

Bivariate Right Fractional Pseudo-Polynomial Monotone Approximation

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

In this article we deal with the following general two-dimensional problem: Let f be a two variable continuously differentiable real valued function of a given order, let \bar{L} be a linear right fractional mixed partial differential operator and suppose that $\bar{L}(f) \geq 0$ on a critical region. Then for sufficiently large $n, m \in \mathbb{N}$, we can find a sequence of pseudo-polynomials $Q_{n,m}^*$ in two variables with the property $\bar{L}(Q_{n,m}^*) \geq 0$ on this critical region such that f is approximated with rates right fractionally and simultaneously by $Q_{n,m}^*$ in the uniform norm on the whole domain of f . This restricted approximation is given via inequalities involving the mixed modulus of smoothness $\omega_{s,q}$, $s, q \in \mathbb{N}$, of highest order integer partial derivative of f .

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1 Introduction

The topic of monotone approximation started in [10] has become a major trend in approximation theory. A typical problem in this subject is: given a positive integer k , approximate a given function whose k th derivative is ≥ 0 by polynomials having this property.

In [3] the authors replaced the k th derivative with a linear differential operator of order k . We mention this motivating result.

Theorem 1 Let h, k, p be integers, $0 \leq h \leq k \leq p$ and let f be a real function, $f^{(p)}$ continuous in $[-1, 1]$ with modulus of continuity $\omega_1(f^{(p)}, x)$ there. Let $a_j(x)$, $j = h, h+1, \dots, k$ be real functions, defined and bounded on $[-1, 1]$ and assume $a_h(x)$ is either \geq some number $\alpha > 0$ or \leq some number $\beta < 0$ throughout $[-1, 1]$. Consider the operator

$$L = \sum_{j=h}^k a_j(x) \left[\frac{d^j}{dx^j} \right] \quad (1)$$

and suppose, throughout $[-1, 1]$,

$$L(f) \geq 0. \quad (2)$$

Then, for every integer $n \geq 1$, there is a real polynomial $Q_n(x)$ of degree $\leq n$ such that

$$L(Q_n) \geq 0 \text{ throughout } [-1, 1] \quad (3)$$

and

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq C n^{k-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right), \quad (4)$$

where C is independent of n or f .

Next let $n, m \in \mathbb{Z}_+$, P_θ denote the space of algebraic polynomials of degree $\leq \theta$. Consider the tensor product spaces $P_n \otimes C([-1, 1])$, $C([-1, 1]) \otimes P_m$ and their sum $P_n \otimes C([-1, 1]) + C([-1, 1]) \otimes P_m$, that is

$$P_n \otimes C([-1, 1]) + C([-1, 1]) \otimes P_m = \left\{ \sum_{i=0}^n x^i A_i(y) + \sum_{j=0}^m B_j(x) y^j; A_i, B_j \in C([-1, 1]), x, y \in [-1, 1] \right\}. \quad (5)$$

This is the space of pseudo-polynomials of degree $\leq (n, m)$, first introduced by A. Marchaud in 1924-1927 (see [7], [8]). Here $f^{(k,l)}$ denotes $\frac{\partial^{k+l} f}{\partial x^k \partial y^l}$, the (k, l) -partial derivative of f .

In this section we consider the space $C^{r,p}([-1, 1]^2) = \{f : [-1, 1]^2 \rightarrow \mathbb{R}; f^{(k,l)} \text{ is continuous for } 0 \leq k \leq r, 0 \leq l \leq p\}$. Let $f \in C([-1, 1]^2)$; for $\delta_1, \delta_2 \geq 0$, define the mixed modulus of smoothness of order (s, q) , $s, q \in \mathbb{N}$ (see [9], pp. 516-517) by

$$\omega_{s,q}(f; \delta_1, \delta_2) \equiv \sup \left\{ |{}_x \Delta_{h_1}^s \circ_y \Delta_{h_2}^q f(x, y)| : (x, y), \quad (6)$$

$$(x + sh_1, y + qh_2) \in [-1, 1]^2, |h_i| \leq \delta_i, i = 1, 2 \right\}.$$

Here

$${}_x\Delta_{h_1}^s \circ_y \Delta_{h_2}^q f(x, y) \equiv \sum_{\sigma=0}^s \sum_{\mu=0}^q (-1)^{s+q-\sigma-\mu} \binom{s}{\sigma} \binom{q}{\mu} f(x + \sigma h_1, y + \mu h_2) \quad (7)$$

is a mixed difference of order (s, q) .

We mention

Theorem 2 (H.H. Gonska [4]). *Let $r, p \in \mathbb{Z}_+$, $s, q \in \mathbb{N}$, and $f \in C^{r,p}([-1, 1]^2)$. Let $n, m \in \mathbb{N}$ with $n \geq \max\{4(r+1), r+s\}$ and $m \geq \max\{4(p+1), p+q\}$. Then there exists a linear operator $Q_{n,m}$ from $C^{r,p}([-1, 1]^2)$ into the space of pseudopolynomials $(P_n \otimes C([-1, 1]) + C([-1, 1]) \otimes P_m)$ such that*

$$\left| (f - Q_{n,m}(f))^{(k,l)}(x, y) \right| \leq \quad (8)$$

$$M_{r,s} \cdot M_{p,q} (\Delta_n(x))^{r-k} \cdot (\Delta_m(y))^{p-l} \cdot \omega_{s,q} \left(f^{(r,p)}; \Delta_n(x), \Delta_m(y) \right),$$

for all $(0, 0) \leq (k, l) \leq (r, p)$, $x, y \in [-1, 1]$, where

$$\Delta_\theta(z) = \frac{\sqrt{1-z^2}}{\theta} + \frac{1}{\theta^2}, \quad \theta = n, m; \quad z = x, y \in [-1, 1]. \quad (9)$$

The constants $M_{r,s}$, $M_{p,q}$, are independent of f , (x, y) and (n, m) ; they depend only on (r, s) , (p, q) , respectively.

See also [5], saying that $Q_{n,m}^{(r,p)}(f)$ is continuous on $[-1, 1]^2$.

The need following result which is an easy consequence of the last theorem (see [9], p. 517).

Corollary 3 *Let $r, p \in \mathbb{Z}_+$, $s, q \in \mathbb{N}$, and $f \in C^{r,p}([-1, 1]^2)$. Let $n, m \in \mathbb{N}$ with $n \geq \max\{4(r+1), r+s\}$ and $m \geq \max\{4(p+1), p+q\}$. Then there exists a pseudopolynomial $Q_{n,m} \equiv Q_{n,m}(f) \in (P_n \otimes C([-1, 1]) + C([-1, 1]) \otimes P_m)$ such that*

$$\left\| f^{(k,l)} - Q_{n,m}^{(k,l)} \right\|_\infty \leq \frac{\dot{C}}{n^{r-k} m^{p-l}} \cdot \omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right), \quad (10)$$

for all $(0, 0) \leq (k, l) \leq (r, p)$. Here the constant \dot{C} depends only on r, p, s, q .

Corollary 3 was used in the proof of the main motivational result that follows.

Theorem 4 ([1]) *Let h_1, h_2, v_1, v_2, r, p be integers, $0 \leq h_1 \leq v_1 \leq r$, $0 \leq h_2 \leq v_2 \leq p$ and let $f \in C^{r,p}([-1, 1]^2)$, with $f^{(r,p)}$ having a mixed modulus of smoothness $\omega_{s,q}(f^{(r,p)}; x, y)$ there, $s, q \in \mathbb{N}$. Let $\alpha_{i,j}(x, y)$, $i = h_1, h_1+1, \dots, v_1$;*

$j = h_2, h_2 + 1, \dots, v_2$ be real-valued functions, defined and bounded in $[-1, 1]^2$ and suppose $\alpha_{h_1 h_2}$ is either $\geq \alpha > 0$ or $\leq \beta < 0$ throughout $[-1, 1]^2$. Take the operator

$$L = \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{ij}(x, y) \frac{\partial^{i+j}}{\partial x^i \partial y^j} \quad (11)$$

and assume, throughout $[-1, 1]^2$ that

$$L(f) \geq 0. \quad (12)$$

Then for any integers n, m with $n \geq \max\{4(r+1), r+s\}$, $m \geq \max\{4(p+1), p+q\}$, there exists a pseudopolynomial

$$Q_{n,m} \in (P_n \otimes C([-1, 1]) + C([-1, 1]) \otimes P_m)$$

such that $L(Q_{m,n}) \geq 0$ throughout $[-1, 1]^2$ and

$$\left\| f^{(k,l)} - Q_{n,m}^{(k,l)} \right\|_{\infty} \leq \frac{C}{n^{r-v_1} m^{p-v_2}} \cdot \omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right), \quad (13)$$

for all $(0, 0) \leq (k, l) \leq (h_1, h_2)$. Moreover we get

$$\left\| f^{(k,l)} - Q_{n,m}^{(k,l)} \right\|_{\infty} \leq \frac{C}{n^{r-k} m^{p-l}} \cdot \omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right), \quad (14)$$

for all $(h_1 + 1, h_2 + 1) \leq (k, l) \leq (r, p)$. Also (14) is valid whenever $0 \leq k \leq h_1$, $h_2 + 1 \leq l \leq p$ or $h_1 + 1 \leq k \leq r$, $0 \leq l \leq h_2$. Here C is a constant independent of f and n, m . It depends only on r, p, s, q, L .

We are also motivated by [2].

We need

Definition 5 (see [6]) Let $[-1, 1]^2$; $\alpha_1, \alpha_2 > 0$; $\alpha = (\alpha_1, \alpha_2)$, $f \in C([-1, 1]^2)$, $x = (x_1, x_2)$, $t = (t_1, t_2) \in [-1, 1]^2$. We define the right mixed Riemann-Liouville fractional two dimensional integral of order α

$$(I_{1-}^{\alpha} f)(x) := \frac{1}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \int_{x_1}^1 \int_{x_2}^1 (t_1 - x_1)^{\alpha_1 - 1} (t_2 - x_2)^{\alpha_2 - 1} f(t_1, t_2) dt_1 dt_2, \quad (15)$$

with $x_1, x_2 < 1$.

Notice here that $I_{1-}^{\alpha}(|f|) < \infty$.

Definition 6 Let $\alpha_1, \alpha_2 > 0$ with $[\alpha_1] = m_1$, $[\alpha_2] = m_2$, ($[\cdot]$ ceiling of the number). Let here $f \in C^{m_1, m_2}([-1, 1]^2)$. We consider the right Caputo type fractional partial derivative:

$$D_{1-}^{(\alpha_1, \alpha_2)} f(x) := \frac{(-1)^{m_1 + m_2}}{\Gamma(m_1 - \alpha_1) \Gamma(m_2 - \alpha_2)}.$$

$$\int_{x_1}^1 \int_{x_2}^1 (t_1 - x_1)^{m_1 - \alpha_1 - 1} (t_2 - x_2)^{m_2 - \alpha_2 - 1} \frac{\partial^{m_1 + m_2} f(t_1, t_2)}{\partial t_1^{m_1} \partial t_2^{m_2}} dt_1 dt_2, \quad (16)$$

$\forall x = (x_1, x_2) \in [-1, 1]^2$, where Γ is the gamma function

$$\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt, \quad \nu > 0. \quad (17)$$

We set

$$D_{1-}^{(0,0)} f(x) := f(x), \quad \forall x \in [-1, 1]^2; \quad (18)$$

$$D_{1-}^{(m_1, m_2)} f(x) := (-1)^{m_1 + m_2} \frac{\partial^{m_1 + m_2} f(x)}{\partial x_1^{m_1} \partial x_2^{m_2}}, \quad \forall x \in [-1, 1]^2. \quad (19)$$

Definition 7 We also set

$$D_{1-}^{(0, \alpha_2)} f(x) := \frac{(-1)^{m_2}}{\Gamma(m_2 - \alpha_2)} \int_{x_2}^1 (t_2 - x_2)^{m_2 - \alpha_2 - 1} \frac{\partial^{m_2} f(x_1, t_2)}{\partial t_2^{m_2}} dt_2, \quad (20)$$

$$D_{1-}^{(\alpha_1, 0)} f(x) := \frac{(-1)^{m_1}}{\Gamma(m_1 - \alpha_1)} \int_{x_1}^1 (t_1 - x_1)^{m_1 - \alpha_1 - 1} \frac{\partial^{m_1} f(t_1, x_2)}{\partial t_1^{m_1}} dt_1, \quad (21)$$

and

$$D_{1-}^{(m_1, \alpha_2)} f(x) := \frac{(-1)^{m_2}}{\Gamma(m_2 - \alpha_2)} \int_{x_2}^1 (t_2 - x_2)^{m_2 - \alpha_2 - 1} \frac{\partial^{m_1 + m_2} f(x_1, t_2)}{\partial x_1^{m_1} \partial t_2^{m_2}} dt_2, \quad (22)$$

$$D_{1-}^{(\alpha_1, m_2)} f(x) := \frac{(-1)^{m_1}}{\Gamma(m_1 - \alpha_1)} \int_{x_1}^1 (t_1 - x_1)^{m_1 - \alpha_1 - 1} \frac{\partial^{m_1 + m_2} f(t_1, x_2)}{\partial t_1^{m_1} \partial x_2^{m_2}} dt_1. \quad (23)$$

In this article we extend Theorem 4 to the fractional level. Indeed here L is replaced by \bar{L} , a linear right Caputo fractional mixed partial differential operator. Now the monotonicity property holds true only on the critical square of $[-1, 0]^2$. Simultaneously fractional convergence remains true on all of $[-1, 1]^2$.

2 Main Result

We present

Theorem 8 Let h_1, h_2, v_1, v_2, r, p be integers, $0 \leq h_1 \leq v_1 \leq r$, $0 \leq h_2 \leq v_2 \leq p$ and let $f \in C^{r,p}([-1, 1]^2)$, with $f^{(r,p)}$ having a mixed modulus of smoothness $\omega_{s,q}(f^{(r,p)}; x, y)$ there, $s, q \in \mathbb{N}$. Let $\alpha_{ij}(x, y)$, $i = h_1, h_1 + 1, \dots, v_1$; $j = h_2, h_2 + 1, \dots, v_2$ be real valued functions, defined and bounded in $[-1, 1]^2$ and suppose $\alpha_{h_1 h_2}$ is either $\geq \alpha > 0$ or $\leq \beta < 0$ throughout $[-1, 0]^2$. Assume that $h_1 + h_2 = 2\gamma$, $\gamma \in \mathbb{Z}_+$. Here $n, m \in \mathbb{N} : n \geq \max\{4(r+1), r+s\}$, $m \geq \max\{4(p+1), p+q\}$. Set

$$l_{ij} := \sup_{(x,y) \in [-1,1]^2} |\alpha_{h_1 h_2}^{-1}(x, y) \alpha_{ij}(x, y)| < \infty, \quad (24)$$

for all $h_1 \leq i \leq v_1$, $h_2 \leq j \leq v_2$. Let $\alpha_{1i}, \alpha_{2j} > 0$, $\alpha_{1i}, \alpha_{2j} \notin \mathbb{N}$, with $\lceil \alpha_{1i} \rceil = i$, $\lceil \alpha_{2j} \rceil = j$, $i = 0, 1, \dots, r$; $j = 0, 1, \dots, p$, ($\lceil \cdot \rceil$ ceiling of the number), $\alpha_{10} = 0$, $\alpha_{20} = 0$.

Consider the right fractional bivariate differential operator

$$\bar{L} := \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{ij} (x, y) D_{1-}^{(\alpha_{1i}, \alpha_{2j})}. \quad (25)$$

Assume $\bar{L}f(x, y) \geq 0$, on $[-1, 0]^2$.

Then there exists

$$Q_{n,m}^* \equiv Q_{n,m}^*(f) \in (P_n \otimes C([-1, 1]) + C([-1, 1]) \otimes P_m)$$

such that $\bar{L}Q_{n,m}^*(x, y) \geq 0$, on $[-1, 0]^2$.

Furthermore it holds:

1)

$$\begin{aligned} & \left\| D_{1-}^{(\alpha_{1i}, \alpha_{2j})} (f) - D_{1-}^{(\alpha_{1i}, \alpha_{2j})} Q_{n,m}^* \right\|_{\infty, [-1, 1]^2} \leq \\ & \frac{\dot{C} 2^{(i+j) - (\alpha_{1i} + \alpha_{2j})}}{\Gamma(i - \alpha_{1i} + 1) \Gamma(j - \alpha_{2j} + 1) n^{r-i} m^{p-j}} \cdot \omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right), \end{aligned} \quad (26)$$

where \dot{C} is a constant that depends only on r, p, s, q ; $(h_1 + 1, h_2 + 1) \leq (i, j) \leq (r, p)$, or $0 \leq i \leq h_1$, $h_2 + 1 \leq j \leq p$, or $h_1 + 1 \leq i \leq r$, $0 \leq j \leq h_2$,

2)

$$\begin{aligned} & \left\| D_{1-}^{(\alpha_{1i}, \alpha_{2j})} (f) - D_{1-}^{(\alpha_{1i}, \alpha_{2j})} Q_{n,m}^* \right\|_{\infty, [-1, 1]^2} \leq \\ & \frac{c_{ij}}{n^{r-v_1} m^{p-v_2}} \cdot \omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right), \end{aligned} \quad (27)$$

for $(1, 1) \leq (i, j) \leq (h_1, h_2)$, where $c_{ij} = \dot{C} A_{ij}$, with

$$A_{ij} :=$$

$$\begin{aligned} & \left\{ \left[\sum_{\tau=h_1}^{v_1} \sum_{\mu=h_2}^{v_2} \frac{l_{\tau\mu} 2^{(\tau+\mu) - (\alpha_{1\tau} + \alpha_{2\mu})}}{\Gamma(\tau - \alpha_{1\tau} + 1) \Gamma(\mu - \alpha_{2\mu} + 1)} \right] \left(\sum_{k=0}^{h_1-i} \frac{2^{h_1 - \alpha_{1i} - k}}{k! \Gamma(h_1 - \alpha_{1i} - k + 1)} \right) \right. \\ & \left. \left(\sum_{\lambda=0}^{h_2-j} \frac{2^{h_2 - \alpha_{2j} - \lambda}}{\lambda! \Gamma(h_2 - \alpha_{2j} - \lambda + 1)} \right) + \frac{2^{(i+j) - (\alpha_{1i} + \alpha_{2j})}}{\Gamma(i - \alpha_{1i} + 1) \Gamma(j - \alpha_{2j} + 1)} \right\}, \end{aligned} \quad (28)$$

3)

$$\|f - Q_{n,m}^*\|_{\infty, [-1, 1]^2} \leq \frac{c_{00}}{n^{r-v_1} m^{p-v_2}} \cdot \omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right), \quad (29)$$

where $c_{00} := \dot{C}A_{00}$, with

$$A_{00} := \frac{1}{h_1!h_2!} \left(\sum_{\tau=h_1}^{v_1} \sum_{\mu=h_2}^{v_2} l_{\tau\mu} \frac{2^{(\tau+\mu)-(\alpha_{1\tau}+\alpha_{2\mu})}}{\Gamma(\tau-a_{1\tau}+1)\Gamma(\mu-\alpha_{2\mu}+1)} \right) + 1,$$

4)

$$\begin{aligned} & \left\| D_{1-}^{(0,\alpha_{2j})} (f) - D_{1-}^{(0,\alpha_{2j})} Q_{n,m}^* \right\|_{\infty,[-1,1]^2} \leq \\ & \frac{c_{0j}}{n^{r-v_1} m^{p-v_2}} \cdot \omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right), \end{aligned} \quad (30)$$

where $c_{0j} = \dot{C}A_{0j}$, $j = 1, \dots, h_2$, with

$$\begin{aligned} A_{0j} := & \left[\frac{1}{h_1!} \left(\sum_{\tau=h_1}^{v_1} \sum_{\mu=h_2}^{v_2} l_{\tau\mu} \frac{2^{(\tau+\mu)-(\alpha_{1\tau}+\alpha_{2\mu})}}{\Gamma(\tau-a_{1\tau}+1)\Gamma(\mu-\alpha_{2\mu}+1)} \right) \right. \\ & \left. \left(\sum_{\lambda=0}^{h_2-j} \frac{2^{h_2-\alpha_{2j}-\lambda}}{\lambda! \Gamma(h_2-\alpha_{2j}-\lambda+1)} \right) + \frac{2^{j-\alpha_{2j}}}{\Gamma(j-\alpha_{2j}+1)} \right], \end{aligned} \quad (31)$$

5)

$$\begin{aligned} & \left\| D_{1-}^{(\alpha_{1i},0)} (f) - D_{1-}^{(\alpha_{1i},0)} Q_{n,m}^* \right\|_{\infty,[-1,1]^2} \leq \\ & \frac{c_{i0}}{n^{r-v_1} m^{p-v_2}} \cdot \omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right), \end{aligned} \quad (32)$$

where $c_{i0} = \dot{C}A_{i0}$, $i = 1, \dots, h_1$, with

$$\begin{aligned} A_{i0} := & \left[\frac{1}{h_2!} \left(\sum_{\tau=h_1}^{v_1} \sum_{\mu=h_2}^{v_2} l_{\tau\mu} \frac{2^{(\tau+\mu)-(\alpha_{1\tau}+\alpha_{2\mu})}}{\Gamma(\tau-a_{1\tau}+1)\Gamma(\mu-\alpha_{2\mu}+1)} \right) \right. \\ & \left. \left(\sum_{k=0}^{h_1-i} \frac{2^{h_1-\alpha_{1i}-k}}{k! \Gamma(h_1-\alpha_{1i}-k+1)} \right) + \frac{2^{i-\alpha_{1i}}}{\Gamma(i-\alpha_{1i}+1)} \right]. \end{aligned} \quad (33)$$

Proof. By Corollary 3 there exists

$$Q_{n,m} \equiv Q_{n,m}(f) \in (P_n \otimes C([-1,1]) + C([-1,1]) \otimes P_m)$$

such that

$$\left\| f^{(i,j)} - Q_{n,m}^{(i,j)} \right\|_{\infty} \leq \frac{\dot{C}}{n^{r-i} m^{p-j}} \cdot \omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right), \quad (34)$$

for all $(0, 0) \leq (i, j) \leq (r, p)$, while $Q_{n,m} \in C^{r,p}([-1, 1])^2$. Here \dot{C} depends only on r, p, s, q , where $n \geq \max\{4(r+1), r+s\}$ and $m \geq \max\{4(p+1), p+q\}$, with $r, p \in \mathbb{Z}_+$, $s, q \in \mathbb{N}$, $f \in C^{r,p}([-1, 1]^2)$.

Indeed by [5] we have that $Q_{n,m}^{(r,p)}$ is continuous on $[-1, 1]^2$.

We observe the following ($i = 1, \dots, r; j = 1, \dots, p$)

$$\left| D_{1-}^{(\alpha_{1i}, \alpha_{2j})} f(x_1, x_2) - D_{1-}^{(\alpha_{1i}, \alpha_{2j})} Q_{n,m}(x_1, x_2) \right| = \frac{1}{\Gamma(i - \alpha_{1i}) \Gamma(j - \alpha_{2j})} \cdot \left| \int_{x_1}^1 \int_{x_2}^1 (t_1 - x_1)^{i - \alpha_{1i} - 1} (t_2 - x_2)^{j - \alpha_{2j} - 1} \cdot \left(\frac{\partial^{i+j} f(t_1, t_2)}{\partial t_1^i \partial t_2^j} - \frac{\partial^{i+j} Q_{n,m}(t_1, t_2)}{\partial t_1^i \partial t_2^j} \right) dt_1 dt_2 \right| \leq \quad (35)$$

$$\frac{1}{\Gamma(i - \alpha_{1i}) \Gamma(j - \alpha_{2j})} \int_{x_1}^1 \int_{x_2}^1 (t_1 - x_1)^{i - \alpha_{1i} - 1} (t_2 - x_2)^{j - \alpha_{2j} - 1} \cdot \quad (36)$$

$$\left| \frac{\partial^{i+j} f(t_1, t_2)}{\partial t_1^i \partial t_2^j} - \frac{\partial^{i+j} Q_{n,m}(t_1, t_2)}{\partial t_1^i \partial t_2^j} \right| dt_1 dt_2 \leq (by(10))$$

$$\frac{1}{\Gamma(i - \alpha_{1i}) \Gamma(j - \alpha_{2j})} \left(\int_{x_1}^1 \int_{x_2}^1 (t_1 - x_1)^{i - \alpha_{1i} - 1} (t_2 - x_2)^{j - \alpha_{2j} - 1} \right) \cdot \quad (37)$$

$$\frac{\dot{C}}{n^{r-i} m^{p-j}} \cdot \omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right) =$$

$$\frac{1}{\Gamma(i - \alpha_{1i}) \Gamma(j - \alpha_{2j})} \frac{(1 - x_1)^{i - \alpha_{1i}}}{i - \alpha_{1i}} \frac{(1 - x_2)^{j - \alpha_{2j}}}{j - \alpha_{2j}} \frac{\dot{C}}{n^{r-i} m^{p-j}} \cdot \omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right) \quad (38)$$

$$= \frac{(1 - x_1)^{i - \alpha_{1i}}}{\Gamma(i - \alpha_{1i} + 1)} \frac{(1 - x_2)^{j - \alpha_{2j}}}{\Gamma(j - \alpha_{2j} + 1)} \frac{\dot{C}}{n^{r-i} m^{p-j}} \cdot \omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right).$$

That is there exists $Q_{n,m}$:

$$\left| D_{1-}^{(\alpha_{1i}, \alpha_{2j})} f(x_1, x_2) - D_{1-}^{(\alpha_{1i}, \alpha_{2j})} Q_{n,m}(x_1, x_2) \right| \leq \quad (39)$$

$$\frac{(1 - x_1)^{i - \alpha_{1i}} (1 - x_2)^{j - \alpha_{2j}}}{\Gamma(i - \alpha_{1i} + 1) \Gamma(j - \alpha_{2j} + 1)} \frac{\dot{C}}{n^{r-i} m^{p-j}} \cdot \omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right),$$

$i = 1, \dots, r, j = 1, \dots, p, \forall (x_1, x_2) \in [-1, 1]^2$.

We proved there exists $Q_{n,m}$ such that

$$\left\| D_{1-}^{(\alpha_{1i}, \alpha_{2j})} (f) - D_{1-}^{(\alpha_{1i}, \alpha_{2j})} Q_{n,m}^* \right\|_{\infty} \leq$$

$$\frac{2^{(i+j)-(\alpha_{1i}+\alpha_{2j})}\dot{C}}{\Gamma(i-\alpha_{1i}+1)\Gamma(j-\alpha_{2j}+1)n^{r-i}m^{p-j}} \cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right), \quad (40)$$

$i = 0, 1, \dots, r, j = 0, 1, \dots, p.$

Define

$$\rho_{n,m} \equiv \dot{C}\omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right).$$

$$\left[\sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \left(l_{ij} \frac{2^{(i+j)-(\alpha_{1i}+\alpha_{2j})}}{\Gamma(i-\alpha_{1i}+1)\Gamma(j-\alpha_{2j}+1)} n^{i-r} m^{j-p} \right) \right]. \quad (41)$$

I. Suppose, throughout $[-1, 0]^2$, $\alpha_{h_1 h_2}(x, y) \geq \alpha > 0$. Let $Q_{n,m}^*(x, y)$, $(x, y) \in [-1, 1]^2$, as in (40), so that

$$\left\| D_{1-}^{(\alpha_{1i}, \alpha_{2j})} \left(f(x, y) + \rho_{n,m} \frac{x^{h_1} y^{h_2}}{h_1! h_2!} \right) - D_{1-}^{(\alpha_{1i}, \alpha_{2j})} Q_{n,m}^*(x, y) \right\|_{\infty} \leq \quad (42)$$

$$\frac{2^{(i+j)-(\alpha_{1i}+\alpha_{2j})}\dot{C}}{\Gamma(i-\alpha_{1i}+1)\Gamma(j-\alpha_{2j}+1)n^{r-i}m^{p-j}} \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right) =: T_{ij},$$

$i = 0, 1, \dots, r; j = 0, 1, \dots, p.$

If $(h_1 + 1, h_2 + 1) \leq (i, j) \leq (r, p)$, or $0 < i \leq h_1, h_2 + 1 \leq j \leq p$, or $h_1 + 1 \leq i \leq r, 0 < j \leq h_2$ we get from the last

$$\left\| D_{1-}^{(\alpha_{1i}, \alpha_{2j})} (f) - D_{1-}^{(\alpha_{1i}, \alpha_{2j})} Q_{n,m}^* \right\|_{\infty} \leq$$

$$\frac{2^{(i+j)-(\alpha_{1i}+\alpha_{2j})}\dot{C}}{\Gamma(i-\alpha_{1i}+1)\Gamma(j-\alpha_{2j}+1)n^{r-i}m^{p-j}} \cdot \omega_{s,q}\left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m}\right), \quad (43)$$

proving (26).

If $(0, 0) < (i, j) \leq (h_1, h_2)$, we get

$$\begin{aligned} & \left\| D_{1-}^{(\alpha_{1i}, \alpha_{2j})} f(x, y) + \rho_{n,m} D_{1-}^{\alpha_{1i}} \left(\frac{x^{h_1}}{h_1!} \right) D_{1-}^{\alpha_{2j}} \left(\frac{y^{h_2}}{h_2!} \right) \right. \\ & \quad \left. - D_{1-}^{(\alpha_{1i}, \alpha_{2j})} Q_{n,m}^*(x, y) \right\|_{\infty} \leq T_{ij}. \end{aligned} \quad (44)$$

That is for $i = 1, \dots, h_1; j = 1, \dots, h_2$, we obtain

$$\begin{aligned} & \left\| D_{1-}^{(\alpha_{1i}, \alpha_{2j})} f(x, y) + \rho_{n,m} \left((-1)^{h_1} \sum_{k=0}^{h_1-i} \frac{(-1)^k (1-x)^{h_1-\alpha_{1i}-k}}{k! \Gamma(h_1-\alpha_{1i}-k+1)} \right) \right. \\ & \quad \left. \left((-1)^{h_2} \sum_{\lambda=0}^{h_2-j} \frac{(-1)^\lambda (1-y)^{h_2-\alpha_{2j}-\lambda}}{\lambda! \Gamma(h_2-\alpha_{2j}-\lambda+1)} \right) - D_{1-}^{(\alpha_{1i}, \alpha_{2j})} Q_{n,m}^*(x, y) \right\|_{\infty} \leq T_{ij}. \end{aligned} \quad (45)$$

Hence for $(1, 1) \leq (i, j) \leq (h_1, h_2)$, we have

$$\begin{aligned} & \left\| D_{1-}^{(\alpha_{1i}, \alpha_{2j})} f - D_{1-}^{(\alpha_{1i}, \alpha_{2j})} Q_{n,m}^* \right\|_{\infty} \leq \\ & \rho_{n,m} \left(\sum_{k=0}^{h_1-i} \frac{2^{h_1-\alpha_{1i}-k}}{k! \Gamma(h_1 - \alpha_{1i} - k + 1)} \right) \left(\sum_{\lambda=0}^{h_2-j} \frac{2^{h_2-\alpha_{2j}-\lambda}}{\lambda! \Gamma(h_2 - \alpha_{2j} - \lambda + 1)} \right) + T_{ij} = \end{aligned} \quad (46)$$

$$\dot{C}\omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right) \left[\sum_{\bar{i}=h_1}^{v_1} \sum_{\bar{j}=h_2}^{v_2} l_{\bar{i}\bar{j}} \frac{2^{(\bar{i}+\bar{j})-(\alpha_{1\bar{i}}+\alpha_{2\bar{j}})}}{\Gamma(\bar{i} - \alpha_{1\bar{i}} + 1) \Gamma(\bar{j} - \alpha_{2\bar{j}} + 1)} \frac{1}{n^{r-\bar{i}}} \frac{1}{m^{p-\bar{j}}} \right] \quad (47)$$

$$\begin{aligned} & \left(\sum_{k=0}^{h_1-i} \frac{2^{h_1-\alpha_{1i}-k}}{k! \Gamma(h_1 - \alpha_{1i} - k + 1)} \right) \left(\sum_{\lambda=0}^{h_2-j} \frac{2^{h_2-\alpha_{2j}-\lambda}}{\lambda! \Gamma(h_2 - \alpha_{2j} - \lambda + 1)} \right) + \\ & \frac{2^{(i+j)-(\alpha_{1i}+\alpha_{2j})} \dot{C}\omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right)}{\Gamma(i - \alpha_{1i} + 1) \Gamma(j - \alpha_{2j} + 1) n^{r-i} m^{p-j}} \leq \\ & \dot{C}\omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right) \frac{1}{n^{r-v_1} m^{p-v_2}} A_{ij}, \end{aligned} \quad (48)$$

where

$$\begin{aligned} A_{ij} & := \left\{ \left[\sum_{\bar{i}=h_1}^{v_1} \sum_{\bar{j}=h_2}^{v_2} l_{\bar{i}\bar{j}} \frac{2^{(\bar{i}+\bar{j})-(\alpha_{1\bar{i}}+\alpha_{2\bar{j}})}}{\Gamma(\bar{i} - \alpha_{1\bar{i}} + 1) \Gamma(\bar{j} - \alpha_{2\bar{j}} + 1)} \right] \cdot \right. \\ & \left. \left(\sum_{k=0}^{h_1-i} \frac{2^{h_1-\alpha_{1i}-k}}{k! \Gamma(h_1 - \alpha_{1i} - k + 1)} \right) \left(\sum_{\lambda=0}^{h_2-j} \frac{2^{h_2-\alpha_{2j}-\lambda}}{\lambda! \Gamma(h_2 - \alpha_{2j} - \lambda + 1)} \right) \right. \\ & \left. + \frac{2^{(i+j)-(\alpha_{1i}+\alpha_{2j})}}{\Gamma(i - \alpha_{1i} + 1) \Gamma(j - \alpha_{2j} + 1)} \right\}. \end{aligned} \quad (49)$$

(Set $c_{ij} := \dot{C}A_{ij}$)

We proved, for $(1, 1) \leq (i, j) \leq (h_1, h_2)$, that

$$\left\| D_{1-}^{(\alpha_{1i}, \alpha_{2j})} f - D_{1-}^{(\alpha_{1i}, \alpha_{2j})} Q_{n,m}^* \right\|_{\infty} \leq \frac{c_{ij}}{n^{r-v_1} m^{p-v_2}} \omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right). \quad (50)$$

So that (27) is established.

When $i = j = 0$ from (42) we obtain

$$\left\| f(x, y) + \rho_{n,m} \frac{x^{h_1} y^{h_2}}{h_1! h_2!} - Q_{n,m}^*(x, y) \right\|_{\infty} \leq \frac{\dot{C}}{n^r m^p} \omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right). \quad (51)$$

Hence

$$\|f - Q_{n,m}^*\|_{\infty} \leq \frac{\rho_{n,m}}{h_1! h_2!} + \frac{\dot{C}}{n^r m^p} \omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right) = \quad (52)$$

$$\frac{\dot{C}}{h_1!h_2!}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right).$$

$$\left[\sum_{\bar{i}=h_1}^{v_1}\sum_{\bar{j}=h_2}^{v_2}l_{\bar{i}\bar{j}}\frac{2^{(\bar{i}+\bar{j})-(\alpha_{1\bar{i}}+\alpha_{2\bar{j}})}}{\Gamma(\bar{i}-\alpha_{1\bar{i}}+1)\Gamma(\bar{j}-\alpha_{2\bar{j}}+1)}\frac{1}{n^{r-\bar{i}}}\frac{1}{m^{p-\bar{j}}}\right] \quad (53)$$

$$+\frac{\dot{C}}{n^r m^p}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)\leq\frac{\dot{C}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)}{n^{r-v_1}m^{p-v_2}}A_{00}, \quad (54)$$

where

$$A_{00}:=\left[\frac{1}{h_1!h_2!}\sum_{\bar{i}=h_1}^{v_1}\sum_{\bar{j}=h_2}^{v_2}\frac{l_{\bar{i}\bar{j}}2^{(\bar{i}+\bar{j})-(\alpha_{1\bar{i}}+\alpha_{2\bar{j}})}}{\Gamma(\bar{i}-\alpha_{1\bar{i}}+1)\Gamma(\bar{j}-\alpha_{2\bar{j}}+1)}+1\right]. \quad (55)$$

(Set $c_{00}=\dot{C}A_{00}$).

Then

$$\|f-Q_{n,m}^*\|_\infty\leq\frac{c_{00}}{n^{r-v_1}m^{p-v_2}}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right). \quad (56)$$

So that (29) is established.

Next case of $i=0, j=1, \dots, h_2$, from (42) we get

$$\left\|D_{1-}^{(0,\alpha_{2j})}f(x,y)+\rho_{n,m}\frac{x^{h_1}}{h_1!}\left((-1)^{h_2}\sum_{\lambda=0}^{h_2-j}\frac{(-1)^\lambda(1-y)^{h_2-\alpha_{2j}-\lambda}}{\lambda!\Gamma(h_2-\alpha_{2j}-\lambda+1)}\right)\right. \\ \left.-D_{1-}^{(0,\alpha_{2j})}Q_{n,m}^*(x,y)\right\|_\infty\leq T_{0j}. \quad (57)$$

Then

$$\left\|D_{1-}^{(0,\alpha_{2j})}f-D_{1-}^{(0,\alpha_{2j})}Q_{n,m}^*\right\|_\infty\leq \\ \frac{\rho_{n,m}}{h_1!}\left(\sum_{\lambda=0}^{h_2-j}\frac{2^{h_2-\alpha_{2j}-\lambda}}{\lambda!\Gamma(h_2-\alpha_{2j}-\lambda+1)}\right)+T_{0j}= \quad (58)$$

$$\frac{\dot{C}}{h_1!}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)\left[\sum_{\bar{i}=h_1}^{v_1}\sum_{\bar{j}=h_2}^{v_2}l_{\bar{i}\bar{j}}\frac{2^{(\bar{i}+\bar{j})-(\alpha_{1\bar{i}}+\alpha_{2\bar{j}})}}{\Gamma(\bar{i}-\alpha_{1\bar{i}}+1)\Gamma(\bar{j}-\alpha_{2\bar{j}}+1)}\frac{1}{n^{r-\bar{i}}}\frac{1}{m^{p-\bar{j}}}\right] \quad (59)$$

$$\left(\sum_{\lambda=0}^{h_2-j}\frac{2^{h_2-\alpha_{2j}-\lambda}}{\lambda!\Gamma(h_2-\alpha_{2j}-\lambda+1)}\right)+\frac{2^{j-\alpha_{2j}}\dot{C}}{\Gamma(j-\alpha_{2j}+1)n^r m^{p-j}}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)\leq \\ \frac{\dot{C}\omega_{s,q}\left(f^{(r,p)};\frac{1}{n},\frac{1}{m}\right)}{n^{r-v_1}m^{p-v_2}}A_{0j}, \quad (60)$$

where

$$A_{0j} := \left[\frac{1}{h_1!} \left(\sum_{\bar{i}=h_1}^{v_1} \sum_{\bar{j}=h_2}^{v_2} l_{\bar{i}\bar{j}} \frac{2^{(\bar{i}+\bar{j})-(\alpha_{1\bar{i}}+\alpha_{2\bar{j}})}}{\Gamma(\bar{i}-\alpha_{1\bar{i}}+1)\Gamma(\bar{j}-\alpha_{2\bar{j}}+1)} \right) \right. \\ \left. \left(\sum_{\lambda=0}^{h_2-j} \frac{2^{h_2-\alpha_{2j}-\lambda}}{\lambda! \Gamma(h_2-\alpha_{2j}-\lambda+1)} \right) + \frac{2^{j-\alpha_{2j}}}{\Gamma(j-\alpha_{2j}+1)} \right]. \quad (61)$$

(Set $c_{0j} := \dot{C}A_{0j}$)

We proved that (case of $i = 0, j = 1, \dots, h_2$)

$$\left\| D_{1-}^{(0, \alpha_{2j})} f - D_{1-}^{(0, \alpha_{2j})} Q_{n,m}^* \right\|_{\infty} \leq \frac{c_{0j}}{n^{r-v_1} m^{p-v_2}} \omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right). \quad (62)$$

establishing (30).

Similarly we get for $i = 1, \dots, h_1, j = 0$, that

$$\left\| D_{1-}^{(\alpha_{1i}, 0)} f - D_{1-}^{(\alpha_{1i}, 0)} Q_{n,m}^* \right\|_{\infty} \leq \frac{c_{i0}}{n^{r-v_1} m^{p-v_2}} \omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right), \quad (63)$$

where $c_{i0} := \dot{C}A_{i0}$, with

$$A_{i0} := \left[\frac{1}{h_2!} \left(\sum_{\bar{i}=h_1}^{v_1} \sum_{\bar{j}=h_2}^{v_2} l_{\bar{i}\bar{j}} \frac{2^{(\bar{i}+\bar{j})-(\alpha_{1\bar{i}}+\alpha_{2\bar{j}})}}{\Gamma(\bar{i}-\alpha_{1\bar{i}}+1)\Gamma(\bar{j}-\alpha_{2\bar{j}}+1)} \right) \right. \\ \left. \left(\sum_{k=0}^{h_1-i} \frac{2^{h_1-\alpha_{1i}-k}}{k! \Gamma(h_1-\alpha_{1i}-k+1)} \right) + \frac{2^{i-\alpha_{1i}}}{\Gamma(i-\alpha_{1i}+1)} \right], \quad (64)$$

deriving (32).

So if $(x, y) \in [-1, 0]^2$, then

$$\alpha_{h_1 h_2}^{-1}(x, y) \bar{L}(Q_{n,m}^*(x, y)) = \alpha_{h_1 h_2}^{-1}(x, y) \bar{L}(f(x, y)) + \\ \rho_{n,m} \frac{(1-x)^{h_1-\alpha_{1i}}}{\Gamma(h_1-\alpha_{1i}+1)} \frac{(1-y)^{h_2-\alpha_{2j}}}{\Gamma(h_2-\alpha_{2j}+1)} + \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{h_1 h_2}^{-1}(x, y) \alpha_{ij}(x, y). \quad (65)$$

$$\left[D_{1-}^{(\alpha_{1i}, \alpha_{2j})} Q_{n,m}^*(x, y) - D_{1-}^{(\alpha_{1i}, \alpha_{2j})} f(x, y) - \rho_{n,m} D_{1-}^{(\alpha_{1i}, \alpha_{2j})} \left(\frac{x^{h_1} y^{h_2}}{h_1! h_2!} \right) \right] \stackrel{(42)}{\geq} \\ \rho_{n,m} \frac{(1-x)^{h_1-\alpha_{1i}}}{\Gamma(h_1-\alpha_{1i}+1)} \frac{(1-y)^{h_2-\alpha_{2j}}}{\Gamma(h_2-\alpha_{2j}+1)} - \quad (66)$$

$$\left[\sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} l_{ij} \frac{2^{(i+j)-(\alpha_{1i}+\alpha_{2j})}}{\Gamma(i-\alpha_{1i}+1)\Gamma(j-\alpha_{2j}+1)} \frac{\dot{C}}{n^{r-i} m^{p-j}} \omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right) \right] =$$

$$\rho_{n,m} \left[\frac{(1-x)^{h_1-\alpha_{1i}}}{\Gamma(h_1-\alpha_{1i}+1)} \frac{(1-y)^{h_2-\alpha_{2j}}}{\Gamma(h_2-\alpha_{2j}+1)} - 1 \right] \geq \quad (67)$$

$$\rho_{n,m} \left[\frac{1}{\Gamma(h_1-\alpha_{1i}+1)\Gamma(h_2-\alpha_{2j}+1)} - 1 \right] = \quad (68)$$

$$\rho_{n,m} \left[\frac{1-\Gamma(h_1-\alpha_{1i}+1)\Gamma(h_2-\alpha_{2j}+1)}{\Gamma(h_1-\alpha_{1i}+1)\Gamma(h_2-\alpha_{2j}+1)} \right] \geq 0.$$

Explanation: we have that $\Gamma(1) = 1$, $\Gamma(2) = 1$, and Γ is convex on $(0, \infty)$ and positive there, here $0 \leq h_1 - \alpha_{1h_1}, h_2 - \alpha_{2h_2} < 1$ and $1 \leq h_1 - \alpha_{1h_1} + 1, h_2 - \alpha_{2h_2} + 1 < 2$. Thus $0 < \Gamma(h_1 - \alpha_{1h_1} + 1), \Gamma(h_2 - \alpha_{2h_2} + 1) \leq 1$, and

$$0 \leq \Gamma(h_1 - \alpha_{1h_1} + 1)\Gamma(h_2 - \alpha_{2h_2} + 1) \leq 1. \quad (69)$$

And

$$1 - \Gamma(h_1 - \alpha_{1h_1} + 1)\Gamma(h_2 - \alpha_{2h_2} + 1) \geq 0. \quad (70)$$

Therefore it holds

$$\bar{L}(Q_{n,m}^*(x, y)) \geq 0, \quad \forall (x, y) \in [-1, 0]^2. \quad (71)$$

II. Suppose, throughout $[-1, 0]^2$, $\alpha_{h_1 h_2}(x, y) \leq \beta < 0$. Let $Q_{n,m}^{**}(x, y)$, $(x, y) \in [-1, 1]^2$, as in (40), so that

$$\left\| D_{1-}^{(\alpha_{1i}, \alpha_{2j})} \left(f(x, y) - \rho_{n,m} \frac{x^{h_1} y^{h_2}}{h_1! h_2!} \right) - D_{1-}^{(\alpha_{1i}, \alpha_{2j})} Q_{n,m}^{**}(x, y) \right\|_{\infty} \leq \frac{2^{(i+j)-(\alpha_{1i}+\alpha_{2j})} \dot{C}}{\Gamma(i-\alpha_{1i}+1)\Gamma(j-\alpha_{2j}+1) n^{r-i} m^{p-j}} \omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right), \quad (72)$$

$i = 0, 1, \dots, r, j = 0, 1, \dots, p$.

As earlier we produce the same convergence inequalities (26), (27), (29), (30), and (32).

So for $(x, y) \in [-1, 0]^2$ we get

$$\alpha_{h_1 h_2}^{-1}(x, y) \bar{L}(Q_{n,m}^{**}(x, y)) = \alpha_{h_1 h_2}^{-1}(x, y) \bar{L}(f(x, y)) - \rho_{n,m} \frac{(1-x)^{h_1-\alpha_{1i}}}{\Gamma(h_1-\alpha_{1i}+1)} \frac{(1-y)^{h_2-\alpha_{2j}}}{\Gamma(h_2-\alpha_{2j}+1)} + \sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} \alpha_{h_1 h_2}^{-1}(x, y) \alpha_{ij}(x, y). \quad (73)$$

$$\left[D_{1-}^{(\alpha_{1i}, \alpha_{2j})} Q_{n,m}^{**}(x, y) - D_{1-}^{(\alpha_{1i}, \alpha_{2j})} f(x, y) + \rho_{n,m} D_{1-}^{(\alpha_{1i}, \alpha_{2j})} \left(\frac{x^{h_1} y^{h_2}}{h_1! h_2!} \right) \right] \stackrel{(72)}{\leq} -\rho_{n,m} \frac{(1-x)^{h_1-\alpha_{1i}}}{\Gamma(h_1-\alpha_{1i}+1)} \frac{(1-y)^{h_2-\alpha_{2j}}}{\Gamma(h_2-\alpha_{2j}+1)} + \quad (74)$$

$$\begin{aligned}
& \left[\sum_{i=h_1}^{v_1} \sum_{j=h_2}^{v_2} l_{ij} \frac{2^{(i+j)-(\alpha_{1i}+\alpha_{2j})}}{\Gamma(i-\alpha_{1i}+1)\Gamma(j-\alpha_{2j}+1)} \frac{C}{n^{r-i}m^{p-j}} \omega_{s,q} \left(f^{(r,p)}; \frac{1}{n}, \frac{1}{m} \right) \right] = \\
& \rho_{n,m} \left[1 - \frac{(1-x)^{h_1-\alpha_{1i}}}{\Gamma(h_1-\alpha_{1i}+1)} \frac{(1-y)^{h_2-\alpha_{2j}}}{\Gamma(h_2-\alpha_{2j}+1)} \right] = \\
& \rho_{n,m} \left[\frac{\Gamma(h_1-\alpha_{1i}+1)\Gamma(h_2-\alpha_{2j}+1) - (1-x)^{h_1-\alpha_{1i}}(1-y)^{h_2-\alpha_{2j}}}{\Gamma(h_1-\alpha_{1i}+1)\Gamma(h_2-\alpha_{2j}+1)} \right] \leq \\
& \rho_{n,m} \left[\frac{1 - (1-x)^{h_1-\alpha_{1i}}(1-y)^{h_2-\alpha_{2j}}}{\Gamma(h_1-\alpha_{1i}+1)\Gamma(h_2-\alpha_{2j}+1)} \right] \leq 0. \tag{75}
\end{aligned}$$

Explanation: for $x, y \in [-1, 0]$ we get that $1-x, 1-y \geq 1$, and $0 \leq h_1 - \alpha_{1h_1}$, $h_2 - \alpha_{2h_2} < 1$. Hence $(1-x)^{h_1-\alpha_{1h_1}}, (1-y)^{h_2-\alpha_{2h_2}} \geq 1$, and then

$$(1-x)^{h_1-\alpha_{1h_1}}(1-y)^{h_2-\alpha_{2h_2}} \geq 1,$$

so that

$$1 - (1-x)^{h_1-\alpha_{1h_1}}(1-y)^{h_2-\alpha_{2h_2}} \leq 0. \tag{76}$$

Hence again

$$\bar{L}(Q_{n,m}^{**}(x, y)) \geq 0, \text{ for } (x, y) \in [-1, 0]^2. \tag{77}$$

■

References

- [1] G.A. Anastassiou, *Monotone approximation by pseudopolynomials*, Approximation Theory, Academic Press, New York, 1991, 5-11.
- [2] G.A. Anastassiou, *Bivariate Monotone Approximation*, Proc. of AMS, vol. 112, No. 4 (1991), 959-963.
- [3] G.A. Anastassiou, O. Shisha, *Monotone approximation with linear differential operators*, J. Approx. Theory 44 (1985), 391-393.
- [4] H.H. Gonska, *Simultaneously approximation by algebraic blending functions*, Alfred Haar Memorial Conference, Budapest 1985, Colloquia Mathematica Soc. Janos Bolyai, 49, North-Holand, Amsterdam, 1985, 363-382.
- [5] H.H. Gonska, personal communication with author, 2-24-2014.
- [6] S. Iqbal, K. Krulic, J. Pecaric, On an inequality of H.G. Hardy, J. of Inequalities and Applications, Volume 2010, Article ID 264347, 23 pages.
- [7] A. Marchaud, *Differences et deerivees d'une fonction de deux variables*, C.R. Acad. Sci. 178 (1924), 1467-1470.

- [8] A. Marchaud, *Sur les derivees et sur les differences des fonctions de variables reelles*, J. Math. Pures Appl. 6 (1927), 337-425.
- [9] L.L. Schumaker, *Spline Functions: Basic Theory*, Wiley, New York, 1981.
- [10] O. Shisha, *Monotone approximation*, Pacific J. Math. 15 (1965), 667-671.