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SOME INEQUALITIES ON THE s-GODUNOVA-LEVIN TYPE FUNCTIONS

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ABSTRACT. In this paper, firstly, we obtained two new $s-$Godunova Levin type inequalities about "the mean value Theorem for integrals". Secondly, some inequalities were proved for mappings $q-th$ powers of first derivatives belong to the class $Q_s(I)$ by using the Čebyšev's inequality, Hölder inequality, Power means inequality and some other classical inequalities. The results obtained are euryhythmic by means of literature. Finally, some error estimates for the Trapezoidal formula are also given.

1. INTRODUCTION

One of the most famous inequalities for convex functions is Hadamard's inequality. This double inequality is stated as follows(see for example [7] and [8]): Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

For several recent results concerning the inequality (1.1) we refer the interested reader to ([7], [8], [16]-[18]).

Definition 1. *([15])We say that $f : I \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class Q(I) if f is non-negative and for all $x, y \in I$ and $t \in (0,1)$ we have*

$$f(tx+(1-t)y) \leq \frac{1}{t}f(x) + \frac{1}{1-t}f(y). \tag{1.2}$$

Some further properties of this class of functions can be found in [9]-[14]. Among others, its has been noted that non-negative monotone and non-negative convex functions belong to this class of functions. The above concept can be extended for functions $f : C \subseteq X \to [0,\infty)$ where C is a convex subset of the real or complex linear space X and the inequality (1.2) is satisfied for any vectors $x, y \in C$ and $t \in (0,1)$. If the function $f : C \subseteq X \to \mathbb{R}$ is non-negative and convex, then it is of Godunova-Levin type.

Definition 2. *Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex (in the second sense)*

$$f(tx+(1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0,\infty)$ and $t \in [0,1]$.

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For some properties of this class of functions see [21]-[24], [26].

This concept can be extended for functions defined on convex subsets of linear spaces in the same way as above replacing the interval I be the corresponding convex subset C of the linear space X:

Definition 3. *([27])We say that the function $f : C \subset X \to [0,\infty)$ is of $s-$Godunova-Levin type, with $s \in [0,1]$, if*

$$f(tx+(1-t)y) \leq \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y)$$

for all $t \in (0,1)$ and $x, y \in C$.

We denote by $Q_s(C)$ the class of $s-$Godunova-Levin functions defined on C.

We observe that for $s = 0$, we obtain the class of p-functions while for $s = 1$ we obtain the class of Godunova-Levin. Thus

$$P(C) = Q_0(C) \subseteq Q_{s_1}(C) \subseteq Q_{s_2}(C) \subseteq Q_1(C) = Q(C)$$

for $0 \leq s_1 \leq s_2 \leq 1$.

We recall the well-known Hölder's integral inequality which can be stated as follows, see [29].

Theorem 1. *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on $[a,b]$ and if $|f|^p$ and $|g|^q$ are integrable functions on $[a,b]$, then*

$$\int_a^b |f(x)g(x)|\, dx \leq \left(\int_a^b |f(x)|^p\, dx \right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q\, dx \right)^{\frac{1}{q}}$$

with equality holding if and only if $A|f(x)|^p = B|g(x)|^q$ almost everywhere, where A and B are constants.

Theorem 2. *(Power Mean Inequality)(see [30]) Let $x = (x_i)$, $p = (p_i)$ be two positive n-tubles and let $r \in \mathbb{R} \cup \{+\infty, -\infty\}$, $i = 1, 2, ..., n$. Then, on putting $p_n = \sum_{k=1}^{n} p_k$, the r th power mean of x with weights p is defined by*

$$M_n^{[r]} = \begin{cases} \left(\frac{1}{p_n} \sum_{k=1}^{n} p_k x_k^r \right)^{\frac{1}{r}}, & r \neq +\infty, 0, -\infty \\ \left(\prod_{k=1}^{n} x_k^{p_k} \right)^{\frac{1}{p_n}}, & r = 0 \\ \min(x_1, x_2, ..., x_n), & r = -\infty \\ \max(x_1, x_2, ..., x_n), & r = +\infty \end{cases}$$

Note that if $-\infty \leq r < s \leq \infty$, then

$$M_n^{[r]} \leq M_n^{[s]}$$

(see, e.g., [28]).

Theorem 3. *([7]) Let $f \in Q(I)$, $a, b \in I$ with $a < b$ and $f \in L_1[a,b]$. Then one has the inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{4}{b-a} \int_a^b f(x)dx. \tag{1.3}$$

Theorem 4. *([7]) Let $f \in P(I)$, $a, b \in I$ with $a < b$ and $f \in L_1[a,b]$. Then one has the inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_a^b f(x)dx \leq 2(f(a)+f(b)). \tag{1.4}$$

Both inequalities are best possible.

We need the following inequalities:

Theorem 5. *(see [28]) Let $f, g : [a,b] \to \mathbb{R}$ be integrable functions, both increasing or both decreasing. Furthermore, let $p : [a,b] \to \mathbb{R}_+$ be an integrable function. Then*

$$\int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx \leq \int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx. \tag{1.5}$$

If one of the functions f or g is nonincreasing and the other nondecreasing then the inequality in (1.5) is reversed. Inequality (1.5) is known in the literature as Čebyšev's inequality and so are the following special cases of (1.5):

$$\frac{1}{b-a}\int_a^b f(x)dx \int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx$$

and

$$\int_0^1 f(x)dx \int_0^1 g(x)dx \leq \int_0^1 f(x)g(x)dx.$$

Now, we give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper, see([5]).

Definition 4. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a^+}^{\alpha}f$ and $J_{b^-}^{\alpha}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a^+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a$$

and

$$J_{b^-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad x < b$$

respectively where $\Gamma(\alpha) = \int\limits_0^{\infty} e^{-u}u^{\alpha-1}du$. Here is $J_{a^+}^{0}f(x) = J_{b^-}^{0}f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.
For some recent results connected with fractional integral inequalities see ([1]-[6], [19], [20]).
In [25], Özdemir *et. al.* proved the following result for fractional integrals.

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I with $a < r$, $a, r \in I$. If $f' \in L[a,r]$, then the following equality for fractional integrals holds:*

$$\begin{aligned} &\frac{f(a)+f(r)}{2} - \frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J_{r^-}^{\alpha}f(a) + J_{a^+}^{\alpha}f(r)\right] \\ = \ &\frac{r-a}{2}\int_0^1 \left[(1-t)^{\alpha} - t^{\alpha}\right] f'(r+(a-r)t)dt. \end{aligned}$$

The main of this study is to obtain the inequalities for class of s-Godunova-Levin type functions.

2. MAIN RESULTS

Theorem 6. *Let $f \in Q_s(C)$, with $a < b$ and $f \in L_1[a,b]$, $C = [a,b]$, $s \in [0,1]$. Then one has the inequalities*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{s+1}}{b-a}\int_a^b f(x)dx \tag{2.1}$$

$$\frac{\Gamma((1+s)^2)}{\Gamma(2+2s)(b-a)}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1} \tag{2.2}$$

and

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{1-s}, \quad s \in [0,1) \tag{2.3}$$

Proof. $f \in Q_s(C)$, we have for all $x, y \in C = [a,b]$ with $t = \frac{1}{2}$;

$$2^s(f(x)+f(y)) \geq f\left(\frac{x+y}{2}\right)$$

Now, if we choose $x = ta + (1-t)b$, $y = (1-t)a + tb$, we have

$$2^s(f(ta+(1-t)b) + f((1-t)a+tb)) \geq f\left(\frac{a+b}{2}\right).$$

By integrating, we have that

$$2^s\left[\int_0^1 f(ta+(1-t)b)dt + \int_0^1 f((1-t)a+tb)dt\right] \geq f\left(\frac{a+b}{2}\right) \tag{2.4}$$

On the other hand

$$\begin{aligned}\int_0^1 f(ta+(1-t)b)dt &= \int_0^1 f((1-t)a+tb)dt \\ &= \frac{1}{b-a}\int_0^1 f(x)dx\end{aligned}$$

we get the inequality (2.1) from (2.4).

For the proof of (2.2), if $f \in Q_s(C)$ for all $a, b \in C$ and $t \in (0,1)$, it yields

$$t^s(1-t)^s f(ta+(1-t)b) \leq (1-t)^s f(a) + t^s f(b)$$

and

$$t^s(1-t)^s f((1-t)a+tb) \leq t^s f(a) + (1-t)^s f(b).$$

By adding these inequalities and integrating over [0,1], we find that

$$\int_0^1 t^s(1-t)^s\left[f(ta+(1-t)b) + f((1-t)a+tb)\right]dt \leq \frac{2}{s+1}\left[f(a)+f(b)\right].$$

Now, by a simple computation, we have

$$\int_0^1 t^s(1-t)^s f(ta+(1-t)b)dt \text{ and } \int_0^1 t^s(1-t)^s f((1-t)a+tb)dt.$$

Let be $g(t)=t^s(1-t)^s$. We take symetric the function $g(t)$ and f, respectively $\frac{1}{2}$ and $\frac{a+b}{2}$. Also, let the functions f and g both be either increasing or decreasing. By applying Čebyšev's inequality, then we have

$$\begin{aligned}\int_0^1 t^s(1-t)^s f(ta+(1-t)b)dt &= \int_0^1 t^s(1-t)^s f((1-t)a+tb)dt \\ &\geq \int_0^1 t^s(1-t)^s dt \int_0^1 f((1-t)a+tb)dt \\ &= \frac{\Gamma((1+s)^2)}{\Gamma(2+2s)(b-a)}\int_a^b f(x)dx.\end{aligned}$$

To obtain the inequality (2.3), as $f\in Q_s(C)$, we have

$$f(ta+(1-t)b)\leq t^{-s}f(a)+(1-t)^{-s}f(b)$$

integrating this inequality on $[0,1]$, we get

$$\begin{aligned}\int_0^1 f(ta+(1-t)b)dt &\leq f(a)\int_0^1 t^{-s}dt+f(b)\int_0^1(1-t)^{-s}dt \\ &= \frac{f(a)+f(b)}{1-s}, \qquad (s\in[0,1))\end{aligned}$$

As the change of variable $x=ta+(1-t)b$ gives us that

$$\int_0^1 f(ta+(1-t)b)dt=\frac{1}{b-a}\int_a^b f(x)dx$$

which completes the proof of the inequality (2.3). □

Theorem 7. *Combining the inequalities* (2.1) *and* (2.3) *under the conditions of Theorem 6 except for* $s\in[0,1)$ *we get*

$$f\left(\frac{a+b}{2}\right)\leq\frac{2^{s+1}}{b-a}\int_a^b f(x)dx\leq 2^{s+1}\frac{f(a)+f(b)}{1-s}.$$

Remark 1. *If we choose $s=1$ in (2.1) and (2.2), respectively, we obtain the inequality (1.3) and right hand side of (1.4), respectively.*

Theorem 8. *Let $f:I\subseteq\mathbb{R}\to[0,\infty)$ be a differentiable mapping on I, $a,r\in I$ and $a<r$. If $|f'|\in Q_\alpha(I)$ with $\alpha\in[0,1)$, $t\in(0,1)$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|\frac{f(a)+f(r)}{2}-\frac{\Gamma(\alpha+1)}{2(r-a)^\alpha}\left[J_{r^-}^\alpha f(a)+J_{a^+}^\alpha f(r)\right]\right| \\ &\leq \frac{(r-a)}{2}\left\{\pi\alpha\cos ec(\pi\alpha)\left[|f'(r)|+|f'(a)|\right]+2\beta_{\frac{1}{2}}(1-\alpha,1+\alpha)\right\}\end{aligned} \tag{2.5}$$

where $\beta_x(y,z)=\int_0^x t^{y-1}(1-t)^{z-1}dt$, $0\leq x\leq 1$, is incomplete Beta function.

Proof. Using the Lemma 1 and $|f'|\in Q_\alpha(I)$, it follows that

$$\begin{aligned}&\left|\frac{f(a)+f(r)}{2}-\frac{\Gamma(\alpha+1)}{2(r-a)^\alpha}\left[J_{r^-}^\alpha f(a)+J_{a^+}^\alpha f(r)\right]\right| \\ &\leq \frac{(r-a)}{2}\int_0^1\left|(1-t)^\alpha-t^\alpha\right|\left|f'(r+(a-r)t)\right|dt.\end{aligned}$$

Since

$$\left|f'(r+(a-r)t)\right| = \left|f'(ta+(1-t)r)\right| \leq \frac{1}{t^{\alpha}}\left|f'(a)\right| + \frac{1}{(1-t)^{\alpha}}\left|f'(r)\right|$$

$$\begin{aligned}
&\left|\frac{f(a)+f(r)}{2} - \frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J_{r-}^{\alpha}f(a)+J_{a+}^{\alpha}f(r)\right]\right| \\
\leq\ & \frac{(r-a)}{2}\int_0^1 \left|(1-t)^{\alpha}-t^{\alpha}\right|\left[\frac{1}{t^{\alpha}}\left|f'(a)\right| + \frac{1}{(1-t)^{\alpha}}\left|f'(r)\right|\right]dt \\
=\ & \frac{(r-a)}{2}\left\{\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]\left[\frac{1}{t^{\alpha}}\left|f'(a)\right| + \frac{1}{(1-t)^{\alpha}}\left|f'(r)\right|\right]dt\right. \\
&\left. + \int_{\frac{1}{2}}^1\left[t^{\alpha}-(1-t)^{\alpha}\right]\left[\frac{1}{t^{\alpha}}\left|f'(a)\right| + \frac{1}{(1-t)^{\alpha}}\left|f'(r)\right|\right]dt\right\}
\end{aligned}$$

On the other hand; by a simple computation, we have

$$\begin{aligned}
\int_0^{\frac{1}{2}}\left[\frac{(1-t)^{\alpha}-t^{\alpha}}{t^{\alpha}}\right]dt &= \beta_{\frac{1}{2}}(1-\alpha,1+\alpha)-\frac{1}{2} \\
\int_0^{\frac{1}{2}}\left[\frac{(1-t)^{\alpha}-t^{\alpha}}{(1-t)^{\alpha}}\right]dt &= \frac{1}{2}-\beta_{\frac{1}{2}}(\alpha+1,1-\alpha) \\
\int_{\frac{1}{2}}^1\left[\frac{t^{\alpha}-(1-t)^{\alpha}}{t^{\alpha}}\right]dt &= \frac{1}{2}-\pi\alpha\cos ec(\pi\alpha)+\beta_{\frac{1}{2}}(\alpha+1,1-\alpha)
\end{aligned}$$

and

$$\int_{\frac{1}{2}}^1\left[\frac{t^{\alpha}-(1-t)^{\alpha}}{(1-t)^{\alpha}}\right]dt = \pi\alpha\cos ec(\pi\alpha)-\beta_{\frac{1}{2}}(1+\alpha,1-\alpha)-\frac{1}{2}.$$

Since, finally

$$\left|f'(r)\right|-\left|f'(a)\right| \leq \left|f'(r)\right|+\left|f'(a)\right|.$$

Hence, we obtain inequalty (2.5). □

Theorem 9. *Let $f : I \subset \mathbb{R} \to [0,\infty)$ be a differentiable mapping an I, $a, r \in I$ and $a < r$. If $|f'|^q \in Q_s(I)$ with $\alpha \in [0,1]$, $t \in (0,1)$, then the following inequality for fractional integrals hold:*

$$\begin{aligned}
&\left|\frac{f(a)+f(r)}{2} - \frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J_{r-}^{\alpha}f(a)+J_{a+}^{\alpha}f(r)\right]\right| \\
\leq\ & \frac{(r-a)}{2(\alpha p+1)^{\frac{1}{p}}(1-s)^{\frac{1}{q}}}\left(\left|f'(a)\right|^q+\left|f'(r)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $s \in [0,1)$, $\Gamma(.)$ is Gamma function.

Proof. From Lemma 1 and using Hölder inequality with properties of modulus, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(r)}{2}-\frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J_{r^-}^{\alpha}f(a)+J_{a^+}^{\alpha}f(r)\right]\right| \\
\leq\ & \frac{(r-a)}{2}\int_0^1 \left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'(r+(a-r)t)\right|dt \\
\leq\ & \frac{(r-a)}{2}\left(\int_0^1 \left|(1-t)^{\alpha}-t^{\alpha}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(r+(a-r)t)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

We know that for $\alpha\in(0,1]$ and $\forall t_1,t_2\in(0,1)$,

$$\left|t_1^{\alpha}-t_2^{\alpha}\right|\leq\left|t_1-t_2\right|^{\alpha},$$

therefore

$$\begin{aligned}
\int_0^1 \left|(1-t)^{\alpha}-t^{\alpha}\right|^p dt &\leq \int_0^1 |1-2t|^{\alpha p}dt \\
&= \int_0^{\frac{1}{2}}(1-2t)^{\alpha p}dt+\int_{\frac{1}{2}}^1 (2t-1)^{\alpha p}dt \\
&= \frac{1}{\alpha p+1}.
\end{aligned}$$

Since $|f'|\in Q_s(I)$, we obtain

$$\left|f'(r+(a-r)t)\right|=\left|f'(ta+(1-t)r)\right|\leq\frac{1}{t^s}\left|f'(a)\right|+\frac{1}{(1-t)^s}\left|f'(r)\right|.$$

Hence we get

$$\begin{aligned}
&\left|\frac{f(a)+f(r)}{2}-\frac{\Gamma(\alpha+1)}{2(r-a)^{\alpha}}\left[J_{r^-}^{\alpha}f(a)+J_{a^+}^{\alpha}f(r)\right]\right| \\
\leq\ & \frac{(r-a)}{2(\alpha p+1)^{\frac{1}{p}}(1-s)^{\frac{1}{q}}}\left(\left|f'(a)\right|^q+\left|f'(r)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

which completes the proof. □

Corollary 1. *In Theorem 9, if we choose $\alpha=1$ and $s=0$, then we have*

(2.6)

$$\begin{aligned}
\left|\frac{f(a)+f(r)}{2}-\frac{1}{(r-a)}\int_a^r f(t)dt\right| &\leq \frac{(r-a)}{2(p+1)^{\frac{1}{p}}(1-s)^{\frac{1}{q}}}\left(\left|f'(a)\right|^q+\left|f'(r)\right|^q\right)^{\frac{1}{q}} \\
&\leq \frac{(r-a)}{2}\left(\left|f'(a)\right|+\left|f'(r)\right|\right).
\end{aligned}$$

Proof. Let $a_1=\left|f'(a)\right|^q$, $b_1=\left|f'(r)\right|^q$, $0<\frac{1}{q}<1$ for $q>1$. Using the fact

$$\sum_{i=1}^n (a_i+b_i)^r\leq\sum_{i=1}^n {a_i}^r+\sum_{i=1}^n {b_i}^r$$

for $a_1,a_2,...,a_n\geq 0$, $b_1,b_2,...,b_n\geq 0$, we obtain the inequality (2.6) and since $\frac{1}{2}\leq(\frac{1}{p+1})^{\frac{1}{p}}\leq 1$, $p\in(0,1)$. □

Theorem 10. *Let $f : I \subset \mathbb{R} \to [0, \infty)$ be a differentiable mapping an I, $a, r \in I$, $a < r$ and $q \geq 1$. If $|f'|^q \in Q_\alpha(I)$ with $\alpha \in [0, 1)$ $t \in (0, 1)$, then the following inequality for fractional integrals hold:*

(2.7)

$$\left| \frac{f(a) + f(r)}{2} - \frac{\Gamma(\alpha+1)}{2(r-a)^\alpha} \left[J_{r-}^\alpha f(a) + J_{a+}^\alpha f(r) \right] \right|$$
$$\leq \frac{(r-a)}{2^{\frac{1}{q}} (\alpha+1)^{1-\frac{1}{q}}} \left(1 - \frac{1}{2^\alpha}\right)^{1-\frac{1}{q}} \left[(2\beta_{\frac{1}{2}}(1-\alpha, 1+\alpha) - \pi\alpha \cos ec(\pi\alpha)) \right]^{\frac{1}{q}} \left(|f'(a)| + |f'(r)| \right).$$

Proof. From Lemma 1 and using the well known power-mean inequality, we have

$$\left| \frac{f(a) + f(r)}{2} - \frac{\Gamma(\alpha+1)}{2(r-a)^\alpha} \left[J_{r-}^\alpha f(a) + J_{a+}^\alpha f(r) \right] \right|$$
$$\leq \frac{(r-a)}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| \, |f'(r + (a-r)t)| \, dt$$
$$\leq \frac{(r-a)}{2} \left(\int_0^1 |(1-t)^\alpha - t^\alpha| \, dt \right)^{1-\frac{1}{q}} \left(\int_0^1 |(1-t)^\alpha - t^\alpha| \, |f'(r + (a-r)t)|^q \, dt \right)^{\frac{1}{q}}.$$

On the other hand, we have

$$\int_0^1 |(1-t)^\alpha - t^\alpha| \, dt = \int_0^{\frac{1}{2}} [(1-t)^\alpha - t^\alpha] \, dt + \int_{\frac{1}{2}}^1 [t^\alpha - (1-t)^\alpha] \, dt$$
$$= \frac{2}{\alpha+1} \left(1 - \frac{1}{2^\alpha}\right).$$

Since $|f'| \in Q_\alpha(I)$, we have

$$|f'(r + (a-r)t)|^q = |f'(ta + (1-t)r)|^q \leq \frac{1}{t^\alpha} |f'(a)|^q + \frac{1}{(1-t)^\alpha} |f'(r)|^q$$

and

$$\int_0^1 |(1-t)^\alpha - t^\alpha| \, |f'(r + (a-r)t)|^q \, dt$$
$$\leq \int_0^1 |(1-t)^\alpha - t^\alpha| \left[\frac{1}{t^\alpha} |f'(a)|^q + \frac{1}{(1-t)^\alpha} |f'(r)|^q \right] dt$$
$$\leq \int_0^{\frac{1}{2}} [(1-t)^\alpha - t^\alpha] \left[\frac{1}{t^\alpha} |f'(a)|^q + \frac{1}{(1-t)^\alpha} |f'(r)|^q \right] dt$$
$$+ \int_{\frac{1}{2}}^1 [t^\alpha - (1-t)^\alpha] \left[\frac{1}{t^\alpha} |f'(a)|^q + \frac{1}{(1-t)^\alpha} |f'(r)|^q \right] dt.$$

and since $\sum_{i=1}^n (a_i + b_i)^r \leq \sum_{i=1}^n a_i{}^r + \sum_{i=1}^n b_i{}^r$, we obtain the required inequality (2.7). □

3. APPLICATIONS TO NUMERICAL INTEGRATION

We may not be given a formula for $f(x)$ as a function of x. For instance, $f(x)$ may be an unknown function whose values are at certain points of the interval $[a, b]$. In this cases, we investigate the problem of approximating the value of the integral $I = \int_a^b f(x)dx$ using only the values of $f(x)$ at finetely many points of $[a, b]$. Obtaining such an approximation is called numerical integration. That is why, there are three methods for evaluating definite integrals numerically. One of them is Trapezoid Rule.

Let d be a division of the interval $[a, r]$, i.e., $d : a = x_0 < x_1 < ... < x_{n-1} < x_n = r$, and consider the trapezoidal formula

$$T_n(f, d) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i).$$

So, the following approximation of the integral $\int_a^b f(x)dx$ holds:

$$\int_a^b f(x)dx \approxeq T_n(f, d) + E_n(f, d)$$

where the aproximation error $E_n(f, d)$ of the integral $\int_a^b f(x)dx$ by the trapezoidal formula $T_n(f, d)$ satisfies

$$|E_n(f, d)| \leq \frac{M}{12} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^3.$$

We shall propose some new estimates of the remainder term $E_n(f, d)$.

Proposition 1. *Let f be a differentiable mapping on I°, $a, r \in I^\circ$ with $a < r$. If $|f'|$ is p-convex on $[a, r]$, then for every division d of $[a, r]$,the following holds:*

$$\begin{aligned} |E_n(f, d)| &\leq \frac{1}{2} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left(|f'(x_i)| + |f'(x_{i+1})|\right) \\ &\leq \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \max\left\{|f'(a)|, |f'(r)|\right\}. \end{aligned}$$

Proof. Applying Corollary 1 on the subinterval $[x_i, x_{i+1}]$ $(i = 0, 1, 2, ..., n-1)$ of the divison d, we get

$$\left| \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i) - \int_{x_i}^{x_{i+1}} f(x)dx \right| \leq \frac{(x_{i+1} - x_i)^2 \left(|f'(x_i)| + |f'(x_{i+1})|\right)}{2}.$$

Suming over i from 0 to $n - 1$ on taking into account that $|f'|$ is p-convex, we deduce, by the triangle inequality that

$$\begin{aligned} \left| T_n(f, d) - \int_a^b f(x)dx \right| &\leq \frac{1}{2} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left(|f'(x_i)| + |f'(x_{i+1})|\right) \\ &\leq \max\left\{|f'(x_i)|, |f'(x_{i+1})|\right\} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \\ &\leq \max\left\{|f'(a)|, |f'(r)|\right\} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2. \end{aligned}$$

□

Proposition 2. *Let f be a differentiable mapping on $I^\circ \subset I$, $a, r \in I^\circ$ with $a < r$ and let $\frac{1}{p} + \frac{1}{q} = 1$. If $|f'|^q \in Q_s(I^\circ)$ with $\alpha = 1$, $t \in (0, 1)$.Then for every division* *of* $[a,r]$*, the following holds:*

$$\begin{aligned} |E_n(f,d)| &\leq \frac{1}{2(p+1)^{\frac{1}{q}}(1-s)^{\frac{1}{q}}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left[|f'(x_i)|^q+|f'(x_{i+1})|^q\right]^{\frac{1}{q}} \\ &\leq \frac{\max\{|f'(a)|,|f'(r)|\}}{2(p+1)^{\frac{1}{q}}(1-s)^{\frac{1}{q}}}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2. \end{aligned}$$

Proof. If we apply the Theorem 9 for $\alpha=1$, the proof is similar to that of Proposition 1. □

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