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GENERAL LEBESGUE INTEGRAL INEQUALITIES OF JENSEN AND OSTROWSKI TYPE FOR DIFFERENTIABLE FUNCTIONS WHOSE DERIVATIVES IN ABSOLUTE VALUE ARE h-CONVEX AND APPLICATIONS

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Abstract. Some inequalities related to Jensen and Ostrowski inequalities for general Lebesgue integral of differentiable functions whose derivatives in absolute value are h-convex are obtained. Applications for f-divergence measure are provided as well.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ − algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu = 1$. Consider the Lebesgue space

$$L(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)|\, d\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(t)\, d\mu(t)$.

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, S.S. Dragomir obtained in 2002 [30] the following result:

Theorem 1. *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m, M) and $f : \Omega \to [m, M]$ so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) \cdot f \in L(\Omega, \mu)$. Then we have the inequality:*

$$\begin{aligned} 0 &\le \int_\Omega \Phi \circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \\ &\le \int_\Omega f \cdot (\Phi' \circ f)\, d\mu - \int_\Omega \Phi' \circ f d\mu \int_\Omega f d\mu \\ &\le \frac{1}{2}\left[\Phi'(M) - \Phi'(m)\right] \int_\Omega \left| f - \int_\Omega f d\mu \right| d\mu. \end{aligned} \tag{1.1}$$

In the case of discrete measure, we have:

Corollary 1. *Let $\Phi : [m, M] \to \mathbb{R}$ be a differentiable convex function on (m, M). If $x_i \in [m, M]$ and $w_i \ge 0$ $(i = 1, \dots, n)$ with $W_n := \sum_{i=1}^n w_i = 1$, then one has*

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the counterpart of Jensen's weighted discrete inequality:

$$0 \leq \sum_{i=1}^{n} w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} w_i x_i\right) \leq \sum_{i=1}^{n} w_i \Phi'(x_i) x_i - \sum_{i=1}^{n} w_i \Phi'(x_i) \sum_{i=1}^{n} w_i x_i \leq \frac{1}{2}[\Phi'(M) - \Phi'(m)] \sum_{i=1}^{n} w_i \left| x_i - \sum_{j=1}^{n} w_j x_j \right|. \tag{1.2}$$

Remark 1. *We notice that the inequality between the first and the second term in (1.2) was proved in 1994 by Dragomir & Ionescu, see* [56]*.*

If $f, g : \Omega \to \mathbb{R}$ are $\mu-$measurable functions and $f,\ g,\ fg \in L(\Omega, \mu)$, then we may consider the *Čebyšev functional*

$$T(f,g) := \int_{\Omega} fg d\mu - \int_{\Omega} f d\mu \int_{\Omega} g d\mu. \tag{1.3}$$

The following result is known in the literature as the *Grüss inequality*

$$|T(f,g)| \leq \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta), \tag{1.4}$$

provided

$$-\infty < \gamma \leq f(t) \leq \Gamma < \infty, \quad -\infty < \delta \leq g(t) \leq \Delta < \infty \tag{1.5}$$

for $\mu -$ a.e. $t \in \Omega$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

If we assume that $-\infty < \gamma \leq f(t) \leq \Gamma < \infty$ for $\mu -$ a.e. $t \in \Omega$, then by the Grüss inequality for $g = f$ and by the Schwarz's integral inequality, we have

$$\int_{\Omega} \left| f - \int_{\Omega} f d\mu \right| d\mu \leq \left[\int_{\Omega} f^2 d\mu - \left(\int_{\Omega} f d\mu \right)^2 \right]^{\frac{1}{2}} \leq \frac{1}{2}(\Gamma - \gamma). \tag{1.6}$$

On making use of the results (1.1) and (1.6), we can state the following string of reverse inequalities

$$0 \leq \int_{\Omega} \Phi \circ f d\mu - \Phi\left(\int_{\Omega} f d\mu\right) \leq \int_{\Omega} f \cdot (\Phi' \circ f) d\mu - \int_{\Omega} \Phi' \circ f d\mu \int_{\Omega} f d\mu \leq \frac{1}{2}[\Phi'(M) - \Phi'(m)] \int_{\Omega} \left| f - \int_{\Omega} f d\mu \right| d\mu \leq \frac{1}{2}[\Phi'(M) - \Phi'(m)] \left[\int_{\Omega} f^2 d\mu - \left(\int_{\Omega} f d\mu \right)^2 \right]^{\frac{1}{2}} \leq \frac{1}{4}[\Phi'(M) - \Phi'(m)](M - m), \tag{1.7}$$

provided that $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ is a differentiable convex function on (m, M) and $f : \Omega \to [m, M]$ so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ f \cdot (\Phi' \circ f) \in L(\Omega, \mu)$, with $\int_{\Omega} d\mu = 1$.

The following reverse of the Jensen's inequality also holds [44]:

Theorem 2. *Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, where $\mathring{I}$ is the interior of I. If $f : \Omega \to \mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty < m \leq f(t) \leq M < \infty \text{ for } \mu\text{-a.e. } t \in \Omega$$

and such that f, $\Phi \circ f \in L(\Omega, \mu)$, then

$$\begin{aligned} 0 &\leq \int_\Omega \Phi \circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \\ &\leq \left(M - \int_\Omega f d\mu\right)\left(\int_\Omega f d\mu - m\right)\frac{\Phi'_-(M) - \Phi'_+(m)}{M - m} \\ &\leq \frac{1}{4}(M - m)\left[\Phi'_-(M) - \Phi'_+(m)\right], \end{aligned} \tag{1.8}$$

where Φ'_- is the left and Φ'_+ is the right derivative of the convex function Φ.

For other reverse of Jensen inequality and applications to divergence measures see [44].

In 1938, A. Ostrowski [84], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b \Phi(t)\,dt$ and the value $\Phi(x)$, $x \in [a, b]$.

For various results related to Ostrowski's inequality see [13]-[16], [32]-[63], [68] and the references therein.

Theorem 3. *Let $\Phi : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) such that $\Phi' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e., $\|\Phi'\|_\infty := \sup_{t \in (a,b)} |\Phi'(t)| < \infty$. Then*

$$\left|\Phi(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right]\|\Phi'\|_\infty (b - a), \tag{1.9}$$

for all $x \in [a, b]$ and the constant $\frac{1}{4}$ is the best possible.

Now, for $\gamma, \Gamma \in \mathbb{C}$ and $[a, b]$ an interval of real numbers, define the sets of complex-valued functions [55]

$$\bar{U}_{[a,b]}(\gamma, \Gamma)$$
$$:= \left\{f : [a, b] \to \mathbb{C} \mid \operatorname{Re}\left[(\Gamma - f(t))\left(\overline{f(t)} - \overline{\gamma}\right)\right] \geq 0 \text{ for almost every } t \in [a, b]\right\}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma, \Gamma) := \left\{f : [a, b] \to \mathbb{C} \mid \left|f(t) - \frac{\gamma + \Gamma}{2}\right| \leq \frac{1}{2}|\Gamma - \gamma| \text{ for a.e. } t \in [a, b]\right\}.$$

The following representation result may be stated [55].

Proposition 1. *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have that $\bar{U}_{[a,b]}(\gamma, \Gamma)$ and $\bar{\Delta}_{[a,b]}(\gamma, \Gamma)$ are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma, \Gamma) = \bar{\Delta}_{[a,b]}(\gamma, \Gamma). \tag{1.10}$$

On making use of the complex numbers field properties we can also state that:

Corollary 2. *For any* $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, *we have that*

$$\bar{U}_{[a,b]}(\gamma,\Gamma) = \{f : [a,b] \to \mathbb{C} \mid (\operatorname{Re}\Gamma - \operatorname{Re} f(t))(\operatorname{Re} f(t) - \operatorname{Re}\gamma) + (\operatorname{Im}\Gamma - \operatorname{Im} f(t))(\operatorname{Im} f(t) - \operatorname{Im}\gamma) \geq 0 \text{ for a.e. } t \in [a,b]\}. \tag{1.11}$$

Now, if we assume that $\operatorname{Re}(\Gamma) \geq \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \geq \operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\bar{S}_{[a,b]}(\gamma,\Gamma) := \{f : [a,b] \to \mathbb{C} \mid \operatorname{Re}(\Gamma) \geq \operatorname{Re} f(t) \geq \operatorname{Re}(\gamma) \text{ and } \operatorname{Im}(\Gamma) \geq \operatorname{Im} f(t) \geq \operatorname{Im}(\gamma) \text{ for a.e. } t \in [a,b]\}. \tag{1.12}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma,\Gamma)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\gamma,\Gamma) \subseteq \bar{U}_{[a,b]}(\gamma,\Gamma). \tag{1.13}$$

The following result holds [55]:

Theorem 4. *Let* $\Phi : I \to \mathbb{C}$ *be an absolutely continuous functions on* $[a,b] \subset \mathring{I}$*, the interior of* I*. For some* $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, *assume that* $\Phi' \in \bar{U}_{[a,b]}(\gamma,\Gamma)\left(= \bar{\Delta}_{[a,b]}(\gamma,\Gamma)\right)$. *If* $g : \Omega \to [a,b]$ *is Lebesgue* μ*-measurable on* Ω *and such that* $\Phi \circ g$, $g \in L(\Omega,\mu)$, *then we have the inequality*

$$\left| \int_\Omega \Phi \circ g d\mu - \Phi(x) - \frac{\gamma+\Gamma}{2}\left(\int_\Omega g d\mu - x\right) \right| \leq \frac{1}{2}|\Gamma-\gamma| \int_\Omega |g-x| d\mu \tag{1.14}$$

for any $x \in [a,b]$.

In particular, we have

$$\begin{aligned} &\left| \int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) - \frac{\gamma+\Gamma}{2}\left(\int_\Omega g d\mu - \frac{a+b}{2}\right) \right| \\ &\leq \frac{1}{2}|\Gamma-\gamma| \int_\Omega \left| g - \frac{a+b}{2} \right| d\mu \leq \frac{1}{4}(b-a)|\Gamma-\gamma| \end{aligned} \tag{1.15}$$

and

$$\begin{aligned} \left| \int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu\right) \right| &\leq \frac{1}{2}|\Gamma-\gamma| \int_\Omega \left| g - \int_\Omega g d\mu \right| d\mu \\ &\leq \frac{1}{2}|\Gamma-\gamma| \left(\int_\Omega g^2 d\mu - \left(\int_\Omega g d\mu\right)^2 \right)^{1/2} \\ &\leq \frac{1}{4}(b-a)|\Gamma-\gamma|. \end{aligned} \tag{1.16}$$

Motivated by the above results, in this paper we provide more upper bounds for the quantity

$$\left| \int_\Omega \Phi \circ g d\mu - \Phi(x) \right|, \; x \in [a,b],$$

under various assumptions on the absolutely continuous function Φ, which in the particular case of $x = \int_\Omega g d\mu$ provides some results connected with Jensen's inequality while in the general case provides some generalizations of Ostrowski's inequality. Applications for divergence measures are provided as well.

2. Preliminary Facts

2.1. **Some Identities.** The following result holds [55]:

Lemma 1. *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a,b] \subset \mathring{I}$, the interior of I. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g$, $g \in L(\Omega, \mu)$, then we have the equality*

$$\begin{aligned}(2.1)\qquad & \int_\Omega \Phi \circ g d\mu - \Phi(x) - \lambda \left(\int_\Omega g d\mu - x \right) \\ & = \int_\Omega \left[(g - x) \int_0^1 (\Phi' ((1-s)x + sg) - \lambda) ds \right] d\mu\end{aligned}$$

for any $\lambda \in \mathbb{C}$ and $x \in [a,b]$.

In particular, we have

$$(2.2)\qquad \int_\Omega \Phi \circ g d\mu - \Phi(x) = \int_\Omega \left[(g - x) \int_0^1 \Phi' ((1-s)x + sg) ds \right] d\mu,$$

for any $x \in [a,b]$.

Remark 2. *With the assumptions of Lemma 1 we have*

$$\begin{aligned}(2.3)\qquad & \int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right) \\ & = \int_\Omega \left[\left(g - \frac{a+b}{2} \right) \int_0^1 \Phi' \left((1-s)\frac{a+b}{2} + sg \right) ds \right] d\mu.\end{aligned}$$

Corollary 3. *With the assumptions of Lemma 1 we have*

$$\begin{aligned}(2.4)\qquad & \int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu\right) \\ & = \int_\Omega \left[\left(g - \int_\Omega g d\mu \right) \int_0^1 \Phi' \left((1-s)\int_\Omega g d\mu + sg \right) ds \right] d\mu.\end{aligned}$$

Proof. We observe that since $g : \Omega \to [a,b]$ and $\int_\Omega d\mu = 1$ then $\int_\Omega g d\mu \in [a,b]$ and by taking $x = \int_\Omega g d\mu$ in (2.2) we get (2.4). □

Corollary 4. *With the assumptions of Lemma 1 we have*

$$\begin{aligned}(2.5)\qquad & \int_\Omega \Phi \circ g d\mu - \frac{1}{b-a}\int_a^b \Phi(x)dx - \lambda\left(\int_\Omega g d\mu - \frac{a+b}{2}\right) \\ & = \int_\Omega \left\{ \frac{1}{b-a} \int_a^b \left[(g-x) \int_0^1 (\Phi'((1-s)x + sg) - \lambda) ds \right] dx \right\} d\mu.\end{aligned}$$

Proof. Follows by integrating the identity (2.1) over $x \in [a,b]$, dividing by $b - a > 0$ and using Fubini's theorem. □

Corollary 5. *Let $\Phi : I \to \mathbb{C}$ be an absolutely continuous functions on $[a,b] \subset \mathring{I}$, the interior of I. If $g, h : \Omega \to [a,b]$ are Lebesgue μ-measurable on Ω and such that*

$\Phi\circ g,\ \Phi\circ h,\ g,\ h\in L(\Omega,\mu)$, then we have the equality

$$\int_\Omega \Phi\circ g d\mu-\int_\Omega \Phi\circ h d\mu-\lambda\left(\int_\Omega g d\mu-\int_\Omega h d\mu\right) = \int_\Omega\int_\Omega\left[(g(t)-h(\tau))\int_0^1(\Phi'((1-s)h(\tau)+sg(t))-\lambda)ds\right] \times d\mu(t)\,d\mu(\tau) \tag{2.6}$$

for any $\lambda\in\mathbb{C}$ and $x\in[a,b]$.

In particular, we have

$$\int_\Omega \Phi\circ g d\mu-\int_\Omega \Phi\circ h d\mu = \int_\Omega\int_\Omega\left[(g(t)-h(\tau))\int_0^1\Phi'((1-s)h(\tau)+sg(t))ds\right]d\mu(t)\,d\mu(\tau), \tag{2.7}$$

for any $x\in[a,b]$.

Remark 3. *The above inequality (2.6) can be extended for two measures as follows*

$$\int_{\Omega_1} \Phi\circ g d\mu_1-\int_{\Omega_2} \Phi\circ h d\mu_2-\lambda\left(\int_{\Omega_1} g d\mu_1-\int_{\Omega_2} h d\mu_2\right) = \int_{\Omega_1}\int_{\Omega_2}\left[(g(t)-h(\tau))\int_0^1(\Phi'((1-s)h(\tau)+sg(t))-\lambda)ds\right] \times d\mu_1(t)\,d\mu_2(\tau), \tag{2.8}$$

for any $\lambda\in\mathbb{C}$ and $x\in[a,b]$ and provided that $\Phi\circ g,\ g\in L(\Omega_1,\mu_1)$ while $\Phi\circ h, h\in L(\Omega_2,\mu_2)$.

Remark 4. *If $w\geq 0$ μ-almost everywhere (μ-a.e.) on Ω with $\int_\Omega wd\mu>0$, then by replacing $d\mu$ with $\frac{wd\mu}{\int_\Omega wd\mu}$ in (2.1) we have the weighted equality*

$$\frac{1}{\int_\Omega wd\mu}\int_\Omega w(\Phi\circ g)d\mu-\Phi(x)-\lambda\left(\frac{1}{\int_\Omega wd\mu}\int_\Omega wgd\mu-x\right) = \frac{1}{\int_\Omega wd\mu}\int_\Omega w\cdot\left[(g-x)\int_0^1(\Phi'((1-s)x+sg)-\lambda)ds\right]d\mu \tag{2.9}$$

for any $\lambda\in\mathbb{C}$ and $x\in[a,b]$, provided $\Phi\circ g,\ g\in L_w(\Omega,\mu)$ where

$$L_w(\Omega,\mu):=\left\{g|\ \int_\Omega w|g|d\mu<\infty\right\}.$$

The other equalities have similar weighted versions. However the details are omitted.

2.2. ***h*-Convex Functions.** We recall here some concepts of convexity that are well known in the literature.

Let I be an interval in $\mathbb{R}$.

Definition 1 ([65])**.** *We say that $\Phi:I\to\mathbb{R}$ is a Godunova-Levin function or that Φ belongs to the class $Q(I)$ if Φ is non-negative and for all $x,y\in I$ and $t\in(0,1)$ we have*

$$\Phi(tx+(1-t)y)\leq\frac{1}{t}\Phi(x)+\frac{1}{1-t}\Phi(y). \tag{2.10}$$

Some further properties of this class of functions can be found in [48], [49], [51], [83], [87] and [89]. Among others, its has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

The above concept can be extended for functions $\Phi : C \subseteq X \to [0, \infty)$ where C is a convex subset of the real or complex linear space X and the inequality (2.10) is satisfied for any vectors $x, y \in C$ and $t \in (0, 1)$. If the function $\Phi : C \subseteq X \to \mathbb{R}$ is non-negative and convex, then is of Godunova-Levin type.

Definition 2 ([51])**.** *We say that a function $\Phi : I \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0, 1]$ we have*

$$\Phi(tx + (1-t)y) \leq \Phi(x) + \Phi(y). \tag{2.11}$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions,* i. e. nonnegative functions satisfying

$$\Phi(tx + (1-t)y) \leq \max\{\Phi(x), \Phi(y)\} \tag{2.12}$$

for all $x, y \in I$ and $t \in [0, 1]$.

For some results on P-functions see [51] and [85] while for quasi convex functions, the reader can consult [50].

If $\Phi : C \subseteq X \to [0, \infty)$, where C is a convex subset of the real or complex linear space X, then we say that it is of P-type (or quasi-convex) if the inequality (2.11) (or (2.12)) holds true for $x, y \in C$ and $t \in [0, 1]$.

Definition 3 ([10])**.** *Let s be a real number, $s \in (0, 1]$. A function $\Phi : [0, \infty) \to [0, \infty)$ is said to be s-convex (in the second sense) or Breckner s-convex if*

$$\Phi(tx + (1-t)y) \leq t^s \Phi(x) + (1-t)^s \Phi(y)$$

for all $x, y \in [0, \infty)$ and $t \in [0, 1]$.

For some properties of this class of functions see [2], [3], [10], [11], [46], [47], [67], [77] and [95].

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of h-convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}, (0, 1) \subseteq J$ and functions h and Φ are real non-negative functions defined in J and I, respectively.

Definition 4 ([105])**.** *Let $h : J \to [0, \infty)$ with h not identical to 0. We say that $\Phi : I \to [0, \infty)$ is an h-convex function if for all $x, y \in I$ we have*

$$\Phi(tx + (1-t)y) \leq h(t)\Phi(x) + h(1-t)\Phi(y) \tag{2.13}$$

for all $t \in (0, 1)$.

For some results concerning this class of functions see [105], [9], [80], [94], [93] and [103].

We can introduce now another class of functions.

Definition 5. *We say that the function $\Phi : I \to [0, \infty) \to [0, \infty)$ is of s-Godunova-Levin type, with $s \in [0, 1]$, if*

$$\Phi(tx + (1-t)y) \leq \frac{1}{t^s}\Phi(x) + \frac{1}{(1-t)^s}\Phi(y), \tag{2.14}$$

for all $t \in (0, 1)$ and $x, y \in C$.

We observe that for $s=0$ we obtain the class of P-functions while for $s=1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(C)$ the class of s-Godunova-Levin functions defined on C, then we obviously have

$$P(C)=Q_0(C)\subseteq Q_{s_1}(C)\subseteq Q_{s_2}(C)\subseteq Q_1(C)=Q(C)$$

for $0\leq s_1\leq s_2\leq 1$.

For different inequalities related to these classes of functions, see [2]-[5], [9], [13]-[64], [76]-[80] and [85]-[103].

3. Inequalities for $|\Phi'|$ is h-Convex, Quasi-convex or Log-convex

We use the notations

$$\|k\|_{\Omega,p}:=\begin{cases}\left(\int\limits_{\Omega}|k(t)|^p\,d\mu(t)\right)^{1/p}<\infty,\ p\geq 1,\ k\in L_p(\Omega,\mu);\\ \\ ess\sup_{t\in\Omega}|k(t)|<\infty,\ p=\infty,\ k\in L_\infty(\Omega,\mu);\end{cases}$$

and

$$\|\Phi\|_{[0,1],p}:=\begin{cases}\left(\int_0^1|\Phi(s)|^p\,ds\right)^{1/p}<\infty,\ p\geq 1,\ \Phi\in L_p(0,1);\\ \\ ess\sup_{s\in[0,1]}|\Phi(s)|<\infty,\ p=\infty,\ \Phi\in L_\infty(0,1).\end{cases}$$

The following result holds:

Theorem 5. *Let $\Phi:I\to\mathbb{C}$ be a differentiable function on $\mathring{I}$, the interior of I and such that $|\Phi'|$ is h-convex on the interval $[a,b]\subset\mathring{I}$. If $g:\Omega\to[a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi\circ g,\ g\in L(\Omega,\mu)$, then we have the inequality*

$$\left|\int_\Omega\Phi\circ g\,d\mu-\Phi(x)\right| \tag{3.1}$$
$$\leq\int_0^1h(s)\,ds\begin{cases}\|g-x\|_{\Omega,\infty}\left[|\Phi'(x)|+\|\Phi'\circ g\|_{\Omega,1}\right]\\ \text{if } \Phi'\circ g\in L(\Omega,\mu);\\ \\ \|g-x\|_{\Omega,p}\,\||\Phi'(x)|+|\Phi'\circ g|\|_{\Omega,q}\\ \text{if } \Phi'\circ g\in L_q(\Omega,\mu),\ p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ \|g-x\|_{\Omega,1}\left[|\Phi'(x)|+\|\Phi'\circ g\|_{\Omega,\infty}\right]\\ \text{if } \Phi'\circ g\in L_\infty(\Omega,\mu)\end{cases}$$

for any $x\in[a,b]$.

In particular, we have

$$\left|\int_{\Omega}\Phi\circ gd\mu-\Phi\left(\int_{\Omega}gd\mu\right)\right| \tag{3.2}$$

$$\leq\int_{0}^{1}h(s)\,ds\begin{cases}\left\|g-\int_{\Omega}gd\mu\right\|_{\Omega,\infty}\left[\left|\Phi'\left(\int_{\Omega}gd\mu\right)\right|+\left\|\Phi'\circ g\right\|_{\Omega,1}\right], \\ \text{if } \Phi'\circ g\in L(\Omega,\mu); \\ \\ \left\|g-\int_{\Omega}gd\mu\right\|_{\Omega,p}\left\|\left|\Phi'\left(\int_{\Omega}gd\mu\right)\right|+\left|\Phi'\circ g\right|\right\|_{\Omega,q} \\ \text{if } \Phi'\circ g\in L_{q}(\Omega,\mu),\ p>1,\frac{1}{p}+\frac{1}{q}=1; \\ \\ \left\|g-\int_{\Omega}gd\mu\right\|_{\Omega,1}\left[\left|\Phi'\left(\int_{\Omega}gd\mu\right)\right|+\left\|\Phi'\circ g\right\|_{\Omega,\infty}\right] \\ \text{if } \Phi'\circ g\in L_{\infty}(\Omega,\mu)\end{cases}$$

and

$$\left|\int_{\Omega}\Phi\circ gd\mu-\Phi\left(\frac{a+b}{2}\right)\right| \tag{3.3}$$

$$\leq\int_{0}^{1}h(s)\,ds\begin{cases}\left\|g-\frac{a+b}{2}\right\|_{\Omega,\infty}\left[\left|\Phi'\left(\frac{a+b}{2}\right)\right|+\left\|\Phi'\circ g\right\|_{\Omega,1}\right] \\ \text{if } \Phi'\circ g\in L(\Omega,\mu);; \\ \\ \left\|g-\frac{a+b}{2}\right\|_{\Omega,p}\left\|\left|\Phi'\left(\frac{a+b}{2}\right)\right|+\left|\Phi'\circ g\right|\right\|_{\Omega,q} \\ \text{if } \Phi'\circ g\in L_{q}(\Omega,\mu),\ p>1,\frac{1}{p}+\frac{1}{q}=1; \\ \\ \left\|g-\frac{a+b}{2}\right\|_{\Omega,1}\left[\left|\Phi'\left(\frac{a+b}{2}\right)\right|+\left\|\Phi'\circ g\right\|_{\Omega,\infty}\right] \\ \text{if } \Phi'\circ g\in L_{\infty}(\Omega,\mu)\end{cases}$$

$$\leq\frac{1}{2}(b-a)\int_{0}^{1}h(s)\,ds\begin{cases}\left[\left|\Phi'\left(\frac{a+b}{2}\right)\right|+\left\|\Phi'\circ g\right\|_{\Omega,1}\right]; \\ \\ \left\|\left|\Phi'\left(\frac{a+b}{2}\right)\right|+\left|\Phi'\circ g\right|\right\|_{\Omega,q} \\ p>1,\frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\left|\Phi'\left(\frac{a+b}{2}\right)\right|+\left\|\Phi'\circ g\right\|_{\Omega,\infty}\right].\end{cases}$$

Proof. We have from (2.2) that

$$\left|\int_{\Omega}\Phi\circ gd\mu-\Phi(x)\right|\leq\int_{\Omega}|g-x|\left|\int_{0}^{1}\Phi'((1-s)x+sg)\,ds\right|d\mu, \tag{3.4}$$

for any $x\in[a,b]$.

Utilising Hölder's inequality for the μ-measurable functions $F,G:\Omega\to\mathbb{C}$,

$$\left|\int_{\Omega}FGd\mu\right|\leq\left(\int_{\Omega}|F|^{p}\,d\mu\right)^{1/p}\left(\int_{\Omega}|G|^{q}\,d\mu\right)^{1/q},\ p>1,\frac{1}{p}+\frac{1}{q}=1$$

and

$$\left|\int_{\Omega}FGd\mu\right|\leq ess\sup_{t\in\Omega}|F(t)|\int_{\Omega}|G|\,d\mu,$$

we have

$$B:=\int_{\Omega}|g-x|\left|\int_{0}^{1}\Phi'\left((1-s)x+sg\right)ds\right|d\mu \tag{3.5}$$
$$\leq\begin{cases} ess\sup_{t\in\Omega}|g(t)-x|\int_{\Omega}\left|\int_{0}^{1}\Phi'\left((1-s)x+sg\right)ds\right|d\mu; \\ \\ \left(\int_{\Omega}|g-x|^{p}d\mu\right)^{1/p}\left(\int_{\Omega}\left|\int_{0}^{1}\Phi'\left((1-s)x+sg\right)ds\right|^{q}d\mu\right)^{1/q} \\ p>1,\frac{1}{p}+\frac{1}{q}=1; \\ \\ \int_{\Omega}|g-x|\,d\mu\ ess\sup_{t\in\Omega}\left|\int_{0}^{1}\Phi'\left((1-s)x+sg\right)ds\right|, \end{cases}$$

for any $x\in[a,b]$.

Since $|\Phi'|$ is h-convex on the interval $[a,b]$, then we have for any $t\in\Omega$ that

$$\begin{aligned}\left|\int_{0}^{1}\Phi'\left((1-s)x+sg(t)\right)ds\right| &\leq\int_{0}^{1}\left|\Phi'\left((1-s)x+sg(t)\right)\right|ds \\ &\leq|\Phi'(x)|\int_{0}^{1}h(1-s)\,ds+|\Phi'(g(t))|\int_{0}^{1}h(s)\,ds \\ &=\left[|\Phi'(x)|+|\Phi'(g(t))|\right]\int_{0}^{1}h(s)\,ds,\end{aligned}$$

for any $x\in[a,b]$.

This implies that

$$\int_{\Omega}\left|\int_{0}^{1}\Phi'\left((1-s)x+sg\right)ds\right|d\mu\leq\int_{0}^{1}h(s)\,ds\left[|\Phi'(x)|+\int_{\Omega}|\Phi'\circ g|\,d\mu\right] \tag{3.6}$$

for any $x\in[a,b]$.

We have for any $t\in\Omega$ that

$$\begin{aligned}\left|\int_{0}^{1}\Phi'\left((1-s)x+sg(t)\right)ds\right|^{q} &\leq\left[\int_{0}^{1}\left|\Phi'\left((1-s)x+sg(t)\right)\right|ds\right]^{q} \\ &\leq\left[\left[|\Phi'(x)|+|\Phi'(g(t))|\right]\int_{0}^{1}h(s)\,ds\right]^{q} \\ &=\left[\int_{0}^{1}h(s)\,ds\right]^{q}\left[|\Phi'(x)|+|\Phi'(g(t))|\right]^{q}\end{aligned}$$

for any $x\in[a,b]$.

This implies

$$\begin{aligned}&\left(\int_{\Omega}\left|\int_{0}^{1}\Phi'\left((1-s)x+sg\right)ds\right|^{q}d\mu\right)^{1/q} \\ &\leq\int_{0}^{1}h(s)\,ds\left[\int_{\Omega}\left[|\Phi'(x)|+|\Phi'(g(t))|\right]^{q}d\mu\right]^{1/q} \\ &=\int_{0}^{1}h(s)\,ds\left[\int_{\Omega}\left[|\Phi'(x)|+|\Phi'\circ g|\right]^{q}d\mu\right]^{1/q}.\end{aligned} \tag{3.7}$$

Also

$$\begin{aligned}(3.8)\qquad & ess\sup_{t\in\Omega}\left|\int_0^1 \Phi'\left((1-s)\,x+sg\right)ds\right| \\ &\leq \left[\left|\Phi'(x)\right| + ess\sup_{t\in\Omega}\left|\Phi'(g(t))\right|\right]\int_0^1 h(s)\,ds \\ &= \left[\left|\Phi'(x)\right| + ess\sup_{t\in\Omega}\left|\Phi'\circ g\right|\right]\int_0^1 h(s)\,ds\end{aligned}$$

for any $x\in[a,b]$.

Making use of (3.6)-(3.8) we get the desired result (3.1). □

Remark 5. *With the assumptions of Theorem 5 and if $|\Phi'|$ is convex on the interval $[a,b]$, then $\int_0^1 h(s)\,ds=\frac{1}{2}$ and the inequalities (3.1)-(3.3) hold with $\frac{1}{2}$ instead of $\int_0^1 h(s)\,ds$. If $|\Phi'|$ is of s-Godunova-Levin type, with $s\in[0,1)$ on the interval $[a,b]$, then $\int_0^1 \frac{1}{t^s}dt=\frac{1}{1-s}$ and the inequalities (3.1)-(3.3) hold with $\frac{1}{1-s}$ instead of $\int_0^1 h(s)\,ds$.*

Firstly, let us recall the definition of quasi-convex functions.

Definition 6. *The function $h:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is said to be quasi-convex (QC) on the interval I if*

$$h(\lambda x+(1-\lambda)\,y)\leq\max\{h(x),h(y)\} \tag{3.9}$$

for any $x,y\in I$ and $\lambda\in[0,1]$.

Following [57], we say that for an interval $I\subseteq\mathbb{R}$, the mapping $h:I\to\mathbb{R}$ is *quasi-monotone* on I if it is either monotone on $I=[c,d]$ or monotone nonincreasing on a proper subinterval $[c,c']\subset I$ and monotone nondecreasing on $[c',d]$.

The class $QM(I)$ of quasi-monotone functions on I provides an immediate characterization of quasi-convex functions [57].

Proposition 2. *Suppose $I\subseteq\mathbb{R}$. Then the following statements are equivalent for a function $h:I\to\mathbb{R}$:*

(a) $h\in QM(I)$;
(b) *On any subinterval of I, h achieves its supremum at an end point;*
(c) $h\in QC(I)$.

As examples of quasi-convex functions we may consider the class of monotonic functions on an interval I for the class of convex functions on that interval.

Theorem 6. *Let $\Phi:I\to\mathbb{C}$ be a differentiable function on $\mathring{I}$, the interior of I and such that $|\Phi'|$ is quasi-convex on the interval $[a,b]\subset\mathring{I}$. If $g:\Omega\to[a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi\circ g,\ g\in L(\Omega,\mu)$ and $\Phi'\circ g\in L_\infty(\Omega,\mu)$, then we have the inequality*

$$\begin{aligned}(3.10)\qquad \left|\int_\Omega \Phi\circ g d\mu-\Phi(x)\right| &\leq \int_\Omega |g-x|\max\{|\Phi'(x)|,|\Phi'\circ g|\}\,d\mu \\ &\leq \max\left\{|\Phi'(x)|,\|\Phi'\circ g\|_{\Omega,\infty}\right\}\|g-x\|_{\Omega,1}\end{aligned}$$

for any $x\in[a,b]$.

In particular, we have

$$\left|\int_\Omega \Phi\circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| \tag{3.11}$$
$$\leq \int_\Omega \left|g - \int_\Omega g d\mu\right| \max\left\{\left|\Phi'\left(\int_\Omega g d\mu\right)\right|, |\Phi'\circ g|\right\} d\mu$$
$$\leq \max\left\{|\Phi'(x)|, \|\Phi'\circ g\|_{\Omega,\infty}\right\} \left\|g - \int_\Omega g d\mu\right\|_{\Omega,1}$$

and

$$\left|\int_\Omega \Phi\circ g d\mu - \Phi\left(\frac{a+b}{2}\right)\right| \tag{3.12}$$
$$\leq \int_\Omega \left|g - \frac{a+b}{2}\right| \max\left\{\left|\Phi'\left(\frac{a+b}{2}\right)\right|, |\Phi'\circ g|\right\} d\mu$$
$$\leq \max\left\{\left|\Phi'\left(\frac{a+b}{2}\right)\right|, \|\Phi'\circ g\|_{\Omega,\infty}\right\} \left\|g - \frac{a+b}{2}\right\|_{\Omega,1}.$$

Proof. From (3.4) we have

$$\left|\int_\Omega \Phi\circ g d\mu - \Phi(x)\right| \leq \int_\Omega |g-x| \left(\int_0^1 |\Phi'((1-s)x + sg)| ds\right) d\mu \tag{3.13}$$
$$\leq \int_\Omega |g-x| \max\left\{|\Phi'(x)|, |\Phi'\circ g|\right\} d\mu,$$

for any $x \in [a,b]$.

Observe that

$$|(\Phi'\circ g)(t)| \leq \|\Phi'\circ g\|_{\Omega,\infty} \text{ for almost every } t\in\Omega$$

and then

$$\int_\Omega |g-x| \max\left\{|\Phi'(x)|, |\Phi'\circ g|\right\} d\mu \tag{3.14}$$
$$\leq \int_\Omega |g-x| \max\left\{|\Phi'(x)|, \|\Phi'\circ g\|_{\Omega,\infty}\right\} d\mu$$
$$= \max\left\{|\Phi'(x)|, \|\Phi'\circ g\|_{\Omega,\infty}\right\} \int_\Omega |g-x| d\mu,$$

for any $x \in [a,b]$.

Using (3.13) and (3.14) we get the desired result (3.10). □

In what follows, I will denote an interval of real numbers. A function $f : I \to (0,\infty)$ is said to be *log-convex* or *multiplicatively convex* if $\log f$ is convex, or, equivalently, if for any $x, y \in I$ and $t \in [0,1]$ one has the inequality

$$f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t}. \tag{3.15}$$

We note that if f and g are convex and g is increasing, then $g\circ f$ is convex, moreover, since $f = \exp[\log f]$, it follows that a log-convex function is convex, but the converse may not necessarily be true. This follows directly from (3.15) since, by the arithmetic-geometric mean inequality we have

$$[f(x)]^t [f(y)]^{1-t} \leq tf(x) + (1-t) f(y) \tag{3.16}$$

for all $x, y \in I$ and $t \in [0,1]$.

Theorem 7. *Let $\Phi : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$, the interior of I and such that $|\Phi'|$ is log-convex on the interval $[a,b] \subset \mathring{I}$. If $g : \Omega \to [a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi \circ g$, $\Phi' \circ g$, $g \in L(\Omega,\mu)$ then we have the inequality*

$$
\begin{aligned}
&\left|\int_\Omega \Phi \circ g d\mu - \Phi(x)\right| \\
&\le \int_\Omega |g - x| L(|\Phi' \circ g|, |\Phi'(x)|) d\mu \\
&\le \frac{1}{2}\left[|\Phi'(x)| \int_\Omega |g - x| d\mu + \int_\Omega |g - x| |\Phi' \circ g| d\mu\right] \\
&\left(\le \frac{1}{2}\left[|\Phi'(x)| + \|\Phi' \circ g\|_{\Omega,\infty}\right] \|g - x\|_{\Omega,1} \text{ if } \Phi' \circ g \in L_\infty(\Omega,\mu)\right)
\end{aligned} \tag{3.17}
$$

for any $x \in [a,b]$, where $L(\cdot,\cdot)$ is the logarithmic mean, namely for $\alpha, \beta > 0$

$$
L(\alpha,\beta) := \begin{cases} \frac{\alpha-\beta}{\ln\alpha - \ln\beta}, & \alpha \ne \beta \\ \alpha, & \alpha = \beta. \end{cases}
$$

In particular, we have

$$
\begin{aligned}
&\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| \\
&\le \int_\Omega \left|g - \int_\Omega g d\mu\right| L\left(|\Phi' \circ g|, \left|\Phi'\left(\int_\Omega g d\mu\right)\right|\right) d\mu \\
&\le \frac{1}{2}\left[\left|\Phi'\left(\int_\Omega g d\mu\right)\right| \int_\Omega \left|g - \int_\Omega g d\mu\right| d\mu + \int_\Omega \left|g - \int_\Omega g d\mu\right| |\Phi' \circ g| d\mu\right] \\
&\left(\le \frac{1}{2}\left[\left|\Phi'\left(\int_\Omega g d\mu\right)\right| + \|\Phi' \circ g\|_{\Omega,\infty}\right] \left\|g - \int_\Omega g d\mu\right\|_{\Omega,1} \right. \\
&\left. \text{if } \Phi' \circ g \in L_\infty(\Omega,\mu)\right)
\end{aligned} \tag{3.18}
$$

and

$$
\begin{aligned}
&\left|\int_\Omega \Phi \circ g d\mu - \Phi\left(\frac{a+b}{2}\right)\right| \\
&\le \int_\Omega \left|g - \frac{a+b}{2}\right| L\left(|\Phi' \circ g|, \left|\Phi'\left(\frac{a+b}{2}\right)\right|\right) d\mu \\
&\le \frac{1}{2}\left[\left|\Phi'\left(\frac{a+b}{2}\right)\right| \int_\Omega \left|g - \frac{a+b}{2}\right| d\mu + \int_\Omega \left|g - \frac{a+b}{2}\right| |\Phi' \circ g| d\mu\right] \\
&\left(\le \frac{1}{2}\left[\left|\Phi'\left(\frac{a+b}{2}\right)\right| + \|\Phi' \circ g\|_{\Omega,\infty}\right] \left\|g - \frac{a+b}{2}\right\|_{\Omega,1} \right. \\
&\left. \text{if } \Phi' \circ g \in L_\infty(\Omega,\mu)\right).
\end{aligned} \tag{3.19}
$$

Proof. From (3.4) we have

$$\left|\int_{\Omega}\Phi\circ gd\mu-\Phi(x)\right|\leq\int_{\Omega}|g-x|\left(\int_{0}^{1}|\Phi'((1-s)x+sg)|\,ds\right)d\mu \tag{3.20}$$
$$\leq\int_{\Omega}|g-x|\left(\int_{0}^{1}|\Phi'(x)|^{1-s}|\Phi'\circ g|^{s}\,ds\right)d\mu,$$

for any $x\in[a,b]$.

Since, for any $C>0$, one has

$$\int_{0}^{1}C^{\lambda}d\lambda=\frac{C-1}{\ln C},$$

then for any $t\in\Omega$ we have

$$\int_{0}^{1}|\Phi'(x)|^{1-s}|\Phi'(g(t))|^{s}\,ds=|\Phi'(x)|\int_{0}^{1}\left|\frac{\Phi'(g(t))}{\Phi'(x)}\right|^{s}ds \tag{3.21}$$
$$=|\Phi'(x)|\frac{\left|\frac{\Phi'(g(t))}{\Phi'(x)}\right|-1}{\ln\left|\frac{\Phi'(g(t))}{\Phi'(x)}\right|}$$
$$=\frac{|\Phi'(g(t))|-|\Phi'(x)|}{\ln|\Phi'(g(t))|-\ln|\Phi'(x)|}$$
$$=L(|\Phi'(g(t))|,|\Phi'(x)|),$$

for any $x\in[a,b]$.

Making use of (3.20) and (3.21) we get the first inequality in (3.17).

The second inequality in (3.17) follows by the fact that

$$L(\alpha,\beta)\leq\frac{\alpha+\beta}{2}\text{ for any }\alpha,\beta>0.$$

The last inequality in (3.17) is obvious. □

4. Inequalities for $|\Phi'|^{q}$ is h-Convex or Log-convex

We have:

Theorem 8. *Let $\Phi:I\to\mathbb{C}$ be a differentiable function on $\mathring{I}$, the interior of I and such that for $p>1,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, $|\Phi'|^{q}$ is h-convex on the interval $[a,b]\subset\mathring{I}$. If $g:\Omega\to[a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi\circ g$, $g\in L(\Omega,\mu)$ and $\Phi'\circ g\in L_{q}(\Omega,\mu)$ then we have the inequality*

$$\left|\int_{\Omega}\Phi\circ gd\mu-\Phi(x)\right| \tag{4.1}$$
$$\leq\left(\int_{0}^{1}h(s)\,ds\right)^{1/q}\|g-x\|_{\Omega,p}\left(|\Phi'(x)|^{q}+\int_{\Omega}|\Phi'\circ g|^{q}\,d\mu\right)^{1/q}$$
$$\leq\left(\int_{0}^{1}h(s)\,ds\right)^{1/q}\|g-x\|_{\Omega,p}\left(|\Phi'(x)|+\|\Phi'\circ g\|_{\Omega,q}\right)$$

for any $x\in[a,b]$.

In particular, we have

$$\left|\int_{\Omega}\Phi\circ gd\mu-\Phi\left(\int_{\Omega}gd\mu\right)\right| \tag{4.2}$$
$$\leq\left(\int_0^1 h(s)\,ds\right)^{1/q}$$
$$\times\left\|g-\int_{\Omega}gd\mu\right\|_{\Omega,p}\left(\left|\Phi'\left(\int_{\Omega}gd\mu\right)\right|^q+\int_{\Omega}\left|\Phi'\circ g\right|^q d\mu\right)^{1/q}$$
$$\leq\left(\int_0^1 h(s)\,ds\right)^{1/q}$$
$$\times\left\|g-\int_{\Omega}gd\mu\right\|_{\Omega,p}\left(\left|\Phi'\left(\int_{\Omega}gd\mu\right)\right|+\left\|\Phi'\circ g\right\|_{\Omega,q}\right)$$

and

$$\left|\int_{\Omega}\Phi\circ gd\mu-\Phi\left(\frac{a+b}{2}\right)\right| \tag{4.3}$$
$$\leq\left(\int_0^1 h(s)\,ds\right)^{1/q}$$
$$\times\left\|g-\frac{a+b}{2}\right\|_{\Omega,p}\left(\left|\Phi'\left(\frac{a+b}{2}\right)\right|^q+\int_{\Omega}\left|\Phi'\circ g\right|^q d\mu\right)^{1/q}$$
$$\leq\left(\int_0^1 h(s)\,ds\right)^{1/q}$$
$$\times\left\|g-\frac{a+b}{2}\right\|_{\Omega,p}\left(\left|\Phi'\left(\frac{a+b}{2}\right)\right|+\left\|\Phi'\circ g\right\|_{\Omega,q}\right).$$

Proof. From the proof of Theorem 5 we have

$$\left|\int_{\Omega}\Phi\circ gd\mu-\Phi(x)\right| \tag{4.4}$$
$$\leq\int_{\Omega}|g-x|\left|\int_0^1\Phi'((1-s)x+sg)\,ds\right|d\mu$$
$$\leq\left(\int_{\Omega}|g-x|^p d\mu\right)^{1/p}\left(\int_{\Omega}\left|\int_0^1\Phi'((1-s)x+sg)\,ds\right|^q d\mu\right)^{1/q}$$
$$\leq\left(\int_{\Omega}|g-x|^p d\mu\right)^{1/p}\left(\int_{\Omega}\left(\int_0^1\left|\Phi'((1-s)x+sg)\right|^q ds\right)d\mu\right)^{1/q}$$

for $p>1, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $x\in[a,b]$.

Since $|\Phi'|^q$ is h-convex on the interval $[a,b]$, then

$$\int_0^1\left|\Phi'((1-s)x+sg(t))\right|^q ds\leq\left|\Phi'(x)\right|^q\int_0^1 h(1-s)\,ds+\left|\Phi'(g(t))\right|^q\int_0^1 h(s)\,ds$$
$$=\left[\left|\Phi'(x)\right|^q+\left|\Phi'(g(t))\right|^q\right]\int_0^1 h(s)\,ds$$

for any $x\in[a,b]$ and $t\in\Omega$.

Therefore

$$\left(\int_{\Omega}\left(\int_{0}^{1}\left|\Phi'\left((1-s)x+sg\right)\right|^{q}ds\right)d\mu\right)^{1/q} \tag{4.5}$$
$$\leq\left(\int_{\Omega}\left(\left[\left|\Phi'(x)\right|^{q}+\left|\Phi'(g(t))\right|^{q}\right]\int_{0}^{1}h(s)\,ds\right)d\mu\right)^{1/q}$$
$$=\left(\int_{0}^{1}h(s)\,ds\right)^{1/q}\left(\left|\Phi'(x)\right|^{q}+\int_{\Omega}\left|\Phi'\circ g\right|^{q}d\mu\right)^{1/q}$$

for any $x\in[a,b]$.

This proves the first inequality in (4.1).

Now, we observe that the following elementary inequality holds:

$$(\alpha+\beta)^{r}\geq(\leq)\,\alpha^{r}+\beta^{r} \tag{4.6}$$

for any $\alpha,\beta\geq0$ and $r\geq1$ $(0<r<1)$.

Indeed, if we consider the function $f_r:[0,\infty)\to\mathbb{R}$, $f_r(t)=(t+1)^{r}-t^{r}$ we have $f_r'(t)=r\left[(t+1)^{r-1}-t^{r-1}\right]$. Observe that for $r>1$ and $t>0$ we have that $f_r'(t)>0$ showing that f_r is strictly increasing on the interval $[0,\infty)$. Now for $t=\frac{\alpha}{\beta}$ $(\beta>0,\alpha\geq0)$ we have $f_r(t)>f_r(0)$ giving that $\left(\frac{\alpha}{\beta}+1\right)^{r}-\left(\frac{\alpha}{\beta}\right)^{r}>1$, i.e., the desired inequality (2.1).

For $r\in(0,1)$ we have that f_r is strictly decreasing on $[0,\infty)$ which proves the second case in (4.6).

Making use of (4.6) for $r=1/q\in(0,1)$ we have

$$\left(\left|\Phi'(x)\right|^{q}+\int_{\Omega}\left|\Phi'\circ g\right|^{q}d\mu\right)^{1/q}\leq\left|\Phi'(x)\right|+\left(\int_{\Omega}\left|\Phi'\circ g\right|^{q}d\mu\right)^{1/q}$$

and then we get the second part of (4.1). □

Finally, we have:

Theorem 9. *Let $\Phi:I\to\mathbb{C}$ be a differentiable function on $\mathring{I}$, the interior of I and such that for $p>1,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, $\left|\Phi'\right|^{q}$ is log-convex on the interval $[a,b]\subset\mathring{I}$. If $g:\Omega\to[a,b]$ is Lebesgue μ-measurable on Ω and such that $\Phi\circ g$, $g\in L(\Omega,\mu)$ and $\Phi'\circ g\in L_q(\Omega,\mu)$ then we have the inequality*

$$\left|\int_{\Omega}\Phi\circ g\,d\mu-\Phi(x)\right| \tag{4.7}$$
$$\leq\left\|g-x\right\|_{\Omega,p}\left(\int_{\Omega}L\left(\left|\Phi'\circ g\right|^{q},\left|\Phi'(x)\right|^{q}\right)d\mu\right)^{1/q}$$
$$\leq\frac{1}{2^{1/q}}\left\|g-x\right\|_{\Omega,p}\left[\left|\Phi'(x)\right|^{q}+\int_{\Omega}\left|\Phi'\circ g\right|^{q}d\mu\right]^{1/q}$$
$$\leq\frac{1}{2^{1/q}}\left\|g-x\right\|_{\Omega,p}\left[\left|\Phi'(x)\right|+\left\|\Phi'\circ g\right\|_{\Omega,q}\right]$$

for any $x\in[a,b]$.

In particular, we have

$$(4.8)\qquad \begin{aligned} &\left|\int_\Omega \Phi\circ g d\mu - \Phi\left(\int_\Omega g d\mu\right)\right| \\ &\le \left\|g-\int_\Omega g d\mu\right\|_{\Omega,p}\left(\int_\Omega L\left(\left|\Phi'\circ g\right|^q, \left|\Phi'\left(\int_\Omega g d\mu\right)\right|^q\right)d\mu\right)^{1/q} \\ &\le \frac{1}{2^{1/q}}\left\|g-\int_\Omega g d\mu\right\|_{\Omega,p}\left[\left|\Phi'\left(\int_\Omega g d\mu\right)\right|^q + \int_\Omega \left|\Phi'\circ g\right|^q d\mu\right]^{1/q} \\ &\le \frac{1}{2^{1/q}}\left\|g-\int_\Omega g d\mu\right\|_{\Omega,p}\left[\left|\Phi'\left(\int_\Omega g d\mu\right)\right| + \left\|\Phi'\circ g\right\|_{\Omega,q}\right] \end{aligned}$$

and

$$(4.9)\qquad \begin{aligned} &\left|\int_\Omega \Phi\circ g d\mu - \Phi\left(\frac{a+b}{2}\right)\right| \\ &\le \left\|g-\frac{a+b}{2}\right\|_{\Omega,p}\left(\int_\Omega L\left(\left|\Phi'\circ g\right|^q, \left|\Phi'\left(\frac{a+b}{2}\right)\right|^q\right)d\mu\right)^{1/q} \\ &\le \frac{1}{2^{1/q}}\left\|g-\frac{a+b}{2}\right\|_{\Omega,p}\left[\left|\Phi'\left(\frac{a+b}{2}\right)\right|^q + \int_\Omega \left|\Phi'\circ g\right|^q d\mu\right]^{1/q} \\ &\le \frac{1}{2^{1/q}}\left\|g-\frac{a+b}{2}\right\|_{\Omega,p}\left[\left|\Phi'\left(\frac{a+b}{2}\right)\right| + \left\|\Phi'\circ g\right\|_{\Omega,q}\right]. \end{aligned}$$

Proof. Since $|\Phi'|^q$ is log-convex on the interval $[a,b]$, then

$$\begin{aligned} \int_0^1 \left|\Phi'\left((1-s)x + sg(t)\right)\right|^q ds &\le \int_0^1 \left|\Phi'(x)\right|^{q(1-s)}\left|g(t)\right|^{sq} ds \\ &= \left|\Phi'(x)\right|^q \int_0^1 \left|\frac{g(t)}{\Phi'(x)}\right|^{sq} ds \\ &= L\left(\left|\Phi'(g(t))\right|^q, \left|\Phi'(x)\right|^q\right) \end{aligned}$$

for any $x\in[a,b]$ and $t\in\Omega$.

Then

$$\left(\int_\Omega\left(\int_0^1 \left|\Phi'\left((1-s)x+sg\right)\right|^q ds\right)d\mu\right)^{1/q} \le \left(\int_\Omega L\left(\left|\Phi'\circ g\right|^q, \left|\Phi'(x)\right|^q\right)d\mu\right)^{1/q}$$

and by (4.4) we get the first inequality in (4.7).

Since, in general

$$L(\alpha,\beta)\le\frac{\alpha+\beta}{2} \text{ for any } \alpha,\beta>0,$$

then

$$\begin{aligned} \int_\Omega L\left(\left|\Phi'\circ g\right|^q, \left|\Phi'(x)\right|^q\right)d\mu &\le \frac{1}{2}\int_\Omega\left[\left|\Phi'\circ g\right|^q + \left|\Phi'(x)\right|^q\right]d\mu \\ &= \frac{1}{2}\left[\left|\Phi'(x)\right|^q + \int_\Omega\left|\Phi'\circ g\right|^q d\mu\right] \end{aligned}$$

and we get the second inequality in (4.7).

The last part is obvious. □

5. Applications for f-Divergence

One of the important issues in many applications of Probability Theory is finding an appropriate measure of *distance* (or *difference* or *discrimination*) between two probability distributions. A number of divergence measures for this purpose have been proposed and extensively studied by Jeffreys [71], Kullback and Leibler [78], Rényi [91], Havrda and Charvat [69], Kapur [74], Sharma and Mittal [96], Burbea and Rao [12], Rao [90], Lin [79], Csiszár [20], Ali and Silvey [1], Vajda [104], Shioya and Da-te [98] and others (see for example [81] and the references therein).

These measures have been applied in a variety of fields such as: anthropology [90], genetics [81], finance, economics, and political science [97], [101], [102], biology [88], the analysis of contingency tables [66], approximation of probability distributions [18], [75], signal processing [72], [73] and pattern recognition [7], [17]. A number of these measures of distance are specific cases of Csiszár f-divergence and so further exploration of this concept will have a flow on effect to other measures of distance and to areas in which they are applied.

Assume that a set Ω and the σ-finite measure μ are given. Consider the set of all probability densities on μ to be $\mathcal{P} := \left\{p|p : \Omega \to \mathbb{R},\, p(t) \geq 0,\, \int_\Omega p(t)\, d\mu(t) = 1\right\}$. The Kullback-Leibler divergence [78] is well known among the information divergences. It is defined as:

$$D_{KL}(p,q) := \int_\Omega p(t) \ln\left[\frac{p(t)}{q(t)}\right] d\mu(t), \quad p,q \in \mathcal{P}, \tag{5.1}$$

where ln is to base e.

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence. These are the: *variation distance* D_v, *Hellinger distance* D_H [70], χ^2*-divergence* D_{χ^2}, α*-divergence* D_α, *Bhattacharyya distance* D_B [8], *Harmonic distance* D_{Ha}, *Jeffrey's distance* D_J [71], *triangular discrimination* D_Δ [100], etc... They are defined as follows:

$$D_v(p,q) := \int_\Omega |p(t) - q(t)|\, d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.2}$$

$$D_H(p,q) := \int_\Omega \left|\sqrt{p(t)} - \sqrt{q(t)}\right| d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.3}$$

$$D_{\chi^u}(p,q) := \int_\Omega p(t)\left[\left(\frac{q(t)}{p(t)}\right)^r - 1\right] d\mu(t),\ u \geq 2,\ p,q \in \mathcal{P}; \tag{5.4}$$

$$D_\alpha(p,q) := \frac{4}{1-\alpha^2}\left[1 - \int_\Omega [p(t)]^{\frac{1-\alpha}{2}} [q(t)]^{\frac{1+\alpha}{2}} d\mu(t)\right], \quad p,q \in \mathcal{P}; \tag{5.5}$$

$$D_B(p,q) := \int_\Omega \sqrt{p(t)\, q(t)}\, d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.6}$$

$$D_{Ha}(p,q) := \int_\Omega \frac{2p(t)\, q(t)}{p(t) + q(t)} d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.7}$$

$$D_J(p,q) := \int_\Omega [p(t) - q(t)] \ln\left[\frac{p(t)}{q(t)}\right] d\mu(t), \quad p,q \in \mathcal{P}; \tag{5.8}$$

$$D_{\Delta}(p,q) := \int_{\Omega} \frac{[p(t)-q(t)]^2}{p(t)+q(t)} d\mu(t), \quad p,q \in \mathcal{P}. \tag{5.9}$$

For other divergence measures, see the paper [74] by Kapur or the book on line [99] by Taneja.

Csiszár f-divergence is defined as follows [21]

$$I_f(p,q) := \int_{\Omega} p(t) f\left[\frac{q(t)}{p(t)}\right] d\mu(t), \quad p,q \in \mathcal{P}, \tag{5.10}$$

where f is convex on $(0,\infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u=1$. By appropriately defining this convex function, various divergences are derived. Most of the above distances (5.1)-(5.9), are particular instances of Csiszár f-divergence. There are also many others which are not in this class (see for example [99]). For the basic properties of Csiszár f-divergence see [21], [22] and [104].

The following result holds:

Proposition 3. *Let $f:(0,\infty)\to\mathbb{R}$ be a convex function with the property that $f(1)=0$. Assume that $p,q\in\mathcal{P}$ and there exists the constants $0<r<1<R<\infty$ such that*

$$r \le \frac{q(t)}{p(t)} \le R \text{ for } \mu\text{-a.e. } t\in\Omega. \tag{5.11}$$

If $|f'|$ is h-convex on the interval $[r,R]$, then we have the inequalities

$$0 \le I_f(p,q) \le \int_0^1 h(s)\,ds \begin{cases} (R-r)\left[|\Phi'(1)| + I_{|f'|}(p,q)\right], \\ \\ D_v(p,q)\left[|\Phi'(1)| + \|f'\|_{[r,R],\infty}\right]. \end{cases} \tag{5.12}$$

Proof. Applying the inequality (3.2) we have

$$\begin{aligned} &\left|\int_{\Omega} p(t) f\left(\frac{q(t)}{p(t)}\right) d\mu(t) - f(1)\right| \\ &\le \int_0^1 h(s)\,ds \\ &\times \begin{cases} ess\sup_{t\in\Omega}\left|\frac{q(t)}{p(t)}-1\right|\left[|\Phi'(1)| + \int_{\Omega} p(t)\left|f'\left(\frac{q(t)}{p(t)}\right)\right| d\mu(t)\right], \\ \\ \|q-p\|_{\Omega,1}\left[|\Phi'(1)| + ess\sup_{t\in\Omega}\left|f'\left(\frac{q(t)}{p(t)}\right)\right|\right] \end{cases} \\ &\le \int_0^1 h(s)\,ds \\ &\times \begin{cases} (R-r)\left[|\Phi'(1)| + I_{|f'|}(p,q)\right], \\ \\ D_v(p,q)\left[|\Phi'(1)| + ess\sup_{x\in[r,R]}|f'(x)|\right] \end{cases} \end{aligned}$$

and the inequality (5.12) is obtained. □

Consider the convex function $f(x)=x^u-1$, $u\ge 2$. Then $f(1)=0$, $f'(x)=ux^{u-1}$ and $|f'|$ is convex on the interval $[r,R]$ for any $0<r<1<R<\infty$.

Then by (5.12) we have

$$0 \le D_{\chi^u}(p,q) \le \frac{1}{2}u \begin{cases} (R-r)\left[1+D_{\chi^{u-1}}(p,q)\right], \\ \\ D_v(p,q)\left(1+R^{u-1}\right) \end{cases} \tag{5.13}$$

provided

$$r \le \frac{q(t)}{p(t)} \le R \text{ for } \mu\text{-a.e. } t \in \Omega.$$

If we consider the convex function $f:(0,\infty)\to\mathbb{R}$, $f(t)=-\ln t$ then

$$\begin{aligned} I_f(p,q) &:= -\int_\Omega p(t)\ln\left[\frac{q(t)}{p(t)}\right]d\mu(t) = \int_\Omega p(t)\ln\left[\frac{p(t)}{q(t)}\right]d\mu(t) \\ &= D_{KL}(p,q). \end{aligned}$$

We have $f'(t)=-\frac{1}{t}$ and $|f'|$ is convex on the interval $[r,R]$ for any $0<r<1<R<\infty$. If we apply the inequality (5.12) we have

$$0 \le D_{KL}(p,q) \le \frac{1}{2} \begin{cases} (R-r)\left[2+D_{\chi^2}(q,p)\right], \\ \\ \frac{r+1}{r}D_v(p,q), \end{cases} \tag{5.14}$$

provided

$$r \le \frac{q(t)}{p(t)} \le R \text{ for } \mu\text{-a.e. } t \in \Omega.$$

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