1)}{m}, \frac{l(l-1)}{n} \right\} \cdot \left\| f^{(k,l)} \right\|_{\infty}, \quad (3)$$

where $m > k \geq 0$, $n > l \geq 0$ are integers, f is a real-valued function on $[0, 1]^2$ such that $f^{(k,l)}$ is continuous, and t is a positive real-valued function on $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. Here $\|\cdot\|_{\infty}$ is the supremum norm on $[0, 1]^2$.

Denote $C^{r,p}([0, 1]^2) := \{f : [0, 1]^2 \rightarrow \mathbb{R}; f^{(k,l)} \text{ is continuous for } 0 \leq k \leq r, 0 \leq l \leq p\}$.

In [1] the author proved the following main motivational result.

Theorem 5 *Let h_1, h_2, v_1, v_2, r, p be integers, $0 \leq h_1 \leq v_1 \leq r$, $0 \leq h_2 \leq v_2 \leq p$ and let $f \in C^{r,p}([0, 1]^2)$. Let $\alpha_{i,j}(x, y)$, $i = h_1, h_1+1, \dots, v_1$; $j = h_2, h_2+1, \dots, v_2$ be real-valued functions, defined and bounded in $[0, 1]^2$ and assume $\alpha_{h_1 h_2}$ is either $\geq \alpha > 0$ or $\leq \beta < 0$ throughout $[0, 1]^2$. Consider the operator*

$$L = \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{ij}(x, y) \frac{\partial^{i+j}}{\partial x^i \partial y^j} \quad (4)$$

and suppose that throughout $[0, 1]^2$,

$$L(f) \geq 0.$$

Then for integers m, n with $m > r$, $n > p$, there exists a polynomial $Q_{m,n}(x, y)$ of degree (m, n) such that $L(Q_{m,n}(x, y)) \geq 0$ throughout $[0, 1]^2$ and

$$\left\| f^{(k,l)} - Q_{m,n}^{(k,l)} \right\|_{\infty} \leq \frac{P_{m,n}(L, f)}{(h_1 - k)!(h_2 - l)!} + M_{m,n}^{k,l}(f), \quad (5)$$

all $(0, 0) \leq (k, l) \leq (h_1, h_2)$. Furthermore we get

$$\left\| f^{(k,l)} - Q_{m,n}^{(k,l)} \right\|_{\infty} \leq M_{m,n}^{k,l}(f), \quad (6)$$

for all $(h_1 + 1, h_2 + 1) \leq (k, l) \leq (r, p)$. Also (6) is true whenever $0 \leq k \leq h_1$, $h_2 + 1 \leq l \leq p$ or $h_1 + 1 \leq k \leq r$, $0 \leq l \leq h_2$. Here

$$M_{m,n}^{k,l} \equiv M_{m,n}^{k,l}(f) \equiv t(k, l).$$

$$\omega_1 \left(f^{(k,l)}; \frac{1}{\sqrt{m-k}}, \frac{1}{\sqrt{n-l}} \right) + \max \left\{ \frac{k(k-1)}{m}, \frac{l(l-1)}{n} \right\} \cdot \left\| f^{(k,l)} \right\|_{\infty} \quad (7)$$

and

$$P_{m,n} \equiv P_{m,n}(L, f) \equiv \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} l_{ij} \cdot M_{m,n}^{i,j}, \quad (8)$$

where t is a positive real-valued function on \mathbb{Z}_+^2 and

$$l_{ij} \equiv \sup_{(x,y) \in [0,1]^2} |\alpha_{h_1 h_2}^{-1}(x, y) \cdot \alpha_{ij}(x, y)| < \infty. \quad (9)$$

In this article we extend Theorem 5 to the right fractional level. Indeed here L is replaced by \bar{L} , a linear right Caputo fractional mixed partial differential operator. Now the monotonicity property is only true on a critical region of $[0, 1]^2$ that depends on \bar{L} parameters. Simultaneous right fractional convergence remains true on all of $[0, 1]^2$.

We need

Definition 6 (see [4]) Let $\alpha_1, \alpha_2 > 0$; $\alpha = (\alpha_1, \alpha_2)$, $f \in C([0, 1]^2)$ and let $x = (x_1, x_2)$, $t = (t_1, t_2) \in [0, 1]^2$. We define the right mixed Riemann-Liouville fractional two dimensional integral of order α :

$$(I_{1-}^\alpha f)(x) := \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{x_1}^1 \int_{x_2}^1 (t_1 - x_1)^{\alpha_1-1} (t_2 - x_2)^{\alpha_2-1} f(t_1, t_2) dt_1 dt_2,$$

with $x_1, x_2 < 1$. Here Γ stands for the Gamma function.

Notice here

$$I_{1-}^\alpha (|f|) < \infty. \quad (10)$$

Definition 7 Let $\alpha_1, \alpha_2 > 0$ with $[\alpha_1] = m_1$, $[\alpha_2] = m_2$, ($[\cdot]$ ceiling of the number). Let here $f \in C^{m_1, m_2}([0, 1]^2)$. We consider the right (Caputo type) fractional partial derivative:

$$D_{1-}^{(\alpha_1, \alpha_2)} f(x) := \frac{(-1)^{m_1+m_2}}{\Gamma(m_1 - \alpha_1)\Gamma(m_2 - \alpha_2)}.$$

$$\int_{x_1}^1 \int_{x_2}^1 (J_1 - x_1)^{m_1 - \alpha_1 - 1} (J_2 - x_2)^{m_2 - \alpha_2 - 1} \frac{\partial^{m_1+m_2} f(J_1, J_2)}{\partial J_1^{m_1} \partial J_2^{m_2}} dJ_1 dJ_2, \quad (11)$$

$\forall x = (x_1, x_2) \in [0, 1]^2$.

We set

$$\begin{aligned} D_{1-}^{(0,0)} f(x) &:= f(x), \\ D_{1-}^{(m_1, m_2)} f(x) &:= (-1)^{m_1+m_2} \frac{\partial^{m_1+m_2} f(x)}{\partial x_1^{m_1} \partial x_2^{m_2}}, \quad \forall x \in [0, 1]^2. \end{aligned} \quad (12)$$

Definition 8 We also set

$$D_{1-}^{(0,\alpha_2)} f(x) := \frac{(-1)^{m_2}}{\Gamma(m_2 - \alpha_2)} \int_{x_2}^1 (J_2 - x_2)^{m_2 - \alpha_2 - 1} \frac{\partial^{m_2} f(x_1, J_2)}{\partial J_2^{m_2}} dJ_2, \quad (13)$$

$$D_{1-}^{(\alpha_1,0)} f(x) := \frac{(-1)^{m_1}}{\Gamma(m_1 - \alpha_1)} \int_{x_1}^1 (J_1 - x_1)^{m_1 - \alpha_1 - 1} \frac{\partial^{m_1} f(J_1, x_2)}{\partial J_1^{m_1}} dJ_1, \quad (14)$$

and

$$D_{1-}^{(m_1,\alpha_2)} f(x) := \frac{(-1)^{m_2}}{\Gamma(m_2 - \alpha_2)} \int_{x_2}^1 (J_2 - x_2)^{m_2 - \alpha_2 - 1} \frac{\partial^{m_1+m_2} f(x_1, J_2)}{\partial x_1^{m_1} \partial J_2^{m_2}} dJ_2, \quad (15)$$

$$D_{1-}^{(\alpha_1,m_2)} f(x) := \frac{(-1)^{m_1}}{\Gamma(m_1 - \alpha_1)} \int_{x_1}^1 (J_1 - x_1)^{m_1 - \alpha_1 - 1} \frac{\partial^{m_1+m_2} f(J_1, x_2)}{\partial J_1^{m_1} \partial x_2^{m_2}} dJ_1. \quad (16)$$

2 Main Result

We present our main result

Theorem 9 Let h_1, h_2, v_1, v_2, r, p be integers, $0 \leq h_1 \leq v_1 \leq r$, $0 \leq h_2 \leq v_2 \leq p$ and let $f \in C^{r,p}([0, 1]^2)$. Let $\alpha_{ij}(x, y)$, $i = h_1, h_1+1, \dots, v_1$; $j = h_2, h_2+1, \dots, v_2$ be real valued functions, defined and bounded in $[0, 1]^2$ and assume $\alpha_{h_1 h_2}$ is either $\geq \alpha > 0$ or $\leq \beta < 0$ throughout $[0, 1]^2$.

Let integers $\overline{m}_1, \overline{m}_2$ with $\overline{m}_1 > r$, $\overline{m}_2 > p$. Set

$$l_{ij} := \sup_{(x,y) \in [0,1]^2} |\alpha_{h_1 h_2}^{-1}(x, y) \cdot \alpha_{ij}(x, y)| < \infty. \quad (17)$$

Also set ($\lceil \alpha_{1i} \rceil = i$, $\lceil \alpha_{2j} \rceil = j$, $\lceil \cdot \rceil$ ceiling of number)

$$M_{\overline{m}_1, \overline{m}_2}^{i,j} := M_{\overline{m}_1, \overline{m}_2}^{i,j}(f) := \frac{1}{\Gamma(i - \alpha_{1i} + 1) \Gamma(j - \alpha_{2j} + 1)} \left\{ t(i, j) \omega_1 \left(f^{(i,j)}; \frac{1}{\sqrt{\overline{m}_1 - i}}, \frac{1}{\sqrt{\overline{m}_2 - j}} \right) + \max \left\{ \frac{i(i-1)}{\overline{m}_1}, \frac{j(j-1)}{\overline{m}_2} \right\} \cdot \|f^{(i,j)}\|_\infty \right\}, \quad (18)$$

$i = h_1, \dots, v_1$; $j = h_2, \dots, v_2$.

Here t is a positive real-valued function on \mathbb{Z}_+^2 , $\|\cdot\|_\infty$ is the supremum norm on $[0, 1]^2$. Call

$$P_{\overline{m}_1, \overline{m}_2} := P_{\overline{m}_1, \overline{m}_2}(f) = \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} l_{ij} \cdot M_{\overline{m}_1, \overline{m}_2}^{i,j}. \quad (19)$$

Let $0 \leq \alpha_{1h_1} \leq h_1 < \alpha_{11} < h_1 + 1 < \alpha_{12} < h_1 + 2 < \alpha_{13} < h_1 + 3 < \dots < \alpha_{1v_1} < v_1 < \dots < \alpha_{1r} < r$, with $\lceil \alpha_{1h_1} \rceil = h_1$; $0 \leq \alpha_{2h_2} \leq h_2 < \alpha_{21} < h_2 + 1 < \alpha_{22} < h_2 + 2 < \alpha_{23} < h_2 + 3 < \dots < \alpha_{2v_2} < v_2 < \dots < \alpha_{2p} < p$, with $\lceil \alpha_{2h_2} \rceil = h_2$. Here $h_1 + h_2 = 2m$, $m = 0, 1, 2, \dots$.

Consider the right fractional bivariate operator

$$\bar{L} := \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{ij} (x, y) D_{1-}^{(\alpha_{1i}, \alpha_{2j})}. \quad (20)$$

Then there exists a polynomial $Q_{\bar{m}_1, \bar{m}_2}(x, y)$ of degree (\bar{m}_1, \bar{m}_2) on $[0, 1]^2$ such that

$$\begin{aligned} & \left\| D_{1-}^{(\alpha_{1k}, \alpha_{2l})} (f) - D_{1-}^{(\alpha_{1k}, \alpha_{2l})} (Q_{\bar{m}_1, \bar{m}_2}) \right\|_{\infty} \leq \\ & \frac{P_{\bar{m}_1, \bar{m}_2}}{(h_1 - k)! (h_2 - l)!} \left\{ \left[\sum_{\theta=0}^{h_1-k} \binom{h_1-k}{\theta} \frac{\Gamma(h_1 - k - \theta + 1)}{\Gamma(h_1 - \alpha_{1k} - \theta + 1)} \right] \right. \\ & \left. \left[\sum_{\rho=0}^{h_2-l} \binom{h_2-l}{\rho} \frac{\Gamma(h_2 - l - \rho + 1)}{\Gamma(h_2 - \alpha_{2l} - \rho + 1)} \right] \right\} + M_{\bar{m}_1, \bar{m}_2}^{k,l}, \end{aligned} \quad (21)$$

for $(0, 0) \leq (k, l) \leq (h_1, h_2)$.

If $(h_1 + 1, h_2 + 1) \leq (k, l) \leq (r, p)$, or $0 \leq k \leq h_1$, $h_2 + 1 \leq l \leq p$, or $h_1 + 1 \leq k \leq r$, $0 \leq l \leq h_2$, then

$$\left\| D_{1-}^{(\alpha_{1k}, \alpha_{2l})} (f) - D_{1-}^{(\alpha_{1k}, \alpha_{2l})} (Q_{\bar{m}_1, \bar{m}_2}) \right\|_{\infty} \leq M_{\bar{m}_1, \bar{m}_2}^{k,l}. \quad (22)$$

If $\bar{L}(f(0, 0)) \geq 0$, then $\bar{L}(Q_{\bar{m}_1, \bar{m}_2}(0, 0)) \geq 0$.

Let $0 < x < 1$, $0 < y < 1$, with $\alpha_{1h_1} \neq h_1$ and $\alpha_{2h_2} \neq h_2$, such that

$$1 - x \geq \Gamma(h_1 - \alpha_{1h_1} + 1)^{\frac{1}{(h_1 - \alpha_{1h_1})}}, \quad (23)$$

$$1 - y \geq \Gamma(h_2 - \alpha_{2h_2} + 1)^{\frac{1}{(h_2 - \alpha_{2h_2})}},$$

(equivalently,

$$x \leq 1 - \Gamma(h_1 - \alpha_{1h_1} + 1)^{\frac{1}{(h_1 - \alpha_{1h_1})}}, \quad (24)$$

$$y \leq 1 - \Gamma(h_2 - \alpha_{2h_2} + 1)^{\frac{1}{(h_2 - \alpha_{2h_2})}},$$

and

$$\bar{L}(f(x, y)) \geq 0.$$

Then

$$\bar{L}(Q_{\bar{m}_1, \bar{m}_2}(x, y)) \geq 0.$$

To prove Theorem 9 it takes some preparation. We need

Definition 10 Let f be a real-valued function defined on $[0, 1]^2$ and let $\overline{m}_1, \overline{m}_2 \in \mathbb{N}$. Let $B_{\overline{m}_1, \overline{m}_2}$ be the Bernstein (polynomial) operator of order $(\overline{m}_1, \overline{m}_2)$ given by

$$B_{\overline{m}_1, \overline{m}_2}(f; x_1, x_2) := \sum_{i_1=0}^{\overline{m}_1} \sum_{i_2=0}^{\overline{m}_2} f\left(\frac{i_1}{\overline{m}_1}, \frac{i_2}{\overline{m}_2}\right) \binom{\overline{m}_1}{i_1} \binom{\overline{m}_2}{i_2} x_1^{i_1} (1-x_1)^{\overline{m}_1-i_1} x_2^{i_2} (1-x_2)^{\overline{m}_2-i_2}. \quad (25)$$

We need the following simultaneous approximation result.

Theorem 11 (I. Badea, C. Badea [3]). It holds that

$$\begin{aligned} \left\| f^{(k_1, k_2)} - (B_{\overline{m}_1, \overline{m}_2} f)^{(k_1, k_2)} \right\|_{\infty} &\leq t(k_1, k_2) \omega_1\left(f^{(k_1, k_2)}; \frac{1}{\sqrt{\overline{m}_1 - k_1}}, \frac{1}{\sqrt{\overline{m}_2 - k_2}}\right) \\ &\quad + \max\left\{ \frac{k_1(k_1 - 1)}{\overline{m}_1}, \frac{k_2(k_2 - 1)}{\overline{m}_2} \right\} \cdot \left\| f^{(k_1, k_2)} \right\|_{\infty}, \end{aligned} \quad (26)$$

where $\overline{m}_1 > k_1 \geq 0$, $\overline{m}_2 > k_2 \geq 0$ are integers, f is a real-valued function on $[0, 1]^2$, such that $f^{(k_1, k_2)}$ is continuous, and t is a positive real-valued function on \mathbb{Z}_+^2 . Here $\|\cdot\|_{\infty}$ is the supremum norm on $[0, 1]^2$.

Remark 12 We assume that $\overline{m}_1 > m_1 = \lceil \alpha_1 \rceil$, $\overline{m}_2 > m_2 = \lceil \alpha_2 \rceil$, where $\alpha_1, \alpha_2 > 0$.

We consider also

$$\begin{aligned} D_{1-}^{(\alpha_1, \alpha_2)}(B_{\overline{m}_1, \overline{m}_2} f)(x_1, x_2) &= \frac{(-1)^{m_1+m_2}}{\Gamma(m_1 - \alpha_1) \Gamma(m_2 - \alpha_2)}. \quad (27) \\ \int_{x_1}^1 \int_{x_2}^1 (t_1 - x_1)^{m_1 - \alpha_1 - 1} (t_2 - x_2)^{m_2 - \alpha_2 - 1} \frac{\partial^{m_1+m_2}(B_{\overline{m}_1, \overline{m}_2} f)(t_1, t_2)}{\partial t_1^{m_1} \partial t_2^{m_2}} dt_1 dt_2, \end{aligned}$$

$\forall (x_1, x_2) \in [0, 1]^2$.

Proposition 13 Let $\alpha_1, \alpha_2 > 0$ with $\lceil \alpha_1 \rceil = m_1$, $\lceil \alpha_2 \rceil = m_2$, $f \in C^{m_1, m_2}([0, 1]^2)$, where $\overline{m}_1, \overline{m}_2 \in \mathbb{N}$: $\overline{m}_1 > m_1$, $\overline{m}_2 > m_2$. Then

$$\begin{aligned} \left\| D_{1-}^{(\alpha_1, \alpha_2)} f - D_{1-}^{(\alpha_1, \alpha_2)}(B_{\overline{m}_1, \overline{m}_2} f) \right\|_{\infty} &\leq \frac{1}{\Gamma(m_1 - \alpha_1 + 1) \Gamma(m_2 - \alpha_2 + 1)}. \\ &\quad \left\{ t(m_1, m_2) \omega_1\left(f^{(m_1, m_2)}; \frac{1}{\sqrt{\overline{m}_1 - m_1}}, \frac{1}{\sqrt{\overline{m}_2 - m_2}}\right) + \right. \\ &\quad \left. \max\left\{ \frac{m_1(m_1 - 1)}{\overline{m}_1}, \frac{m_2(m_2 - 1)}{\overline{m}_2} \right\} \cdot \left\| f^{(m_1, m_2)} \right\|_{\infty} \right\}, \end{aligned} \quad (28)$$

Proof. We observe the following

$$\left| D_{1-}^{(\alpha_1, \alpha_2)} f(x_1, x_2) - D_{1-}^{(\alpha_1, \alpha_2)} (B_{\bar{m}_1, \bar{m}_2} f)(x_1, x_2) \right| = \frac{1}{\Gamma(m_1 - \alpha_1) \Gamma(m_2 - \alpha_2)}. \quad (29)$$

$$\begin{aligned} & \left| \int_{x_1}^1 \int_{x_2}^1 (t_1 - x_1)^{m_1 - \alpha_1 - 1} (t_2 - x_2)^{m_2 - \alpha_2 - 1} \right. \\ & \quad \left. \left(\frac{\partial^{m_1 + m_2} f(t_1, t_2)}{\partial t_1^{m_1} \partial t_2^{m_2}} - \frac{\partial^{m_1 + m_2} (B_{\bar{m}_1, \bar{m}_2} f)(t_1, t_2)}{\partial t_1^{m_1} \partial t_2^{m_2}} \right) dt_1 dt_2 \right| \\ & \leq \frac{1}{\Gamma(m_1 - \alpha_1) \Gamma(m_2 - \alpha_2)} \int_{x_1}^1 \int_{x_2}^1 (t_1 - x_1)^{m_1 - \alpha_1 - 1} (t_2 - x_2)^{m_2 - \alpha_2 - 1}. \quad (30) \end{aligned}$$

$$\begin{aligned} & \left| \frac{\partial^{m_1 + m_2} f(t_1, t_2)}{\partial t_1^{m_1} \partial t_2^{m_2}} - \frac{\partial^{m_1 + m_2} (B_{\bar{m}_1, \bar{m}_2} f)(t_1, t_2)}{\partial t_1^{m_1} \partial t_2^{m_2}} \right| dt_1 dt_2 \stackrel{(26)}{\leq} \\ & \quad \left\{ t(m_1, m_2) \omega_1 \left(f^{(m_1, m_2)}; \frac{1}{\sqrt{\bar{m}_1} - m_1}, \frac{1}{\sqrt{\bar{m}_2} - m_2} \right) + \right. \\ & \quad \left. \max \left\{ \frac{m_1(m_1 - 1)}{\bar{m}_1}, \frac{m_2(m_2 - 1)}{\bar{m}_2} \right\} \cdot \|f^{(m_1, m_2)}\|_{\infty} \right\} \\ & \cdot \frac{1}{\Gamma(m_1 - \alpha_1) \Gamma(m_2 - \alpha_2)} \int_{x_1}^1 \int_{x_2}^1 (t_1 - x_1)^{m_1 - \alpha_1 - 1} (t_2 - x_2)^{m_2 - \alpha_2 - 1} dt_1 dt_2 = \quad (31) \end{aligned}$$

$$\begin{aligned} & \frac{(1 - x_1)^{m_1 - \alpha_1} (1 - x_2)^{m_2 - \alpha_2}}{\Gamma(m_1 - \alpha_1 + 1) \Gamma(m_2 - \alpha_2 + 1)} \\ & \quad \left\{ t(m_1, m_2) \omega_1 \left(f^{(m_1, m_2)}; \frac{1}{\sqrt{\bar{m}_1} - m_1}, \frac{1}{\sqrt{\bar{m}_2} - m_2} \right) \right. \\ & \quad \left. + \max \left\{ \frac{m_1(m_1 - 1)}{\bar{m}_1}, \frac{m_2(m_2 - 1)}{\bar{m}_2} \right\} \|f^{(m_1, m_2)}\|_{\infty} \right\}, \quad (32) \end{aligned}$$

$\forall (x_1, x_2) \in [0, 1]^2$. ■

Proof. of Theorem 9.

We need for $(0, 0) \leq (k, l) \leq (h_1, h_2)$ to calculate

$$\begin{aligned} & D_{1-}^{(\alpha_{1k}, \alpha_{2l})} \left(\frac{x^{h_1} y^{h_2}}{h_1! h_2!} \right) = \frac{(-1)^{k+l}}{\Gamma(k - \alpha_{1k}) \Gamma(l - \alpha_{2l})} \\ & \int_x^1 \int_y^1 (J_1 - x)^{k - \alpha_{1k} - 1} (J_2 - y)^{l - \alpha_{2l} - 1} \frac{J_1^{h_1 - k}}{(h_1 - k)!} \frac{J_2^{h_2 - l}}{(h_2 - l)!} dJ_1 dJ_2 = \quad (33) \end{aligned}$$

$$\begin{aligned} & \frac{(-1)^{k+l}}{(h_1 - k)! (h_2 - l)!} \left\{ \left(\frac{1}{\Gamma(k - \alpha_{1k})} \int_x^1 (J_1 - x)^{k - \alpha_{1k} - 1} J_1^{h_1 - k} dJ_1 \right) \cdot \right. \\ & \quad \left. \left(\frac{1}{\Gamma(l - \alpha_{2l})} \int_y^1 (J_2 - y)^{l - \alpha_{2l} - 1} J_2^{h_2 - l} dJ_2 \right) \right\}. \quad (34) \end{aligned}$$

We find that

$$\begin{aligned}
& \int_x^1 J_1^{h_1-k} (J_1 - x)^{k-\alpha_{1k}-1} dJ_1 = \\
& (-1)^{h_1-k} \int_x^1 (-J_1)^{h_1-k} (J_1 - x)^{k-\alpha_{1k}-1} dJ_1 = \\
& (-1)^{h_1-k} \int_x^1 ((1 - J_1) - 1)^{h_1-k} (J_1 - x)^{k-\alpha_{1k}-1} dJ_1 = \\
& (-1)^{h_1-k} \int_x^1 \left(\sum_{\theta=0}^{h_1-k} \binom{h_1-k}{\theta} (1 - J_1)^{h_1-k-\theta} (-1)^\theta \right) (J_1 - x)^{k-\alpha_{1k}-1} dJ_1 = \\
& \sum_{\theta=0}^{h_1-k} \binom{h_1-k}{\theta} (-1)^{h_1-k+\theta} \int_x^1 (1 - J_1)^{(h_1-k-\theta+1)-1} (J_1 - x)^{k-\alpha_{1k}-1} dJ_1 = \\
& \sum_{\theta=0}^{h_1-k} \binom{h_1-k}{\theta} (-1)^{h_1-k+\theta} \left\{ \frac{\Gamma(h_1 - k - \theta + 1) \Gamma(k - \alpha_{1k})}{\Gamma(h_1 - \alpha_{1k} - \theta + 1)} (1 - x)^{h_1 - \alpha_{1k} - \theta} \right\}.
\end{aligned} \tag{35}$$

Similarly we find

$$\begin{aligned}
& \int_y^1 (J_2 - y)^{l-\alpha_{2l}-1} J_2^{h_2-l} dJ_2 = \\
& \sum_{\rho=0}^{h_2-l} \binom{h_2-l}{\rho} (-1)^{h_2-l+\rho} \left\{ \frac{\Gamma(h_2 - l - \rho + 1) \Gamma(l - \alpha_{2l})}{\Gamma(h_2 - \alpha_{2l} - \rho + 1)} (1 - y)^{h_2 - \alpha_{2l} - \rho} \right\}.
\end{aligned} \tag{37}$$

Therefore

$$\begin{aligned}
& D_{1-}^{(\alpha_{1k}, \alpha_{2l})} \left(\frac{x^{h_1} y^{h_2}}{h_1! h_2!} \right) = \frac{1}{(h_1 - k)! (h_2 - l)!} \\
& \left\{ \left[\sum_{\theta=0}^{h_1-k} \binom{h_1-k}{\theta} (-1)^{h_1+\theta} \left\{ \frac{\Gamma(h_1 - k - \theta + 1)}{\Gamma(h_1 - \alpha_{1k} - \theta + 1)} (1 - x)^{h_1 - \alpha_{1k} - \theta} \right\} \right] \right. \\
& \left. \left[\sum_{\rho=0}^{h_2-l} \binom{h_2-l}{\rho} (-1)^{h_2+\rho} \left\{ \frac{\Gamma(h_2 - l - \rho + 1)}{\Gamma(h_2 - \alpha_{2l} - \rho + 1)} (1 - y)^{h_2 - \alpha_{2l} - \rho} \right\} \right] \right\}.
\end{aligned} \tag{38}$$

Here we use a lot Proposition 13.

Case (i). Assume that on $[0, 1]^2$, $\alpha_{h_1 h_2} \geq \alpha > 0$.

Call

$$Q_{\overline{m_1}, \overline{m_2}}(x, y) := B_{\overline{m_1}, \overline{m_2}}(f; x, y) + P_{\overline{m_1}, \overline{m_2}} \frac{x^{h_1} y^{h_2}}{h_1! h_2!}. \tag{39}$$

Then by (28) we get

$$\left\| D_{1-}^{(\alpha_{1k}, \alpha_{2l})} \left(f + P_{\overline{m_1}, \overline{m_2}} \frac{x^{h_1} y^{h_2}}{h_1! h_2!} \right) - D_{1-}^{(\alpha_{1k}, \alpha_{2l})} (Q_{\overline{m_1}, \overline{m_2}}(x, y)) \right\|_{\infty} \leq M_{\overline{m_1}, \overline{m_2}}^{k, l}, \tag{40}$$

all $0 \leq k \leq r, 0 \leq l \leq p$.

When $(0, 0) \leq (k, l) \leq (h_1, h_2)$, using (38) inequality (40) becomes

$$\begin{aligned} & \left\| D_{1-}^{(\alpha_{1k}, \alpha_{2l})} (f) + P_{\overline{m_1}, \overline{m_2}} \frac{1}{(h_1 - k)! (h_2 - l)!} \right. \\ & \left. \left\{ \left[\sum_{\theta=0}^{h_1-k} \binom{h_1-k}{\theta} (-1)^{h_1+\theta} \left\{ \frac{\Gamma(h_1-k-\theta+1)}{\Gamma(h_1-\alpha_{1k}-\theta+1)} (1-x)^{h_1-\alpha_{1k}-\theta} \right\} \right] \right. \right. \\ & \left. \left. \left[\sum_{\rho=0}^{h_2-l} \binom{h_2-l}{\rho} (-1)^{h_2+\rho} \left\{ \frac{\Gamma(h_2-l-\rho+1)}{\Gamma(h_2-\alpha_{2l}-\rho+1)} (1-y)^{h_2-\alpha_{2l}-\rho} \right\} \right] \right\} \right. \\ & \left. - D_{1-}^{(\alpha_{1k}, \alpha_{2l})} (Q_{\overline{m_1}, \overline{m_2}}(x, y)) \right\|_{\infty} \leq M_{\overline{m_1}, \overline{m_2}}^{k, l}, \end{aligned} \quad (41)$$

all $(0, 0) \leq (k, l) \leq (h_1, h_2), x, y \in [0, 1]$.

Using (41) and triangle inequality we obtain for $(0, 0) \leq (k, l) \leq (h_1, h_2)$ that

$$\begin{aligned} & \left\| D_{1-}^{(\alpha_{1k}, \alpha_{2l})} (f) - D_{1-}^{(\alpha_{1k}, \alpha_{2l})} (Q_{\overline{m_1}, \overline{m_2}}) \right\|_{\infty} \leq \frac{P_{\overline{m_1}, \overline{m_2}}}{(h_1 - k)! (h_2 - l)!} \\ & \left\{ \left[\sum_{\theta=0}^{h_1-k} \binom{h_1-k}{\theta} \left\{ \frac{\Gamma(h_1-k-\theta+1)}{\Gamma(h_1-\alpha_{1k}-\theta+1)} \right\} \right] \right. \\ & \left. \left[\sum_{\rho=0}^{h_2-l} \binom{h_2-l}{\rho} \left\{ \frac{\Gamma(h_2-l-\rho+1)}{\Gamma(h_2-\alpha_{2l}-\rho+1)} \right\} \right] \right\} + M_{\overline{m_1}, \overline{m_2}}^{k, l} \end{aligned} \quad (42)$$

proving (21).

Next if $(h_1 + 1, h_2 + 1) \leq (k, l) \leq (r, p)$, or $0 \leq k \leq h_1, h_2 + 1 \leq l \leq p$, or $h_1 + 1 \leq k \leq r, 0 \leq l \leq h_2$, we get by (40) that

$$\left\| D_{1-}^{(\alpha_{1k}, \alpha_{2l})} (f) - D_{1-}^{(\alpha_{1k}, \alpha_{2l})} (Q_{\overline{m_1}, \overline{m_2}}) \right\|_{\infty} \leq M_{\overline{m_1}, \overline{m_2}}^{k, l}, \quad (43)$$

proving (22).

Furthermore, if (x, y) in critical region, see (23), we get

$$\begin{aligned} & \alpha_{h_1 h_2}^{-1} (x, y) \overline{L} (Q_{\overline{m_1}, \overline{m_2}}(x, y)) = \alpha_{h_1 h_2}^{-1} (x, y) \overline{L} (f(x, y)) + \quad (44) \\ & P_{\overline{m_1}, \overline{m_2}} \frac{(1-x)^{h_1-\alpha_{1h_1}} (1-y)^{h_2-\alpha_{2h_2}}}{\Gamma(h_1-\alpha_{1h_1}+1) \Gamma(h_2-\alpha_{2h_2}+1)} + \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{h_1 h_2}^{-1} (x, y) \alpha_{ij} (x, y) \cdot \\ & D_{1-}^{(\alpha_{1i}, \alpha_{2j})} \left[Q_{\overline{m_1}, \overline{m_2}}(x, y) - f(x, y) - P_{\overline{m_1}, \overline{m_2}} \frac{x^{h_1} y^{h_2}}{h_1! h_2!} \right] \stackrel{(40)}{\geq} \\ & P_{\overline{m_1}, \overline{m_2}} \frac{(1-x)^{h_1-\alpha_{1h_1}} (1-y)^{h_2-\alpha_{2h_2}}}{\Gamma(h_1-\alpha_{1h_1}+1) \Gamma(h_2-\alpha_{2h_2}+1)} - \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} l_{ij} M_{\overline{m_1}, \overline{m_2}}^{i, j} = \end{aligned}$$

$$P_{\overline{m_1}, \overline{m_2}} \frac{(1-x)^{h_1-\alpha_1 h_1} (1-y)^{h_2-\alpha_2 h_2}}{\Gamma(h_1-\alpha_1 h_1+1) \Gamma(h_2-\alpha_2 h_2+1)} - P_{\overline{m_1}, \overline{m_2}} = \quad (45)$$

$$P_{\overline{m_1}, \overline{m_2}} \left[\frac{(1-x)^{h_1-\alpha_1 h_1} (1-y)^{h_2-\alpha_2 h_2}}{\Gamma(h_1-\alpha_1 h_1+1) \Gamma(h_2-\alpha_2 h_2+1)} - 1 \right] = P_{\overline{m_1}, \overline{m_2}}.$$

$$\left[\frac{(1-x)^{h_1-\alpha_1 h_1} (1-y)^{h_2-\alpha_2 h_2} - \Gamma(h_1-\alpha_1 h_1+1) \Gamma(h_2-\alpha_2 h_2+1)}{\Gamma(h_1-\alpha_1 h_1+1) \Gamma(h_2-\alpha_2 h_2+1)} \right] =: (*). \quad (46)$$

We know $\Gamma(1) = 1$, $\Gamma(2) = 1$, and Γ is convex and positive on $(0, \infty)$. Here $0 \leq h_1 - \alpha_1 h_1 < 1$ and $0 \leq h_2 - \alpha_2 h_2 < 1$, hence $1 \leq h_1 - \alpha_1 h_1 + 1 < 2$, $1 \leq h_2 - \alpha_2 h_2 + 1 < 2$. Thus $0 < \Gamma(h_1 - \alpha_1 h_1 + 1)$, $\Gamma(h_2 - \alpha_2 h_2 + 1) \leq 1$, and $1 - \Gamma(h_1 - \alpha_1 h_1 + 1) \Gamma(h_2 - \alpha_2 h_2 + 1) \geq 0$. Clearly acting as in (44)-(46), when $\overline{L}(f(0,0)) \geq 0$, we get $\overline{L}(Q_{\overline{m_1}, \overline{m_2}}(0,0)) \geq 0$.

Also clearly here on the critical region (23) we have $(*) \geq 0$. That is there $\overline{L}(Q_{\overline{m_1}, \overline{m_2}}(x,y)) \geq 0$.

Case (ii). Assume that throughout $[0, 1]^2$, $\alpha_{h_1 h_2} \leq \beta < 0$. Consider

$$Q_{\overline{m_1}, \overline{m_2}}^-(x, y) := B_{\overline{m_1}, \overline{m_2}}(f; x, y) - P_{\overline{m_1}, \overline{m_2}} \frac{x^{h_1}}{h_1!} \frac{y^{h_2}}{h_2!}.$$

Then by (28) we get

$$\left\| D_{1-}^{(\alpha_{1k}, \alpha_{2l})} \left(f - P_{\overline{m_1}, \overline{m_2}} \frac{x^{h_1}}{h_1!} \frac{y^{h_2}}{h_2!} \right) - D_{1-}^{(\alpha_{1k}, \alpha_{2l})} \left(Q_{\overline{m_1}, \overline{m_2}}^-(x, y) \right) \right\|_{\infty} \leq M_{\overline{m_1}, \overline{m_2}}^{k,l}, \quad (47)$$

all $0 \leq k \leq r$, $0 \leq l \leq p$.

Working similarly as earlier in this proof we derive again (21), (22).

Furthermore, if (x, y) in critical region, see (23), we get

$$\begin{aligned} \alpha_{h_1 h_2}^{-1}(x, y) \overline{L} \left(Q_{\overline{m_1}, \overline{m_2}}^-(x, y) \right) &= \alpha_{h_1 h_2}^{-1}(x, y) \overline{L}(f(x, y)) - \\ P_{\overline{m_1}, \overline{m_2}} \frac{(1-x)^{h_1-\alpha_1 h_1} (1-y)^{h_2-\alpha_2 h_2}}{\Gamma(h_1-\alpha_1 h_1+1) \Gamma(h_2-\alpha_2 h_2+1)} &+ \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{h_1 h_2}^{-1}(x, y) \alpha_{ij}(x, y). \end{aligned} \quad (48)$$

$$\begin{aligned} D_{1-}^{(\alpha_{1i}, \alpha_{2j})} \left[Q_{\overline{m_1}, \overline{m_2}}^-(x, y) - f(x, y) + P_{\overline{m_1}, \overline{m_2}} \frac{x^{h_1}}{h_1!} \frac{y^{h_2}}{h_2!} \right] &\stackrel{(47)}{\leq} \\ -P_{\overline{m_1}, \overline{m_2}} \frac{(1-x)^{h_1-\alpha_1 h_1} (1-y)^{h_2-\alpha_2 h_2}}{\Gamma(h_1-\alpha_1 h_1+1) \Gamma(h_2-\alpha_2 h_2+1)} &+ \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} l_{ij} M_{\overline{m_1}, \overline{m_2}}^{i,j} = \end{aligned} \quad (49)$$

$$P_{\overline{m_1}, \overline{m_2}} \left(1 - \frac{(1-x)^{h_1-\alpha_1 h_1} (1-y)^{h_2-\alpha_2 h_2}}{\Gamma(h_1-\alpha_1 h_1+1) \Gamma(h_2-\alpha_2 h_2+1)} \right) = P_{\overline{m_1}, \overline{m_2}}. \quad (50)$$

$$\left(\frac{\Gamma(h_1 - \alpha_{1h_1} + 1) \Gamma(h_2 - \alpha_{2h_2} + 1) - (1-x)^{h_1 - \alpha_{1h_1}} (1-y)^{h_2 - \alpha_{2h_2}}}{\Gamma(h_1 - \alpha_{1h_1} + 1) \Gamma(h_2 - \alpha_{2h_2} + 1)} \right) =: (**). \quad (51)$$

We know $\Gamma(1) = 1$, $\Gamma(2) = 1$, and Γ is convex and positive on $(0, \infty)$. Here $0 \leq h_1 - \alpha_{1h_1} < 1$ and $0 \leq h_2 - \alpha_{2h_2} < 1$, hence $1 \leq h_1 - \alpha_{1h_1} + 1 < 2$, $1 \leq h_2 - \alpha_{2h_2} + 1 < 2$. Thus $0 < \Gamma(h_1 - \alpha_{1h_1} + 1)$, $\Gamma(h_2 - \alpha_{2h_2} + 1) \leq 1$, and $\Gamma(h_1 - \alpha_{1h_1} + 1) \Gamma(h_2 - \alpha_{2h_2} + 1) - 1 \leq 0$. Clearly acting as in (48)-(51), when $\bar{L}(f(0, 0)) \geq 0$, we get $\bar{L}\left(Q_{\overline{m_1}, \overline{m_2}}^-(0, 0)\right) \geq 0$.

Also clearly here on the critical region (23) we get $(**) \leq 0$. That is there $\bar{L}\left(Q_{\overline{m_1}, \overline{m_2}}^-(x, y)\right) \geq 0$. ■

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