Received 17/03/14

Bivariate Left Fractional Pseudo-Polynomial Monotone Approximation

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Abstract

In this article we deal with the following general two-dimensional problem: Let f be a two variable continuously differentiable real valued function of a given order, let L^* be a linear left fractional mixed partial differential operator and suppose that $L^*(f) \geq 0$ on a critical region. Then for sufficiently large $n, m \in \mathbb{N}$, we can find a sequence of pseudo-polynomials $Q^*_{n,m}$ in two variables with the property $L^*\left(Q^*_{n,m}\right) \geq 0$ on this critical region such that f is approximated with rates fractionally and simultaneously by $Q^*_{n,m}$ in the uniform norm on the whole domain of f. This restricted approximation is given via inequalities involving the mixed modulus of smoothness $\omega_{s,q}$, $s, q \in \mathbb{N}$, of highest order integer partial derivative of f.

2010 AMS Mathematics Subject Classification : 26A33, 41A17, 41A25, 41A28, 41A29, 41A63, 41A99.
Keywords and Phrases: Monotone Approximation, Caputo fractional derivative, fractional linear differential operator, mixed modulus of smoothness, pseudo-polynomials.

1 Introduction

The topic of monotone approximation started in [10] has become a major trend in approximation theory. A typical problem in this subject is: given a positive integer k, approximate a given function whose kth derivative is ≥ 0 by polynomials having this property.

In [3] the authors replaced the kth derivative with a linear differential operator of order k. We mention this motivating result.

Theorem 1 *Let h, k, p be integers, $0 \leq h \leq k \leq p$ and let f be a real function, $f^{(p)}$ continuous in $[-1,1]$ with modulus of continuity $\omega_1\left(f^{(p)}, x\right)$ there. Let $a_j(x)$, $j = h, h+1, ..., k$ be real functions, defined and bounded on $[-1,1]$ and assume $a_h(x)$ is either $\geq$ some number $\alpha > 0$ or $\leq$ some number $\beta < 0$ throughout $[-1,1]$. Consider the operator*

$$L = \sum_{j=h}^{k} a_j(x) \left[\frac{d^j}{dx^j}\right] \tag{1}$$

and suppose, throughout $[-1,1]$,

$$L(f) \geq 0. \tag{2}$$

Then, for every integer $n \geq 1$, there is a real polynomial $Q_n(x)$ of degree $\leq n$ such that

$$L(Q_n) \geq 0 \text{ throughout } [-1,1] \tag{3}$$

and

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq C n^{k-p} \omega_1\left(f^{(p)}, \frac{1}{n}\right), \tag{4}$$

where C is independent of n or f.

Next let $n, m \in \mathbb{Z}_+$, P_θ denote the space of algebraic polynomials of degree $\leq \theta$. Consider the tensor product spaces $P_n \otimes C([-1,1])$, $C([-1,1]) \otimes P_m$ and their sum $P_n \otimes C([-1,1]) + C([-1,1]) \otimes P_m$, that is

$$P_n \otimes C([-1,1]) + C([-1,1]) \otimes P_m =$$

$$\left\{\sum_{i=0}^{n} x^i A_i(y) + \sum_{j=0}^{m} B_j(x) y^j;\ A_i, B_j \in C([-1,1]),\ x, y \in [-1,1]\right\}. \tag{5}$$

This is the space of pseudo-polynomials of degree $\leq (n, m)$, first introduced by A. Marchaud in 1924-1927 (see [7], [8]). Here $f^{(k,l)}$ denotes $\frac{\partial^{k+l} f}{\partial x^k \partial y^l}$, the (k,l)-partial derivative of f.

In this section we consider the space $C^{r,p}\left([-1,1]^2\right) = \{f : [-1,1]^2 \to \mathbb{R};$ $f^{(k,l)}$ is continuous for $0 \leq k \leq r$, $0 \leq l \leq p\}$. Let $f \in C\left([-1,1]^2\right)$; for $\delta_1, \delta_2 \geq 0$, define the mixed modulus of smoothness of order (s,q), $s, q \in \mathbb{N}$ (see [9], pp. 516-517) by

$$\omega_{s,q}(f; \delta_1, \delta_2) \equiv \sup\left\{\left|{}_x\Delta_{h_1}^s \circ_y \Delta_{h_2}^q f(x,y)\right| : (x,y),\right. \tag{6}$$

$$\left.(x + sh_1, y + qh_2) \in [-1,1]^2, |h_i| \leq \delta_i,\ i = 1,2\right\}.$$

Here

$$
{}_x\Delta_{h_1}^s \circ_y \Delta_{h_2}^q f(x,y) \equiv \sum_{\sigma=0}^{s}\sum_{\mu=0}^{q}(-1)^{s+q-\sigma-\mu}\binom{s}{\sigma}\binom{q}{\mu} f(x+\sigma h_1, y+\mu h_2) \tag{7}
$$

is a mixed difference of order (s,q).

We mention

Theorem 2 *(H.H. Gonska [4]). Let $r,p \in \mathbb{Z}_+$, $s,q \in \mathbb{N}$, and $f \in C^{r,p}\left([-1,1]^2\right)$. Let $n,m \in \mathbb{N}$ with $n \geq \max\{4(r+1), r+s\}$ and $m \geq \max\{4(p+1), p+q\}$. Then there exists a linear operator $Q_{n,m}$ from $C^{r,p}\left([-1,1]^2\right)$ into the space of pseudopolynomials $(P_n \otimes C([-1,1]) + C([-1,1]) \otimes P_m)$ such that*

$$
\left|(f - Q_{n,m}(f))^{(k,l)}(x,y)\right| \leq \tag{8}
$$

$$
M_{r,s} \cdot M_{p,q}(\Delta_n(x))^{r-k} \cdot (\Delta_m(y))^{p-l} \cdot \omega_{s,q}\left(f^{(r,p)}; \Delta_n(x), \Delta_m(y)\right),
$$

for all $(0,0) \leq (k,l) \leq (r,p)$, $x,y \in [-1,1]$, where

$$
\Delta_\theta(z) = \frac{\sqrt{1-z^2}}{\theta} + \frac{1}{\theta^2}, \quad \theta = n,m; \; z = x,y \in [-1,1]. \tag{9}
$$

The constants $M_{r,s}$, $M_{p,q}$, are independent of f, (x,y) and (n,m); they depend only on (r,s), (p,q), respectively.

See also [5], saying that $Q_{n,m}^{(r,p)}(f)$ is continuous on $[-1,1]^2$.

The need following result which is an easy consequence of the last theorem (see [9], p. 517).

Corollary 3 *Let $r,p \in \mathbb{Z}_+$, $s,q \in \mathbb{N}$, and $f \in C^{r,p}\left([-1,1]^2\right)$. Let $n,m \in \mathbb{N}$ with $n \geq \max\{4(r+1), r+s\}$ and $m \geq \max\{4(p+1), p+q\}$. Then there exists a pseudopolynomial $Q_{n,m} \equiv Q_{n,m}(f) \in (P_n \otimes C([-1,1]) + C([-1,1]) \otimes P_m)$ such that*

$$
\left\|f^{(k,l)} - Q_{n,m}^{(k,l)}\right\|_\infty \leq \frac{\dot{C}}{n^{r-k}m^{p-l}} \cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right), \tag{10}
$$

for all $(0,0) \leq (k,l) \leq (r,p)$. Here the constant $\dot{C}$ depends only on r,p,s,q.

Corollary 3 was used in the proof of the main motivational result that follows.

Theorem 4 *([1]) Let h_1, h_2, v_1, v_2, r, p be integers, $0 \leq h_1 \leq v_1 \leq r$, $0 \leq h_2 \leq v_2 \leq p$ and let $f \in C^{r,p}\left([-1,1]^2\right)$, with $f^{(r,p)}$ having a mixed modulus of smoothness $\omega_{s,q}\left(f^{(r,p)}; x, y\right)$ there, $s,q \in \mathbb{N}$. Let $\alpha_{i,j}(x,y)$, $i = h_1, h_1+1, ..., v_1$;*

$j = h_2, h_2+1, ..., v_2$ be real-valued functions, defined and bounded in $[-1,1]^2$ and suppose $\alpha_{h_1 h_2}$ is either $\geq \alpha > 0$ or $\leq \beta < 0$ throughout $[-1,1]^2$. Take the operator

$$L = \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{ij}(x,y) \frac{\partial^{i+j}}{\partial x^i \partial y^j} \tag{11}$$

and assume, throughout $[-1,1]^2$ that

$$L(f) \geq 0. \tag{12}$$

Then for any integers n, m with $n \geq \max\{4(r+1), r+s\}$, $m \geq \max\{4(p+1), p+q\}$, there exists a pseudopolynomial

$$Q_{n,m} \in (P_n \otimes C([-1,1]) + C([-1,1]) \otimes P_m)$$

such that $L(Q_{m,n}) \geq 0$ throughout $[-1,1]^2$ and

$$\left\| f^{(k,l)} - Q_{n,m}^{(k,l)} \right\|_\infty \leq \frac{C}{n^{r-v_1} m^{p-v_2}} \cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right), \tag{13}$$

for all $(0,0) \leq (k,l) \leq (h_1,h_2)$. Moreover we get

$$\left\| f^{(k,l)} - Q_{n,m}^{(k,l)} \right\|_\infty \leq \frac{C}{n^{r-k} m^{p-l}} \cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right), \tag{14}$$

for all $(h_1+1, h_2+1) \leq (k,l) \leq (r,p)$. Also (14) is valid whenever $0 \leq k \leq h_1$, $h_2+1 \leq l \leq p$ or $h_1+1 \leq k \leq r$, $0 \leq l \leq h_2$. Here C is a constant independent of f and n,m. It depends only on r,p,s,q,L.

We are also motivated by [2].

We need

Definition 5 *(see [6]) Let $[-1,1]^2$; $\alpha_1, \alpha_2 > 0$; $\alpha = (\alpha_1, \alpha_2)$, $f \in C\left([-1,1]^2\right)$, $x = (x_1,x_2)$, $t = (t_1,t_2) \in [-1,1]^2$. We define the left mixed Riemann-Liouville fractional two dimensional integral of order α*

$$\left(I_{-1+}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_{-1}^{x_1} \int_{-1}^{x_2} (x_1-t_1)^{\alpha_1-1} (x_2-t_2)^{\alpha_2-1} f(t_1,t_2)\, dt_1 dt_2, \tag{15}$$

with $x_1, x_2 > -1$.

Notice here that $I_{-1+}^{\alpha}(|f|) < \infty$.

Definition 6 *Let $\alpha_1, \alpha_2 > 0$ with $\lceil \alpha_1 \rceil = m_1$, $\lceil \alpha_2 \rceil = m_2$, ($\lceil \cdot \rceil$ ceiling of the number). Let here $f \in C^{m_1,m_2}\left([-1,1]^2\right)$. We consider the left Caputo type fractional partial derivative:*

$$D_{*(-1)}^{(\alpha_1,\alpha_2)} f(x) := \frac{1}{\Gamma(m_1-\alpha_1)\Gamma(m_2-\alpha_2)} \cdot$$

$$\int_{-1}^{x_1}\int_{-1}^{x_2}(x_1-t_1)^{m_1-\alpha_1-1}(x_2-t_2)^{m_2-\alpha_2-1}\frac{\partial^{m_1+m_2}f(t_1,t_2)}{\partial t_1^{m_1}\partial t_2^{m_2}}dt_1dt_2, \quad (16)$$

$\forall\ x=(x_1,x_2)\in[-1,1]^2$, where Γ is the gamma function

$$\Gamma(\nu)=\int_0^{\infty}e^{-t}t^{\nu-1}dt, \quad \nu>0. \quad (17)$$

We set

$$D_{*(-1)}^{(0,0)}f(x):=f(x), \quad \forall\ x\in[-1,1]^2; \quad (18)$$

$$D_{*(-1)}^{(m_1,m_2)}f(x):=\frac{\partial^{m_1+m_2}f(x)}{\partial x_1^{m_1}\partial x_2^{m_2}}, \quad \forall\ x\in[-1,1]^2. \quad (19)$$

Definition 7 *We also set*

$$D_{*(-1)}^{(0,\alpha_2)}f(x):=\frac{1}{\Gamma(m_2-\alpha_2)}\int_{-1}^{x_2}(x_2-t_2)^{m_2-\alpha_2-1}\frac{\partial^{m_2}f(x_1,t_2)}{\partial t_2^{m_2}}dt_2, \quad (20)$$

$$D_{*(-1)}^{(\alpha_1,0)}f(x):=\frac{1}{\Gamma(m_1-\alpha_1)}\int_{-1}^{x_1}(x_1-t_1)^{m_1-\alpha_1-1}\frac{\partial^{m_1}f(t_1,x_2)}{\partial t_1^{m_1}}dt_1, \quad (21)$$

and

$$D_{*(-1)}^{(m,\alpha_2)}f(x):=\frac{1}{\Gamma(m_2-\alpha_2)}\int_{-1}^{x_2}(x_2-t_2)^{m_2-\alpha_2-1}\frac{\partial^{m_1+m_2}f(x_1,t_2)}{\partial x_1^{m_1}\partial t_2^{m_2}}dt_2, \quad (22)$$

$$D_{*(-1)}^{(\alpha_1,m_2)}f(x):=\frac{1}{\Gamma(m_1-\alpha_1)}\int_{-1}^{x_1}(x_1-t_1)^{m_1-\alpha_1-1}\frac{\partial^{m_1+m_2}f(t_1,x_2)}{\partial t_1^{m_1}\partial x_2^{m_2}}dt_1. \quad (23)$$

In this article we extend Theorem 4 to the fractional level. Indeed here L is replaced by L^*, a linear left Caputo fractional mixed partial differential operator. Now the monotonicity property holds true only on the critical square of $[0,1]^2$. Simultaneously fractional convergence remains true on all of $[-1,1]^2$.

2 Main Result

We present

Theorem 8 *Let h_1,h_2,v_1,v_2,r,p be integers, $0\le h_1\le v_1\le r$, $0\le h_2\le v_2\le p$ and let $f\in C^{r,p}\left([-1,1]^2\right)$, with $f^{(r,p)}$ having a mixed modulus of smoothness $\omega_{s,q}\left(f^{(r,p)};x,y\right)$ there, $s,q\in\mathbb{N}$. Let $\alpha_{ij}(x,y)$, $i=h_1,h_1+1,...,v_1$; $j=h_2,h_2+1,...,v_2$ be real valued functions, defined and bounded in $[-1,1]^2$ and suppose $\alpha_{h_1h_2}$ is either $\ge\alpha>0$ or $\le\beta<0$ throughout $[0,1]^2$. Here $n,m\in\mathbb{N}:n\ge\max\{4(r+1),r+s\}$, $m\ge\max\{4(p+1,p+q)\}$. Set*

$$l_{ij}:=\sup_{(x,y)\in[-1,1]^2}\left|\alpha_{h_1h_2}^{-1}(x,y)\,\alpha_{ij}(x,y)\right|<\infty, \quad (24)$$

for all $h_1 \leq i \leq v_1$, $h_2 \leq j \leq v_2$. *Let* $\alpha_{1i}, \alpha_{2j} \geq 0$ *with* $\lceil \alpha_{1i} \rceil = i$, $\lceil \alpha_{2j} \rceil = j$, $i = 0, 1, ..., r$; $j = 0, 1, ..., p$, *(*$\lceil \cdot \rceil$ *ceiling of the number),* $\alpha_{10} = 0$, $\alpha_{20} = 0$.

Consider the left fractional bivariate differential operator

$$L^* := \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{ij} (x, y) D_{*(-1)}^{(\alpha_{1i}, \alpha_{2j})}. \tag{25}$$

Assume $L^* f (x, y) \geq 0$, *on* $[0, 1]^2$.

Then there exists

$$Q_{n,m}^* \equiv Q_{n,m}^* (f) \in (P_n \otimes C ([-1, 1]) + C ([-1, 1]) \otimes P_m)$$

such that $L^* Q_{n,m}^* (x, y) \geq 0$, *on* $[0, 1]^2$.

Furthermore it holds:

1)

$$\left\| D_{*(-1)}^{(\alpha_{1i}, \alpha_{2j})} (f) - D_{*(-1)}^{(\alpha_{11}, \alpha_{2j})} Q_{n,m}^* \right\|_{\infty, [-1,1]^2} \leq$$

$$\frac{\dot{C} 2^{(i+j) - (\alpha_{1i} + \alpha_{2j})}}{\Gamma (i - \alpha_{1i} + 1) \Gamma (j - \alpha_{2j} + 1) n^{r-i} m^{p-j}} \cdot \omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right), \tag{26}$$

where $\dot{C}$ *is a constant that depends only on* r, p, s, q; $(h_1 + 1, h_2 + 1) \leq (i, j) \leq (r, p)$, *or* $0 \leq i \leq h_1$, $h_2 + 1 \leq j \leq p$, *or* $h_1 + 1 \leq i \leq r$, $0 \leq j \leq h_2$,

2)

$$\left\| D_{*(-1)}^{(\alpha_{1i}, \alpha_{2j})} (f) - D_{*(-1)}^{(\alpha_{1i}, \alpha_{2j})} Q_{n,m}^* \right\|_{\infty, [-1,1]^2} \leq$$

$$\frac{c_{ij}}{n^{r-v_1} m^{p-v_2}} \cdot \omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right), \tag{27}$$

for $(1, 1) \leq (i, j) \leq (h_1, h_2)$, *where* $c_{ij} = \dot{C} A_{ij}$, *with*

$$A_{ij} :=$$

$$\left\{ \left[\sum_{\tau=h_1}^{v_1} \sum_{\mu=h_2}^{v_2} \frac{l_{\tau\mu} 2^{(\tau+\mu) - (\alpha_{1\tau} + \alpha_{2\mu})}}{\Gamma (\tau - a_{1\tau} + 1) \Gamma (\mu - \alpha_{2\mu} + 1)} \right] \left(\sum_{k=0}^{h_1 - i} \frac{2^{h_1 - \alpha_{1i} - k}}{k! \Gamma (h_1 - \alpha_{1i} - k + 1)} \right) \cdot \right. \tag{28}$$

$$\left. \left(\sum_{\lambda=0}^{h_2 - j} \frac{2^{h_2 - \alpha_{2j} - \lambda}}{\lambda! \Gamma (h_2 - \alpha_{2j} - \lambda + 1)} \right) + \frac{2^{(i+j) - (\alpha_{1i} + \alpha_{2j})}}{\Gamma (i - \alpha_{1i} + 1) \Gamma (j - \alpha_{2j} + 1)} \right\},$$

3)

$$\left\| f - Q_{n,m}^* \right\|_{\infty, [-1,1]^2} \leq \frac{c_{00}}{n^{r-v_1} m^{p-v_2}} \cdot \omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right), \tag{29}$$

where $c_{00} := \dot{C}A_{00}$, *with*

$$A_{00} := \frac{1}{h_1!h_2!}\left(\sum_{\tau=h_1}^{v_1}\sum_{\mu=h_2}^{v_2} l_{\tau\mu}\frac{2^{(\tau+\mu)-(\alpha_{1\tau}+\alpha_{2\mu})}}{\Gamma\left(\tau-a_{1\tau}+1\right)\Gamma\left(\mu-\alpha_{2\mu}+1\right)}\right)+1,$$

4)

$$\left\|D_{*(-1)}^{(0,\alpha_{2j})}\left(f\right)-D_{*(-1)}^{(0,\alpha_{2j})}Q_{n,m}^{*}\right\|_{\infty,[-1,1]^2}\leq$$

$$\frac{c_{0j}}{n^{r-v_1}m^{p-v_2}}\cdot\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right), \tag{30}$$

where $c_{0j} = \dot{C}A_{0j}$, $j = 1, ..., h_2$, *with*

$$A_{0j} := \left[\frac{1}{h_1!}\left(\sum_{\tau=h_1}^{v_1}\sum_{\mu=h_2}^{v_2} l_{\tau\mu}\frac{2^{(\tau+\mu)-(\alpha_{1\tau}+\alpha_{2\mu})}}{\Gamma\left(\tau-a_{1\tau}+1\right)\Gamma\left(\mu-\alpha_{2\mu}+1\right)}\right)\right. \tag{31}$$

$$\left.\left(\sum_{\lambda=0}^{h_2-j}\frac{2^{h_2-\alpha_{2j}-\lambda}}{\lambda!\Gamma\left(h_2-\alpha_{2j}-\lambda+1\right)}\right)+\frac{2^{j-\alpha_{2j}}}{\Gamma\left(j-\alpha_{2j}+1\right)}\right],$$

5)

$$\left\|D_{*(-1)}^{(\alpha_{1i},0)}\left(f\right)-D_{*(-1)}^{(\alpha_{1i},0)}Q_{n,m}^{*}\right\|_{\infty,[-1,1]^2}\leq$$

$$\frac{c_{i0}}{n^{r-v_1}m^{p-v_2}}\cdot\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right), \tag{32}$$

where $c_{i0} = \dot{C}A_{i0}$, $i = i, ..., h_1$, *with*

$$A_{i0} := \left[\frac{1}{h_2!}\left(\sum_{\tau=h_1}^{v_1}\sum_{\mu=h_2}^{v_2} l_{\tau\mu}\frac{2^{(\tau+\mu)-(\alpha_{1\tau}+\alpha_{2\mu})}}{\Gamma\left(\tau-a_{1\tau}+1\right)\Gamma\left(\mu-\alpha_{2\mu}+1\right)}\right)\right.\cdot \tag{33}$$

$$\left.\left(\sum_{k=0}^{h_1-i}\frac{2^{h_1-\alpha_{1i}-k}}{k!\Gamma\left(h_1-\alpha_{1i}-k+1\right)}\right)+\frac{2^{i-\alpha_{1i}}}{\Gamma\left(i-\alpha_{1i}+1\right)}\right].$$

Proof. By Corollary 3 there exists

$$Q_{n,m}\equiv Q_{n,m}\left(f\right)\in\left(P_n\otimes C\left([-1,1]\right)+C\left([-1,1]\right)\otimes P_m\right)$$

such that

$$\left\|f^{(i,j)}-Q_{n,m}^{(i,j)}\right\|_{\infty}\leq\frac{\dot{C}}{n^{r-i}m^{p-j}}\cdot\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right), \tag{34}$$

for all $(0,0) \leq (i,j) \leq (r,p)$, while $Q_{n,m} \in C^{r,p}\left([-1,1]\right)^2$. Here $\dot{C}$ depends only on r,p,s,q, where $n \geq \max\{4(r+1), r+s\}$ and $m \geq \max\{4(p+1), p+q\}$, with $r,p \in \mathbb{Z}_+,\ s,q \in \mathbb{N},\ f \in C^{r,p}\left([-1,1]^2\right)$.

Indeed by [5] we have that $Q_{n,m}^{(r,p)}$ is continuous on $[-1,1]^2$.

We observe the following ($i = 0,1,...,r;\ j = 0,1,...,p$)

$$\left|D_{*(-1)}^{(\alpha_{1i},\alpha_{2j})} f(x_1,x_2) - D_{*(-1)}^{(\alpha_{1i},\alpha_{2j})} Q_{n,m}(x_1,x_2)\right| = \frac{1}{\Gamma(i-\alpha_{1i})\Gamma(j-\alpha_{2j})}\cdot$$

$$\left|\int_{-1}^{x_1}\int_{-1}^{x_2}(x_1-t_1)^{i-\alpha_{1i}-1}(x_2-t_2)^{j-\alpha_{2j}-1}\cdot\right.$$

$$\left.\left(\frac{\partial^{i+j}f(t_1,t_2)}{\partial t_1^i \partial t_2^j} - \frac{\partial^{i+j}Q_{n,m}(t_1,t_2)}{\partial t_1^i \partial t_2^j}\right)dt_1 dt_2\right| \leq \tag{35}$$

$$\frac{1}{\Gamma(i-\alpha_{1i})\Gamma(j-\alpha_{2j})}\int_{-1}^{x_1}\int_{-1}^{x_2}(x_1-t_1)^{i-\alpha_{1i}-1}(x_2-t_2)^{j-\alpha_{2j}-1}\cdot \tag{36}$$

$$\left|\frac{\partial^{i+j}f(t_1,t_2)}{\partial t_1^i \partial t_2^j} - \frac{\partial^{i+j}Q_{n,m}(t_1,t_2)}{\partial t_1^i \partial t_2^j}\right| dt_1 dt_2 \leq$$

$$\frac{1}{\Gamma(i-\alpha_{1i})\Gamma(j-\alpha_{2j})}\left(\int_{-1}^{x_1}\int_{-1}^{x_2}(x_1-t_1)^{i-\alpha_{1i}-1}(x_2-t_2)^{j-\alpha_{2j}-1}dt_1 dt_2\right)\cdot \tag{37}$$

$$\frac{\dot{C}}{n^{r-i}m^{p-j}}\cdot\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right) =$$

$$\frac{1}{\Gamma(i-\alpha_{1i})\Gamma(j-\alpha_{2j})}\frac{(x_1+1)^{i-\alpha_{1i}}}{i-\alpha_{1i}}\frac{(x_2+1)^{j-\alpha_{2j}}}{j-\alpha_{2j}}\frac{\dot{C}}{n^{r-i}m^{p-j}}\cdot\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right) \tag{38}$$

$$= \frac{(x_1+1)^{i-\alpha_{1i}}}{\Gamma(i-\alpha_{1i}+1)}\frac{(x_2+1)^{j-\alpha_{2j}}}{\Gamma(j-\alpha_{2j}+1)}\frac{\dot{C}}{n^{r-i}m^{p-j}}\cdot\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right).$$

That is there exists $Q_{n,m}$:

$$\left|D_{*(-1)}^{(\alpha_{1i},\alpha_{2j})} f(x_1,x_2) - D_{*(-1)}^{(\alpha_{1i},\alpha_{2j})} Q_{n,m}(x_1,x_2)\right| \leq \tag{39}$$

$$\frac{(x_1+1)^{i-\alpha_{1i}}(x_2+1)^{j-\alpha_{2j}}}{\Gamma(i-\alpha_{1i}+1)\Gamma(j-\alpha_{2j}+1)}\frac{\dot{C}}{n^{r-i}m^{p-j}}\cdot\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right),$$

$i = 0,1,...,r,\ j = 0,1,...,p,\ \forall (x_1,x_2) \in [-1,1]^2$.

We proved there exists $Q_{n,m}$ such that

$$\left\|D_{*(-1)}^{(\alpha_{1i},\alpha_{2j})}(f) - D_{*(-1)}^{(\alpha_{1i},\alpha_{2j})} Q_{n,m}^*\right\|_\infty \leq$$

$$\frac{2^{(i+j)-(\alpha_{1i}+\alpha_{2j})}\dot{C}}{\Gamma\left(i-\alpha_{1i}+1\right)\Gamma\left(j-\alpha_{2j}+1\right)n^{r-i}m^{p-j}}\cdot\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right), \tag{40}$$

$i=0,1,...,r,\ j=0,1,...,p.$

Define

$$\rho_{n,m}\equiv\dot{C}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)\cdot$$

$$\left[\sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2}\left(l_{ij}\frac{2^{(i+j)-(\alpha_{1i}+\alpha_{2j})}}{\Gamma\left(i-a_{1i}+1\right)\Gamma\left(j-\alpha_{2j}+1\right)}n^{i-r}m^{j-p}\right)\right]. \tag{41}$$

I. Suppose, throughout $[0,1]^2$, $\alpha_{h_1h_2}(x,y)\geq\alpha>0$. Let $Q^*_{n,m}(x,y)$, $(x,y)\in[-1,1]^2$, as in (40), so that

$$\left\|D^{(\alpha_{1i},\alpha_{2j})}_{*(-1)}\left(f(x,y)+\rho_{n,m}\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}\right)-D^{(\alpha_{1i},\alpha_{2j})}_{*(-1)}Q^*_{n,m}(x,y)\right\|_\infty\leq \tag{42}$$

$$\frac{2^{(i+j)-(\alpha_{1i}+\alpha_{2j})}\dot{C}}{\Gamma\left(i-\alpha_{1i}+1\right)\Gamma\left(j-\alpha_{2j}+1\right)n^{r-i}m^{p-j}}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)=:T_{ij},$$

$i=0,1,...,r;\ j=0,1,...,p.$

If $(h_1+1,h_2+1)\leq(i,j)\leq(r,p)$, or $0\leq i\leq h_1$, $h_2+1\leq j\leq p$, or $h_1+1\leq i\leq r$, $0\leq j\leq h_2$ we get from the last

$$\left\|D^{(\alpha_{1i},\alpha_{2j})}_{*(-1)}(f)-D^{(\alpha_{1i},\alpha_{2j})}_{*(-1)}Q^*_{n,m}\right\|_\infty\leq$$

$$\frac{2^{(i+j)-(\alpha_{1i}+\alpha_{2j})}\dot{C}}{\Gamma\left(i-\alpha_{1i}+1\right)\Gamma\left(j-\alpha_{2j}+1\right)n^{r-i}m^{p-j}}\cdot\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right), \tag{43}$$

proving (26).

If $(0,0)\leq(i,j)\leq(h_1,h_2)$, we get

$$\left\|D^{(\alpha_{1i},\alpha_{2j})}_{*(-1)}f(x,y)+\rho_{n,m}D^{\alpha_{1i}}_{*(-1)}\left(\frac{x^{h_1}}{h_1!}\right)D^{\alpha_{2j}}_{*(-1)}\left(\frac{y^{h_2}}{h_2!}\right)\right.$$

$$\left.-D^{(\alpha_{1i},\alpha_{2j})}_{*(-1)}Q^*_{n,m}(x,y)\right\|_\infty\leq T_{ij}. \tag{44}$$

That is for $i=1,...,h_1$; $j=1,...,h_2$, we obtain

$$\left\|D^{(\alpha_{1i},\alpha_{2j})}_{*(-1)}f(x,y)+\rho_{n,m}\left(\sum_{k=0}^{h_1-i}\frac{(-1)^k(x+1)^{h_1-\alpha_{1i}-k}}{k!\Gamma\left(h_1-\alpha_{1i}-k+1\right)}\right)\cdot\right.$$

$$\left.\left(\sum_{\lambda=0}^{h_2-j}\frac{(-1)^\lambda(y+1)^{h_2-\alpha_{2j}-\lambda}}{\lambda!\Gamma\left(h_2-\alpha_{2j}-\lambda+1\right)}\right)-D^{(\alpha_{1i},\alpha_{2j})}_{*(-1)}Q^*_{n,m}(x,y)\right\|_\infty\leq T_{ij}. \tag{45}$$

Hence for $(1,1) \leq (i,j) \leq (h_1,h_2)$, we have

$$\left\| D_{*(-1)}^{(\alpha_{1i},\alpha_{2j})} f - D_{*(-1)}^{(\alpha_{1i},\alpha_{2j})} Q_{n,m}^* \right\|_{\infty} \leq$$

$$\rho_{n,m} \left(\sum_{k=0}^{h_1-i} \frac{2^{h_1-\alpha_{1i}-k}}{k!\Gamma\left(h_1-\alpha_{1i}-k+1\right)} \right) \left(\sum_{\lambda=0}^{h_2-j} \frac{2^{h_2-\alpha_{2j}-\lambda}}{\lambda!\Gamma\left(h_2-\alpha_{2j}-\lambda+1\right)} \right) + T_{ij} = \tag{46}$$

$$\dot{C}\omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right) \left[\sum_{\bar{i}=h_1}^{v_1} \sum_{\bar{j}=h_2}^{v_2} l_{\bar{i}\bar{j}} \frac{2^{\left(\bar{i}+\bar{j}\right)-\left(\alpha_{1\bar{i}}+\alpha_{2\bar{j}}\right)}}{\Gamma\left(\bar{i}-\alpha_{1\bar{i}}+1\right)\Gamma\left(\bar{j}-\alpha_{2\bar{j}}+1\right)} \frac{1}{n^{r-\bar{i}}} \frac{1}{m^{p-\bar{j}}} \right] \tag{47}$$

$$\left(\sum_{k=0}^{h_1-i} \frac{2^{h_1-\alpha_{1i}-k}}{k!\Gamma\left(h_1-\alpha_{1i}-k+1\right)} \right) \left(\sum_{\lambda=0}^{h_2-j} \frac{2^{h_2-\alpha_{2j}-\lambda}}{\lambda!\Gamma\left(h_2-\alpha_{2j}-\lambda+1\right)} \right) +$$

$$\frac{2^{(i+j)-(\alpha_{1i}+\alpha_{2j})}\dot{C}\omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right)}{\Gamma\left(i-\alpha_{1i}+1\right)\Gamma\left(j-\alpha_{2j}+1\right) n^{r-i} m^{p-j}} \leq$$

$$\dot{C}\omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right) \frac{1}{n^{r-v_1} m^{p-v_2}} A_{ij}, \tag{48}$$

where

$$A_{ij} := \left\{ \left[\sum_{\bar{i}=h_1}^{v_1} \sum_{\bar{j}=h_2}^{v_2} l_{\bar{i}\bar{j}} \frac{2^{\left(\bar{i}+\bar{j}\right)-\left(\alpha_{1\bar{i}}+\alpha_{2\bar{j}}\right)}}{\Gamma\left(\bar{i}-\alpha_{1\bar{i}}+1\right)\Gamma\left(\bar{j}-\alpha_{2\bar{j}}+1\right)} \right] \cdot \right.$$

$$\left(\sum_{k=0}^{h_1-i} \frac{2^{h_1-\alpha_{1i}-k}}{k!\Gamma\left(h_1-\alpha_{1i}-k+1\right)} \right) \left(\sum_{\lambda=0}^{h_2-j} \frac{2^{h_2-\alpha_{2j}-\lambda}}{\lambda!\Gamma\left(h_2-\alpha_{2j}-\lambda+1\right)} \right)$$

$$\left. + \frac{2^{(i+j)-(\alpha_{1i}+\alpha_{2j})}}{\Gamma\left(i-\alpha_{1i}+1\right)\Gamma\left(j-\alpha_{2j}+1\right)} \right\}. \tag{49}$$

(Set $c_{ij} := \dot{C}A_{ij}$)

We proved, for $(1,1) \leq (i,j) \leq (h_1,h_2)$, that

$$\left\| D_{*(-1)}^{(\alpha_{1i},\alpha_{2j})} f - D_{*(-1)}^{(\alpha_{1i},\alpha_{2j})} Q_{n,m}^* \right\|_{\infty} \leq \frac{c_{ij}}{n^{r-v_1} m^{p-v_2}} \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right). \tag{50}$$

So that (27) is established.

When $i=j=0$ from (42) we obtain

$$\left\| f(x,y) + \rho_{n,m} \frac{x^{h_1}}{h_1!} \frac{y^{h_2}}{h_2!} - Q_{n,m}^*(x,y) \right\|_{\infty} \leq \frac{\dot{C}}{n^r m^p} \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right). \tag{51}$$

Hence

$$\left\| f - Q_{n,m}^* \right\|_{\infty} \leq \frac{\rho_{n,m}}{h_1! h_2!} + \frac{\dot{C}}{n^r m^p} \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right) = \tag{52}$$

$$\frac{\dot{C}}{h_1!h_2!}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)\cdot$$

$$\left[\sum_{\bar{i}=h_1}^{v_1}\sum_{\bar{j}=h_2}^{v_2} l_{\bar{i}\bar{j}}\frac{2^{\left(\bar{i}+\bar{j}\right)-\left(\alpha_{1\bar{i}}+\alpha_{2\bar{j}}\right)}}{\Gamma\left(\bar{i}-\alpha_{1\bar{i}}+1\right)\Gamma\left(\bar{j}-\alpha_{2\bar{j}}+1\right)}\frac{1}{n^{r-\bar{i}}}\frac{1}{m^{p-\bar{j}}}\right] \tag{53}$$

$$+\frac{\dot{C}}{n^r m^p}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)\leq\frac{\dot{C}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)}{n^{r-v_1}m^{p-v_2}}A_{00}, \tag{54}$$

where

$$A_{00}:=\left[\frac{1}{h_1!h_2!}\sum_{\bar{i}=h_1}^{v_1}\sum_{\bar{j}=h_2}^{v_2}\frac{l_{\bar{i}\bar{j}}2^{\left(\bar{i}+\bar{j}\right)-\left(\alpha_{1\bar{i}}+\alpha_{2\bar{j}}\right)}}{\Gamma\left(\bar{i}-\alpha_{1\bar{i}}+1\right)\Gamma\left(\bar{j}-\alpha_{2\bar{j}}+1\right)}+1\right]. \tag{55}$$

(Set $c_{00}=\dot{C}A_{00}$).

Then

$$\left\|f-Q_{n,m}^{*}\right\|_{\infty}\leq\frac{c_{00}}{n^{r-v_1}m^{p-v_2}}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right). \tag{56}$$

So that (29) is established.

Next case of $i=0$, $j=1,...,h_2$, from (42) we get

$$\left\|D_{*(-1)}^{(0,\alpha_{2j})}f(x,y)+\rho_{n,m}\frac{x^{h_1}}{h_1!}\left(\sum_{\lambda=0}^{h_2-j}\frac{(-1)^{\lambda}(y+1)^{h_2-\alpha_{2j}-\lambda}}{\lambda!\Gamma\left(h_2-\alpha_{2j}-\lambda+1\right)}\right)\right.$$

$$\left.-D_{*(-1)}^{(0,\alpha_{2j})}Q_{n,m}^{*}(x,y)\right\|_{\infty}\leq T_{0j}. \tag{57}$$

Then

$$\left\|D_{*(-1)}^{(0,\alpha_{2j})}f-D_{*(-1)}^{(0,\alpha_{2j})}Q_{n,m}^{*}\right\|_{\infty}\leq$$

$$\frac{\rho_{n,m}}{h_1!}\left(\sum_{\lambda=0}^{h_2-j}\frac{2^{h_2-\alpha_{2j}-\lambda}}{\lambda!\Gamma\left(h_2-\alpha_{2j}-\lambda+1\right)}\right)+T_{0j}= \tag{58}$$

$$\frac{\dot{C}}{h_1!}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)\left[\sum_{\bar{i}=h_1}^{v_1}\sum_{\bar{j}=h_2}^{v_2} l_{\bar{i}\bar{j}}\frac{2^{\left(\bar{i}+\bar{j}\right)-\left(\alpha_{1\bar{i}}+\alpha_{2\bar{j}}\right)}}{\Gamma\left(\bar{i}-\alpha_{1\bar{i}}+1\right)\Gamma\left(\bar{j}-\alpha_{2\bar{j}}+1\right)}\frac{1}{n^{r-\bar{i}}}\frac{1}{m^{p-\bar{j}}}\right] \tag{59}$$

$$\left(\sum_{\lambda=0}^{h_2-j}\frac{2^{h_2-\alpha_{2j}-\lambda}}{\lambda!\Gamma\left(h_2-\alpha_{2j}-\lambda+1\right)}\right)+\frac{2^{j-\alpha_{2j}}\dot{C}}{\Gamma\left(j-\alpha_{2j}+1\right)n^r m^{p-j}}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)\leq$$

$$\frac{\dot{C}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)}{n^{r-v_1}m^{p-v_2}}A_{0j}, \tag{60}$$

where

$$A_{0j} := \left[\frac{1}{h_1!} \left(\sum_{\overline{i}=h_1}^{v_1} \sum_{\overline{j}=h_2}^{v_2} l_{\overline{i}\overline{j}} \frac{2^{\left(\overline{i}+\overline{j}\right)-\left(\alpha_{1\overline{i}}+\alpha_{2\overline{j}}\right)}}{\Gamma\left(\overline{i}-\alpha_{1\overline{i}}+1\right)\Gamma\left(\overline{j}-\alpha_{2\overline{j}}+1\right)} \right) \cdot \right.$$

$$\left. \left(\sum_{\lambda=0}^{h_2-j} \frac{2^{h_2-\alpha_{2j}-\lambda}}{\lambda!\Gamma\left(h_2-\alpha_{2j}-\lambda+1\right)} \right) + \frac{2^{j-\alpha_{2j}}}{\Gamma\left(j-\alpha_{2j}+1\right)} \right]. \tag{61}$$

(Set $c_{0j} := \dot{C} A_{0j}$)

We proved that (case of $i = 0$, $j = 1, ..., h_2$)

$$\left\| D_{*(-1)}^{(0,\alpha_{2j})} f - D_{*(-1)}^{(0,\alpha_{2j})} Q_{n,m}^{*} \right\|_{\infty} \leq \frac{c_{0j}}{n^{r-v_1} m^{p-v_2}} \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right). \tag{62}$$

establishing (30).

Similarly we get for $i = 1, ..., h_1$, $j = 0$, that

$$\left\| D_{*(-1)}^{(\alpha_{1i},0)} f - D_{*(-1)}^{(\alpha_{1i},0)} Q_{n,m}^{*} \right\|_{\infty} \leq \frac{c_{i0}}{n^{r-v_1} m^{p-v_2}} \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right), \tag{63}$$

where $c_{i0} := \dot{C} A_{i0}$, with

$$A_{i0} := \left[\frac{1}{h_2!} \left(\sum_{\overline{i}=h_1}^{v_1} \sum_{\overline{j}=h_2}^{v_2} l_{\overline{i}\overline{j}} \frac{2^{\left(\overline{i}+\overline{j}\right)-\left(\alpha_{1\overline{i}}+\alpha_{2\overline{j}}\right)}}{\Gamma\left(\overline{i}-\alpha_{1\overline{i}}+1\right)\Gamma\left(\overline{j}-\alpha_{2\overline{j}}+1\right)} \right) \cdot \right.$$

$$\left. \left(\sum_{k=0}^{h_1-i} \frac{2^{h_1-\alpha_{1i}-k}}{k!\Gamma\left(h_1-\alpha_{1i}-k+1\right)} \right) + \frac{2^{i-\alpha_{1i}}}{\Gamma\left(i-\alpha_{1i}+1\right)} \Bigg\} \right], \tag{64}$$

deriving (32).

So if $(x, y) \in [0, 1]^2$, then

$$\alpha_{h_1h_2}^{-1}(x,y) L^{*}\left(Q_{n,m}^{*}(x,y)\right) = \alpha_{h_1h_2}^{-1}(x,y) L^{*}(f(x,y)) +$$

$$\rho_{n,m} \frac{(x+1)^{h_1-\alpha_{1i}}}{\Gamma\left(h_1-\alpha_{1i}+1\right)} \frac{(y+1)^{h_2-\alpha_{2j}}}{\Gamma\left(h_2-\alpha_{2j}+1\right)} + \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{h_1h_2}^{-1}(x,y)\, \alpha_{ij}(x,y) \cdot \tag{65}$$

$$\left[D_{*(-1)}^{(\alpha_{1i},\alpha_{2j})} Q_{n,m}^{*}(x,y) - D_{*(-1)}^{(\alpha_{1i},\alpha_{2j})} f(x,y) - \rho_{n,m} D_{*(-1)}^{(\alpha_{1i},\alpha_{2j})} \left(\frac{x^{h_1}}{h_1!} \frac{y^{h_2}}{h_2!} \right) \right] \overset{(42)}{\geq}$$

$$\rho_{n,m} \frac{(x+1)^{h_1-\alpha_{1i}}}{\Gamma\left(h_1-\alpha_{1i}+1\right)} \frac{(y+1)^{h_2-\alpha_{2j}}}{\Gamma\left(h_2-\alpha_{2j}+1\right)} - \tag{66}$$

$$\left[\sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} l_{ij} \frac{2^{(i+j)-(\alpha_{1i}+\alpha_{2j})}}{\Gamma\left(i-\alpha_{1i}+1\right)\Gamma\left(j-\alpha_{2j}+1\right)} \frac{\dot{C}}{n^{r-i} m^{p-j}} \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right) \right] =$$

$$\rho_{n,m}\left[\frac{(x+1)^{h_1-\alpha_{1i}}}{\Gamma(h_1-\alpha_{1i}+1)}\frac{(y+1)^{h_2-\alpha_{2j}}}{\Gamma(h_2-\alpha_{2j}+1)}-1\right]\geq \tag{67}$$

$$\rho_{n,m}\left[\frac{1}{\Gamma(h_1-\alpha_{1i}+1)\Gamma(h_2-\alpha_{2j}+1)}-1\right]= \tag{68}$$

$$\rho_{n,m}\left[\frac{1-\Gamma(h_1-\alpha_{1i}+1)\Gamma(h_2-\alpha_{2j}+1)}{\Gamma(h_1-\alpha_{1i}+1)\Gamma(h_2-\alpha_{2j}+1)}\right]\geq 0.$$

Explanation: we have that $\Gamma(1)=1$, $\Gamma(2)=1$, and Γ is convex on $(0,\infty)$ and positive there, here $0\leq h_1-\alpha_{1h_1}, h_2-\alpha_{2h_2}<1$ and $1\leq h_1-\alpha_{1h_1}+1, h_2-\alpha_{2h_2}+1<2$. Thus $0<\Gamma(h_1-\alpha_{1h_1}+1), \Gamma(h_2-\alpha_{2h_2}+1)\leq 1$, and

$$0\leq \Gamma(h_1-\alpha_{1h_1}+1)\Gamma(h_2-\alpha_{2h_2}+1)\leq 1. \tag{69}$$

And

$$1-\Gamma(h_1-\alpha_{1h_1}+1)\Gamma(h_2-\alpha_{2h_2}+1)\geq 0. \tag{70}$$

Therefore it holds

$$L^*(Q_{n,m}(x,y))\geq 0,\ \forall\ (x,y)\in[0,1]^2. \tag{71}$$

II. Suppose, throughout $[0,1]^2, \alpha_{h_1h_2}(x,y)\leq\beta<0$. Let $Q_{n,m}^{**}(x,y)$, $(x,y)\in[-1,1]^2$, as in (40), so that

$$\left\|D_{*(-1)}^{(\alpha_{1i},\alpha_{2j})}\left(f(x,y)-\rho_{n,m}\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}\right)-D_{*(-1)}^{(\alpha_{1i},\alpha_{2j})}Q_{n,m}^{**}(x,y)\right\|_\infty\leq$$

$$\frac{2^{(i+j)-(\alpha_{1i}+\alpha_{2j})}\dot{C}}{\Gamma(i-\alpha_{1i}+1)\Gamma(j-\alpha_{2j}+1)n^{r-i}m^{p-j}}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right), \tag{72}$$

$i=0,1,...,r,\ j=0,1,...,p.$

As earlier we produce the same convergence inequalities (26), (27), (29), (30), and (32).

So for $(x,y)\in[0,1]^2$ we get

$$\alpha_{h_1h_2}^{-1}(x,y)L^*\left(Q_{n,m}^{**}(x,y)\right)=\alpha_{h_1h_2}^{-1}(x,y)L^*(f(x,y))-$$

$$\rho_{n,m}\frac{(x+1)^{h_1-\alpha_{1i}}}{\Gamma(h_1-\alpha_{1i}+1)}\frac{(y+1)^{h_2-\alpha_{2j}}}{\Gamma(h_2-\alpha_{2j}+1)}+\sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2}\alpha_{h_1h_2}^{-1}(x,y)\alpha_{ij}(x,y)\cdot \tag{73}$$

$$\left[D_{*(-1)}^{(\alpha_{1i},\alpha_{2j})}Q_{n,m}^{**}(x,y)-D_{*(-1)}^{(\alpha_{1i},\alpha_{2j})}f(x,y)+\rho_{n,m}D_{*(-1)}^{(\alpha_{1i},\alpha_{2j})}\left(\frac{x^{h_1}}{h_1!}\frac{y^{h_2}}{h_2!}\right)\right]\overset{(72)}{\leq}$$

$$-\rho_{n,m}\frac{(x+1)^{h_1-\alpha_{1i}}}{\Gamma(h_1-\alpha_{1i}+1)}\frac{(y+1)^{h_2-\alpha_{2j}}}{\Gamma(h_2-\alpha_{2j}+1)}+ \tag{74}$$

$$\left[\sum_{i=h_1}^{v_1}\sum_{j=h_2}^{v_2} l_{ij}\frac{2^{(i+j)-(\alpha_{1i}+\alpha_{2j})}}{\Gamma(i-\alpha_{1i}+1)\Gamma(j-\alpha_{2j}+1)}\frac{\dot{C}}{n^{r-i}m^{p-j}}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)\right]=$$

$$\rho_{n,m}\left[1-\frac{(x+1)^{h_1-\alpha_{1i}}}{\Gamma(h_1-\alpha_{1i}+1)}\frac{(y+1)^{h_2-\alpha_{2j}}}{\Gamma(h_2-\alpha_{2j}+1)}\right]=$$

$$\rho_{n,m}\left[\frac{\Gamma(h_1-\alpha_{1i}+1)\Gamma(h_2-\alpha_{2j}+1)-(x+1)^{h_1-\alpha_{1i}}(y+1)^{h_2-\alpha_{2j}}}{\Gamma(h_1-\alpha_{1i}+1)\Gamma(h_2-\alpha_{2j}+1)}\right]\leq$$

$$\rho_{n,m}\left[\frac{1-(x+1)^{h_1-\alpha_{1i}}(y+1)^{h_2-\alpha_{2j}}}{\Gamma(h_1-\alpha_{1i}+1)\Gamma(h_2-\alpha_{2j}+1)}\right]\leq 0. \tag{75}$$

Explanation: for $x, y \in [0,1]$ we get that $x+1$, $y+1 \geq 1$, and $0 \leq h_1 - \alpha_{1h_1}$, $h_2 - \alpha_{2h_2} < 1$. Hence $(x+1)^{h_1-\alpha_{1i}}$, $(y+1)^{h_2-\alpha_{2j}} \geq 1$, and then

$$(x+1)^{h_1-\alpha_{1i}}(y+1)^{h_2-\alpha_{2j}} \geq 1,$$

so that

$$1-(x+1)^{h_1-\alpha_{1i}}(y+1)^{h_2-\alpha_{2j}} \leq 0. \tag{76}$$

Hence again

$$L^*\left(Q_{n,m}^{**}(x,y)\right) \geq 0, \text{ for } (x,y) \in [0,1]^2. \tag{77}$$

■

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